

A NOTE ON CHEN'S BASIC EQUALITY FOR SUBMANIFOLDS IN A SASAKIAN SPACE FORM

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It is proved that a Riemannian manifold M isometrically immersed in a Sasakian space form $\tilde{M}(c)$ of constant φ -sectional curvature $c < 1$, with the structure vector field ξ tangent to M , satisfies Chen's basic equality if and only if it is a 3-dimensional minimal invariant submanifold.

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1. Introduction. Let \tilde{M} be an m -dimensional almost contact manifold endowed with an almost contact structure (φ, ξ, η) , that is, φ be a $(1, 1)$ -tensor field, ξ be a vector field, and η be a 1-form, such that $\varphi^2 = -I + \eta \otimes \xi$ and $\eta(\xi) = 1$. Then, $\varphi(\xi) = 0$, $\eta \circ \varphi = 0$, and m is an odd positive integer. An almost contact structure is said to be *normal*, if in the product manifold $\tilde{M} \times \mathbb{R}$ the induced almost complex structure J defined by $J(X, \lambda d/dt) = (\varphi X - \lambda \xi, \eta(X)d/dt)$ is integrable, where X is tangent to \tilde{M} , t is the coordinate of \mathbb{R} , and λ is a smooth function on $\tilde{M} \times \mathbb{R}$. The condition for an almost contact structure to be *normal* is equivalent to the vanishing of the torsion tensor $[\varphi, \varphi] + 2d\eta \otimes \xi$, where $[\varphi, \varphi]$ is the Nijenhuis tensor of φ .

Let g be a compatible Riemannian metric with the structure (φ, ξ, η) , that is, $g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$ or equivalently, $g(X, \varphi Y) = -g(\varphi X, Y)$ and $g(X, \xi) = \eta(X)$ for all $X, Y \in T\tilde{M}$. Then, \tilde{M} becomes an almost contact metric manifold equipped with the almost contact metric structure (φ, ξ, η, g) . Moreover, if $g(X, \varphi Y) = d\eta(X, Y)$, then \tilde{M} is said to have a *contact metric structure* (φ, ξ, η, g) , and \tilde{M} is called a *contact metric manifold*. A normal contact metric structure in \tilde{M} is a *Sasakian structure* and \tilde{M} is a *Sasakian manifold*. A necessary and sufficient condition for an almost contact metric structure to be a Sasakian structure is

$$(\tilde{\nabla}_X \varphi)Y = g(X, Y)\xi - \eta(Y)X, \quad X, Y \in T\tilde{M}, \quad (1.1)$$

where $\tilde{\nabla}$ is the Levi-Civita connection of the Riemannian metric g . The manifolds \mathbb{R}^{2n+1} and S^{2n+1} are equipped with standard Sasakian structures. The sectional curvature $\tilde{K}(X \wedge \varphi X)$ of a plane section spanned by a unit vector X orthogonal to ξ is called a φ -*sectional curvature*. If \tilde{M} has a constant

φ -sectional curvature c , then it is called a *Sasakian space form* and is denoted by $\tilde{M}(c)$. For more details, we refer to [2].

Let M be an n -dimensional submanifold immersed in an almost contact metric manifold $\tilde{M}(\varphi, \xi, \eta, g)$. Also let g denote the induced metric on M . We denote by h the second fundamental form of M and by A_N the shape operator associated to any vector N in the normal bundle $T^\perp M$. Then $g(h(X, Y), N) = g(A_N X, Y)$ for all $X, Y \in TM$ and $N \in T^\perp M$. The mean curvature vector is given by $nH = \text{trace}(h)$, and the submanifold M is *minimal* if $H = 0$.

For a vector field X in M , we put $\varphi X = PX + FX$, where $PX \in TM$ and $FX \in T^\perp M$. Thus, P is an endomorphism of the tangent bundle of M and satisfies $g(X, PY) = -g(PX, Y)$ for all $X, Y \in TM$. From now on, let the structure vector field ξ be tangent to M . Then we write the orthogonal direct decomposition $TM = \mathcal{D} \oplus \{\xi\}$. Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of the tangent space $T_p M$. We can define the squared norm of P by $\|P\|^2 = \sum_{i,j=1}^n g(e_i, Pe_j)^2$. For a plane section $\pi \subset T_p M$, we denote the functions $\alpha(\pi)$ and $\beta(\pi)$ of tangent space $T_p M$ into $[0, 1]$ by $\alpha(\pi) = (g(X, PY))^2$ and $\beta(\pi) = (\eta(X))^2 + (\eta(Y))^2$, where π is spanned by any orthonormal vectors X and Y .

