First Passage Times of Two-Dimensional Brownian Motion

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Outline

1. Motivation
2. Joint Laplace Transform of τ_1 and τ_2
3. Relation to A Non-singular Bivariate Exponential Distribution
4. Numerical Results
5. Application to Default Correlation
6. Probability Distribution of $|\tau_1 - \tau_2|$
7. Conclusion

Outline

Model Setup and Notations

- Two-dimensional Brownian motion with drifts

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$$Cov(W_1(t), W_2(t)) = \rho t.$$

- First passage times: let

$$\tau_i = \inf_{t \geq 0}\{t : X_i(t) = 0\}, \quad i = 1, 2$$

be the first passage time of $X_i(t)$ to hit 0.

Motivation

Applications of First Passage Times in two dimensions

- Structural Model in Credit Risk Modeling
  -Pricing of credit default swaps (Haworth et al.(2008))
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'Finding the law of functional of a Brownian path can be either quite straightforward or quite impossible... (The question here) is innocent to state, but surprisingly tricky to answer ' –Rogers and Shepp (2006)

	Cases Studied					
	$P(\min(\tau_1, \tau_2) \leq t)$		$P(\tau_1 \leq t_1, \tau_2 \leq t_2)$		$P(\lvert\tau_1 - \tau_2\rvert \leq t)$	
	$\mu_1 = 0$ $\mu_2 = 0$	arbitrary drifts μ_1, μ_2	$\mu_1 = 0$ $\mu_2 = 0$	arbitrary drifts μ_1, μ_2	$\mu_1 = 0$ $\mu_2 = 0$	arbitrary drifts μ_1, μ_2
Spitzer(1959)	integral transform	N.A.	N.A.	N.A.	N.A.	N.A.
Iyenger(1985)	analytical solution	N.A.	joint density	N.A.	N.A.	N.A.
He et al.(1998)	cond. prob. on terminal values	cond. prob. on terminal values	N.A.	N.A.	N.A.	N.A.
Zhou(2001)	analytical solution	analytical solution	N.A.	N.A.	N.A.	N.A.
Rogers and Shepp(2006)	Laplace transform	N.A.	N.A.	N.A.	N.A.	N.A.
Metzler(2010)	corrects typos in Iyenger(1985)	N.A.	corrects typos in Iyenger(1985)	Monte Carlo simulation	Monte Carlo simulation	Monte Carlo simulation
This Paper	analytical solution	analytical solution	joint Laplace transform	joint Laplace transform	analytical solution	analytical solution

Outline

The Joint Laplace transform

$$L(x_1, x_2) = E[e^{-p_1\tau_1 - p_2\tau_2} | X(0) = (x_1, x_2)], \quad (x_1, x_2) \in \mathcal{R}^2_{++},$$

Numerical algorithms computing $P^{(x_1,x_2)}(\tau_1 \leq t_1, \tau_2 \leq t_2)$:
Double Laplace Inversion

Partial Differential Equation Leading to the Joint Laplace Transform

$$\frac{1}{2}\sigma_1^2\frac{\partial^2 u}{\partial x_1^2} + \rho\sigma_1\sigma_2\frac{\partial^2 u}{\partial x_1 \partial x_2} + \frac{1}{2}\sigma_2^2\frac{\partial^2 u}{\partial x_2^2} + \mu_1\frac{\partial u}{\partial x_1} + \mu_2\frac{\partial u}{\partial x_2} = (p_1 + p_2)u,$$

$$\begin{cases} u(x_1, x_2)|_{x_1=0} = \exp(-\Gamma_2 x_2), & \text{where } \Gamma_2 = \frac{\sqrt{\mu_2^2 + 2p_2\sigma_2^2} + \mu_2}{\sigma_2^2} > 0 \\ u(x_1, x_2)|_{x_2=0} = \exp(-\Gamma_1 x_1), & \text{where } \Gamma_1 = \frac{\sqrt{\mu_1^2 + 2p_1\sigma_1^2} + \mu_1}{\sigma_1^2} > 0 \\ |u| \le C \ \ (C > 1 \text{ is a constant}) \end{cases}$$

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Lemma: Uniqueness of Solution

Any solution to the above, if exists, must be unique and has the following representation

$$u(x_1, x_2) = E^{(x_1, x_2)}\left[e^{-p_1\tau_1 - p_2\tau_2}\right]$$

The PDE problem is Non-trivial...

- It is a modified Helmholtz Equation. Although coefficients are constant, in general solutions can be unbounded.
- It has non-homogeneous boundary conditions (exponential functions instead of zero).
- It is on an unbounded domain, the positive quadrant (instead of, e.g. a unit disk).

Change to Polar Coordinates

We proceed to solve the PDE problem, which is a modified Helmholtz equation in an infinite wedge with non-homogenous conditions.
We perform a sequence of changes of variables to reduce the PDE to a standard form.
Rewrite the Laplacian using polar coordinates r and θ above and replace ρ by α. The PDE becomes

$$\frac{1}{2}\left(\frac{\partial^2 k}{\partial r^2} + \frac{1}{r}\frac{\partial k}{\partial r} + \frac{1}{r^2}\frac{\partial^2 k}{\partial \theta^2}\right) = (p_1 + p_2 + \frac{1}{2}\gamma_1^2 + \frac{1}{2}\gamma_2^2)k$$

such that $k(r,\theta)|_{\theta=0} = \exp(-H_0 r)$, $k(r,\theta)|_{\theta=\alpha} = \exp(-H_1 r)$.

Two Solutions

The first approach is by separable solutions and Kantorovich-Lebedev transform (see Lebedev (1972)). Roger and Shepp (2006) use a same approach to obtain the Laplace transform of τ^* in the case of $\mu_1 = \mu_2 = 0$.

We generalize their method to solve the PDE problem above, that is, in the case where μ_1 and μ_2 are arbitrary.

The second approach is based on finite Fourier transform. More specifically, we consider instead a related PDE problem with nonhomogeneous elliptic PDE problem with homogenous boundary conditions.

Finite Fourior transform is then applied to reduce the PDE problem to an ODE problem, which can be solved rather efficiently.

Method 1: Kontorovich-Lebedev transform.

By separation of variable, we know that for arbitrary constants C_1, C_2,

$$K_{iv}(ar)(C_1 \cosh(\nu\theta) + C_2 \sinh(\nu\theta))$$

is a solution to the partial differential equation.

$$k(r,\theta) = \frac{2}{\pi} \int_0^\infty \frac{K_{i\nu}(ar)}{\sinh(\alpha\nu)} \left[\cosh(\beta_1\nu)\sinh((\alpha-\theta)\nu) + \cosh(\beta_0\nu)\sinh(\theta\nu)\right] d$$

For the boundary condition, the following identities of (inverse) Kontorovich-Lebedev transform

$$\frac{2}{\pi}\int_0^\infty \cosh(\beta_0\nu)K_{i\nu}(ar)d\nu = \exp(-ar\cos\beta_0) = \exp(-H_0 r),$$
$$\frac{2}{\pi}\int_0^\infty \cosh(\beta_1\nu)K_{i\nu}(ar)d\nu = \exp(-ar\cos\beta_1) = \exp(-H_1 r),$$

if $0 \leq \beta_0, \beta_1 < \frac{\pi}{2}$, which holds when both p_1 and p_2 are sufficiently large.

Method 2: Finite Fourier transform.

Recall that we want to solve

$$\frac{1}{2}\left(\frac{\partial^2 k}{\partial r^2}+\frac{1}{r}\frac{\partial k}{\partial r}+\frac{1}{r^2}\frac{\partial^2 k}{\partial \theta^2}\right)=(p_1+p_2+\frac{1}{2}\gamma_1^2+\frac{1}{2}\gamma_2^2)k$$

such that $k(r,\theta)|_{\theta=0}=\exp(-H_0 r)$, $k(r,\theta)|_{\theta=\alpha}=\exp(-H_1 r)$.
Let

$$h(r,\theta)=k(r,\theta)-\exp\left\{-G(\theta)r\right\},$$

where

$$G(\theta)=-\gamma_1\cos\theta-\gamma_2\sin\theta+\Gamma_1\sigma_1\sin(\alpha-\theta)+\Gamma_2\sigma_2\sin(\theta).$$

Then the PDE becomes

$$\begin{aligned}\frac{1}{2}\left(\frac{\partial^2 h}{\partial r^2}+\frac{1}{r}\frac{\partial h}{\partial r}+\frac{1}{r^2}\frac{\partial^2 h}{\partial \theta^2}\right)=&(p_1+p_2+\frac{1}{2}\gamma_1^2+\frac{1}{2}\gamma_2^2)h\\&-\rho\sigma_1\sigma_2\Gamma_1\Gamma_2\exp(-G(\theta)r),\end{aligned}$$

such that $h(r,\theta)|_{\theta=0}=0$, $h(r,\theta)|_{\theta=\alpha}=0$.