The scalar curvature τ at $p \in M$ is given by $\tau = \sum_{i < j} K(e_i \wedge e_j)$, where $K(e_i \wedge e_j)$ is the sectional curvature of the plane section spanned by e_i and e_j . The well-known Chen's invariant δ_M on M is defined by

$$\delta_M = \tau - \inf K, \quad (1.2)$$

where $(\inf K)(p) = \inf\{K(\pi) \mid \pi \text{ is a plane section } \subset T_p M\}$. For a submanifold M in a real space form $\mathbb{R}^m(c)$, Chen [4] gave the following inequality:

$$\delta_M \leq \frac{n^2(n-2)}{2(n-1)} \|H\|^2 + \frac{1}{2}(n+1)(n-2)c. \quad (1.3)$$

He also established in [5] the similar basic inequalities for submanifolds in a complex space form. For an n -dimensional submanifold M in a Sasakian space form $\tilde{M}(c)$ tangential to the structure vector field ξ in [7], the authors established the following Chen's basic inequality.

THEOREM 1.1. *Let M be an n -dimensional ($n \geq 3$) Riemannian manifold isometrically immersed in a Sasakian space form $\tilde{M}(c)$ of constant φ -sectional curvature $c < 1$ with the structure vector field ξ tangent to M . Then,*

$$\delta_M \leq \frac{n^2(n-2)}{2(n-1)} \|H\|^2 + \frac{1}{8}\{n(n-3)c + 3n^2 - n - 8\} \quad (1.4)$$

with equality holding if and only if M admits a quasi-anti-invariant structure of rank $(n-2)$.

For certain inequalities concerned with the invariant $\delta(n_1, \dots, n_k)$, which is a generalization of δ_M , we also refer to [6].

In this note, we prove the following obstruction to the Chen's basic equality.

THEOREM 1.2. *Let M be an n -dimensional Riemannian manifold isometrically immersed in an m -dimensional Sasakian space form $\tilde{M}(c)$ of a constant φ -sectional curvature $c < 1$ with the structure vector field ξ tangent to M . Then, M satisfies the Chen's basic equality*

$$\delta_M = \frac{n^2(n-2)}{2(n-1)} \|H\|^2 + \frac{1}{8} \{n(n-3)c + 3n^2 - n - 8\}, \tag{1.5}$$

if and only if M is a 3-dimensional minimal invariant submanifold. Hence, Chen's basic equality (1.5) becomes

$$\delta_M = 2. \tag{1.6}$$

2. Proof of Theorem 1.2. First, we recall the following theorem [3].

THEOREM 2.1. *Let \tilde{M} be an m -dimensional Sasakian space form $\tilde{M}(c)$. Let M be an n -dimensional ($n \geq 3$) submanifold isometrically immersed in \tilde{M} such that $\xi \in TM$. For each plane section $\pi \subset \mathbb{D}_p$, $p \in M$,*

$$\begin{aligned} \tau - K(\pi) \leq & \frac{n^2(n-2)}{2(n-1)} \|H\|^2 + \frac{1}{8} \{n(n-3)c + 3n^2 - n - 8\} \\ & + \frac{c-1}{8} \{3\|P\|^2 - 6\alpha(\pi)\}. \end{aligned} \tag{2.1}$$

The equality in (2.1) holds at $p \in M$ if and only if there exist an orthonormal basis $\{e_1, \dots, e_n\}$ of T_pM and an orthonormal basis $\{e_{n+1}, \dots, e_m\}$ of $T_p^\perp M$ such that (a) $e_n = \xi$, (b) $\pi = \text{Span}\{e_1, e_2\}$, and (c) the shape operators $A_r \equiv A_{e_r}$, $r = n+1, \dots, m$, take the following forms:

$$A_{n+1} = \begin{pmatrix} h_{11}^{n+1} & 0 & 0 & \cdots & 0 \\ 0 & -h_{11}^{n+1} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{2.2}$$

$$A_r = \begin{pmatrix} h_{11}^r & h_{12}^r & 0 & \cdots & 0 \\ h_{12}^r & -h_{11}^r & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad r = n+2, \dots, m.$$