Method 2: Finite Fourier transform.

To solve the above PDE, we shall perform finite Fourier transform. Let $U_n(r) = \int_0^\alpha \sqrt{\frac{2}{\alpha}} \sin(\nu_n\theta) h(r,\theta) d\theta$, the PDE then becomes an ODE

$$\frac{d^2 U_n}{dr^2} + \frac{1}{r}\frac{dU_n}{dr} - \frac{\nu_n^2}{r^2} U_n = a^2 U_n - 2\rho\sigma_1\sigma_2\Gamma_1\Gamma_2 \int_0^\alpha \sqrt{\frac{2}{\alpha}} \sin(\nu_n\eta) \exp(-G(\eta)r)$$

where a is defined in the statement of the proposition. Add two boundary conditions

$$U_n(r) < \infty, \quad r \to 0, \; r \to \infty.$$

The ODE has a unique solution, and can be solved analytically.

First Representation of Solution: Finite Fourier transform

Theorem

$$L(x_1, x_2) = E^{(x_1,x_2)}(e^{-p_1\tau_1 - p_2\tau_2})$$
$$= e^{-(\gamma_1 \cos\theta + \gamma_2 \sin\theta) r} \left(\sum_{n=1}^{\infty} \sqrt{\frac{2}{\alpha}} \sin(\nu_n \theta) V_n(r) \right) + \exp(-\Gamma_1 x_1 - \Gamma_2 x_2)$$

where

$$V_n(r) = 2\rho\sigma_1\sigma_2\Gamma_1\Gamma_2 \int_{\eta=0}^{\alpha} \sqrt{\frac{2}{\alpha}} \sin(\nu_n \eta) \times$$
$$\left[K_{\nu_n}(ar) \int_{l=0}^{r} \exp(-G(\eta) l) l I_{\nu_n}(al) dl \right.$$
$$\left. + I_{\nu_n}(ar) \int_{l=r}^{\infty} \exp(-G(\eta) l) l K_{\nu_n}(al) dl \right] d\eta;$$

Second Representation of Solution: Kontorovich-Lebedev transform

Involves modified Bessel function of second kind with imaginary order

$$K_{i\nu}(x) = \int_0^\infty \exp(-x\cosh(y))\cos(\nu y)\,dy.$$

Theorem

$$L(x_1, x_2) = E^{(x_1,x_2)}\left(e^{-p_1\tau_1 - p_2\tau_2}\right)$$
$$= \frac{2}{\pi}\exp\left(-(\gamma_1\cos\theta + \gamma_2\sin\theta)r\right)$$
$$\times \int_0^\infty \frac{K_{i\nu}(ar)}{\sinh(\alpha\nu)}\left[\cosh(\beta_1\nu)\sinh((\alpha-\theta)\nu) + \cosh(\beta_0\nu)\sinh(\theta\nu)\right] d\nu$$

Outline

The One Dimensional Case

Define $J(t) = \min_{0 \leq s \leq t} X(s)$, and T_p is an exponential random variable with rate p independent of the Brownian motion. $\tau = \inf\{t : X(t) = 0\}$.

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\begin{aligned}
E^x(e^{-p\tau}) =& P^0(-J(T_p) \geq x) \\
=& \exp\left(-\frac{1}{\sigma^2}\left(\sqrt{\mu^2 + 2\sigma^2 p} + \mu\right) x\right)
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$$= \exp\left(-\frac{1}{\sigma^2}\left(\sqrt{\mu^2 + 2\sigma^2 p} + \mu\right) x\right)$$

In other words, $-J(T_p)$ is exponentially distributed.

Joint Laplace Transform of τ_1 and τ_2 v.s. A Bivariate Exponential Distribution

- Parallel argument holds in two-dimensional case:

$$
\begin{aligned}
L(x_1, x_2) &= E^{(x_1,x_2)}(e^{-p_1\tau_1 - p_2\tau_2}) \\
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- Typical features of a bivariate exponential distribution:
  (a) marginal distributions are exponential;
  (b) non-singularity: absolutely continuous in $\mathbb{R}^2$;
  (c) the lack of memory property holds, namely, for any $x_0, x, y_0, y > 0$,

$$P(\mathcal{E}_1 > x_0 + x, \mathcal{E}_2 > y_0 + y) = P(\mathcal{E}_1 > x, \mathcal{E}_2 > y)P(\mathcal{E}_1 > x_0, \mathcal{E}_2 > y_0).$$

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- $L(x_1, x_2)$ satisfies both (a) and (b), but not (c).

Outline

$P^{(x_1,x_2)}(\tau_1 \leq t_1, \tau_2 \leq t_2)$: High Volatilities

	High Volatilities: $\sigma_1 = \sigma_2 = 0.55$				
	μ_1	μ_2	**FT**	**KL**	**M.C.(std)**
	0.2	0.15	0.3350	0.3358	0.3349(0.0015)
	-0.2	0.15	0.3777	0.3783	0.3771(0.0015)
$\rho = 0.2$	0.2	-0.15	0.3659	0.3653	0.3641(0.0016)
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$\rho = 0.5$	0.2	-0.15	0.4044	0.4068	0.4037(0.0016)
	-0.2	-0.15	0.4515	0.4524	0.4496(0.0015)
	0.2	0.15	0.4266	0.4292	0.4263(0.0016)
	-0.2	0.15	0.4686	0.4703	0.4676(0.0016)
$\rho = 0.8$	0.2	-0.15	0.4552	0.4562	0.4543(0.0016)
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FT: 9-10 mins, **KL**: 77-98 mins, **M.C.**: 5-6 hours

$P^{(x_1,x_2)}(\tau_1 \leq t_1, \tau_2 \leq t_2)$: Low Volatilities

	Low Volatilities: $\sigma_1 = \sigma_2 = 0.2$				
	μ_1	μ_2	**FT**	**KL**	**M.C.(std)**
$\rho = 0.2$	0.2	0.15	0.0204	0.0210	0.0206(0.0004)
	-0.2	0.15	0.0280	0.0293	0.0285(0.0006)
	0.2	-0.15	0.0261	0.0265	0.0256(0.0007)
	-0.2	-0.15	0.0357	0.0366	0.0358(0.0005)
$\rho = 0.5$	0.2	0.15	0.0344	0.0353	0.0342(0.0007)
	-0.2	0.15	0.0464	0.0461	0.0447(0.0008)
	0.2	-0.15	0.0419	0.0423	0.0416(0.0005)
	-0.2	-0.15	0.0548	0.0555	0.0546(0.0006)
$\rho = 0.8$	0.2	0.15	0.0510	0.0511	0.0503(0.0007)
	-0.2	0.15	0.0688	0.0695	0.0685(0.0006)
	0.2	-0.15	0.0606	0.0611	0.0606(0.0007)
	-0.2	-0.15	0.0757	0.0769	0.0755(0.0008)

FT: 9-10 mins, **KL**: 87-109 mins, **M.C.**: 7-8.5 hours

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Application: Default Correlation

In Zhou(2001), default correlation is defined as $Corr(\mathbf{1}(\tau_1 \leq t), \mathbf{1}(\tau_2 \leq t))$.

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- Need to compute $E(1(\tau_1 \leq t)1(\tau_2 \leq t)) = P(\tau_1 \leq t, \tau_2 \leq t)$. Only distribution of $\tau_1 \wedge \tau_2$ is used.
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We can extend the definition to $Corr(\mathbf{1}(\tau_1 \leq t_1), \mathbf{1}(\tau_2 \leq t_2))$, using the numerical values of $P(\tau_1 \leq t_1, \tau_2 \leq t_2)$ for arbitrary drifts μ_1 and μ_2.