A submanifold M of an almost contact metric manifold \tilde{M} with $\xi \in TM$ is called a *semi-invariant submanifold* [1] of \tilde{M} if the distributions $\mathcal{D}^1 = TM \cap \varphi(TM)$ and $\mathcal{D}^0 = TM \cap \varphi(T^\perp M)$ satisfy $TM = \mathcal{D}^1 \oplus \mathcal{D}^0 \oplus \{\xi\}$. In fact, the condition $TM = \mathcal{D}^1 \oplus \mathcal{D}^0 \oplus \{\xi\}$ implies that the endomorphism P is an *f-structure* [9] on M with a $\text{rank}(P) = \dim(\mathcal{D}^1)$. A semi-invariant submanifold of an almost contact metric manifold becomes an *invariant* or an *anti-invariant submanifold* according as the anti-invariant distribution \mathcal{D}^0 is $\{0\}$ (i.e., $F = 0$) or the invariant distribution \mathcal{D}^1 is $\{0\}$ (i.e., $P = 0$) [1].

For each point $p \in M$, we put [3]

$$\delta_M^{\mathcal{D}^0}(p) = \tau(p) - (\inf_{\mathcal{D}^0} K)(p) = \inf \{K(\pi) \mid \text{plane sections } \pi \subset \mathcal{D}_p\}. \tag{2.3}$$

For $c < 1$, we prove the following result.

THEOREM 2.2. *Let M be an n -dimensional ($n \geq 3$) submanifold isometrically immersed in a Sasakian space form $\tilde{M}(c)$ such that the structure vector field ξ is tangent to M . If $c < 1$, then*

$$\delta_M^{\mathcal{D}^0} \leq \frac{n^2(n-2)}{2(n-1)} \|H\|^2 + \frac{1}{8} \{n(n-3)c + 3n^2 - n - 8\}. \tag{2.4}$$

The equality case in (2.4) holds if and only if M is a 3-dimensional minimal invariant submanifold.

PROOF. Since $c < 1$, in order to estimate δ_M , we minimize $\|P\|^2 - 2\alpha(\pi)$ in (2.1). For an orthonormal basis $\{e_1, \dots, e_n = \xi\}$ of $T_p M$ with $\pi = \text{span}\{e_1, e_2\}$, we write

$$\|P\|^2 - 2\alpha(\pi) = \sum_{i,j=3}^n g(e_i, \varphi e_j)^2 + 2 \sum_{j=3}^n \{g(e_1, \varphi e_j)^2 + g(e_2, \varphi e_j)^2\}. \tag{2.5}$$

Thus, the minimum value of $\|P\|^2 - 2\alpha(\pi)$ is 0, provided that

$$\text{span} \{\varphi e_j \mid j = 3, \dots, n\} \tag{2.6}$$

is orthogonal to the tangent space $T_p M$. Thus, we have (2.4) with equality case holding if and only if M is a semi-invariant such that $\text{rank}(P) = 2$. This means that

$$TM = \mathcal{D}^1 \oplus \mathcal{D}^0 \oplus \{\xi\} \tag{2.7}$$

with the $\dim(\mathcal{D}^1) = 2$. From (2.2), we see that M is minimal.

Next, from [8, Proposition 5.2], we have

$$A_{FX}Y - A_{FY}X = \eta(X)Y - \eta(Y)X, \quad X, Y \in \mathcal{D}^0 \oplus \{\xi\}. \quad (2.8)$$

For $X \in \mathcal{D}^0$ and using (2.8), we have

$$g(X, X) = -g(A_{FX}\xi, X), \quad (2.9)$$

which in view of (2.2) becomes zero. Thus $\mathcal{D}^0 = \{0\}$, and M becomes invariant. This completes the proof. \square

From (1.2) and (2.3), it follows that $\delta_M^{\mathcal{D}^0}(p) \leq \delta_M(p)$. Hence in view of Theorem 2.2, we get the proof of Theorem 1.2.

REMARK 2.3. In Theorem 1.1, the phrase “ M admits a quasi-anti-invariant structure of rank $(n-2)$ ” is identical with the statement “ M is a semi-invariant submanifold with $\text{rank}(P) = 2$ or equivalently $\dim(\mathcal{D}^1) = 2$, where \mathcal{D}^1 is the invariant distribution.” Thus, nothing is stated here about the dimension of the anti-invariant distribution \mathcal{D}^0 . But, in the proof of Theorem 2.2, we observe that M becomes minimal and consequently invariant, which makes $\dim(\mathcal{D}^0) = 0$ and $\dim(M) = 3$.

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