Default Correlations (in percentage)

	$t_1 = 2$				
	$t_2 = 1$	$t_2 = 2$	$t_2 = 3$	$t_3 = 4$	$t_2 = 5$
$\mu_1 = \mu_2 = 0$	23.6	24.4	22.9	21.6	20.5
$\mu_1 = \mu_2 = 5\%$	32.2	34.8	34.6	34.3	34.1
$\mu_1 = \mu_2 = -5\%$	13.8	12.4	9.1	6.3	4.1
	$t_1 = 5$				
	$t_2 = 1$	$t_2 = 2$	$t_2 = 3$	$t_3 = 4$	$t_2 = 5$
$\mu_1 = \mu_2 = 0$	18.8	20.5	21.2	21.3	21.2
$\mu_1 = \mu_2 = 5\%$	30.1	34.1	36.1	37.4	38.2
$\mu_1 = \mu_2 = -5\%$	5.5	4.1	2.6	1.1	0.0

Outline

Other Probabilistic Problems

The finite Fourier transform can also be utilized to obtain

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The finite Fourier transform can also be utilized to obtain

- The Lapalce transform and distribution of $\tau_1 \wedge \tau_2)$ for arbitrary drifts μ_1 and μ_2
- The Lapalce transform, distribution, and density of $|\tau_1 - \tau_2|$ for arbitrary drifts μ_1 and μ_2.
- We can prove that the density of $|\tau_1 - \tau_2|)$ at zero is infinity if $\rho > 0$, although $P(|\tau_1 - \tau_2| \leq t) \to 0$, as $t \to 0$.

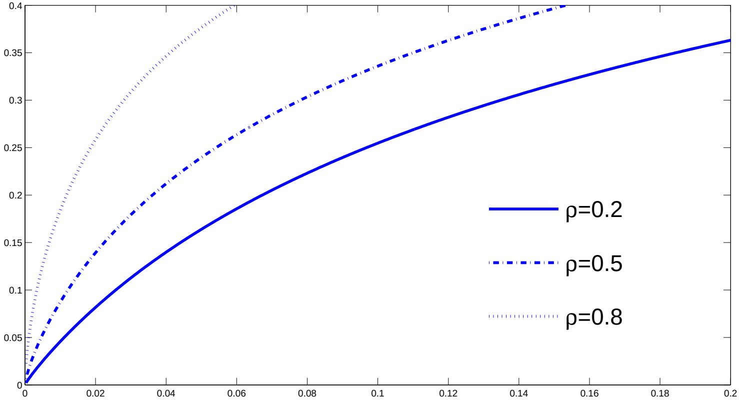

Figure : Distribution functions of $|\tau_2 - \tau_1|$ when $\rho = 0.2, 0.5$ and 0.8. Correlated Brownian motion starts at $x_1 = x_2 = \log(1.2) = 0.1823$. All three distribution functions tend to zero as $t \to 0$.

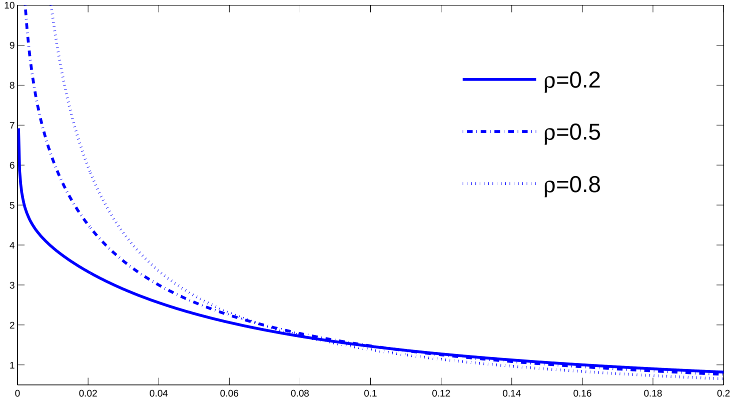

Figure : Density functions of $|\tau_2 - \tau_1|$ when $\rho = 0.2, 0.5$ and 0.8. Correlated Brownian motion starts at $x_1 = x_2 = \log(1.2) = 0.1823$.

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Summary

- Provide an analytical solution of joint Laplace transform of τ_1 and τ_2 for arbitrary drifts μ_1 and μ_2
- Develop a PDE solving procedure leading to an algorithm that is both accurate and efficient.
- Point out a link between the joint Laplace transform to a bivariate exponential distribution which is absolute continuous and does not have memoryless property
- Extend the research work on default correlation
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Thank you for your attention!