SMALE HORSESHOE OF CELLULAR NEURAL NETWORKS

CHENG-HSIUNG HSU*

Department of Mathematics, National Central University,
Chung-Li 32054, Taiwan

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The paper shows the spatial disorder of one-dimensional Cellular Neural Networks (CNN) using the iteration map method. Under certain parameters, the map is two-dimensional and the Smale horseshoe is constructed. Moreover, we also illustrate the variant of CNN, closely related to Henon-type and Belykh maps, and discrete Allen–Cahn equations.

1. Introduction

The paper investigates the complexity of the set of bounded, stable stationary solutions of one-dimensional Cellular Neural Networks (CNN)

\[
\frac{dx_i}{dt} = -x_i + z + \alpha f(x_{i-1}) + af(x_i) + \beta f(x_{i+1}),
\]

(1)

where \( f(x) \) is a piecewise-linear output function defined by

\[
f(x) = f_{\ell,r}(x) = \begin{cases} \ell x + 1 - r & \text{if } x \geq 1, \\ x & \text{if } |x| \leq 1, \\ \ell x + \ell - 1 & \text{if } x \leq -1. \end{cases}
\]

(2)

Here, \( r \) and \( \ell \) are non-negative real constants and the quantity \( z \) is called threshold or bias term, related to independent voltage sources in electric circuits. The coefficients of output function \( \alpha, \alpha \) and \( \beta \) are real constants and called the space-invariant \( A \)-template denoted by

\[
A \equiv [\alpha, \alpha, \beta].
\]

(3)

CNN were first proposed by Chua and Yang [1988a, 1988b]. Since then much work has been done in the electrical engineering community, e.g. [Chua & Roska, 1993; Thiran et al., 1995]. The main applications of CNN are in image processing and pattern recognition, see [Chua, 1998]. A basic and important class of solutions of (1) are the stable stationary solutions of (1). It plays an important role in studying the complexity of stable stationary solutions of (1). Some recent mathematical results about the complexity of stationary solutions and the multiplicity of traveling wave solutions were considered by Juang and Lin [2000; 1997], Hsu and Lin [2000, 1999] and Hsu et al. [1999].

In [Juang & Lin, 2000], they obtained the spatial chaos of (1) with \( f(x) = f_{0,0}(x) \) within certain parameters. If \( r \) and \( \ell \) in (2) are positive, due to the unboundness of \( f \), the method used in [Juang & Lin, 2000] is not effective in dealing with this problem. However, if output \( v = f(x) \) is taken as the unknown variable, i.e. let

\[
v_i = f(x_i) \quad \text{and} \quad u_{i+1} = v_i,
\]

(4)

then if \( f \) is invertible with inverse function \( F \), the stationary solutions of (1) can be written as orbits of one- or two-dimensional iteration maps as follows,

\[
T_1(v) = \frac{1}{\beta} (F(v) - z - av),
\]

(5)

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E-mail: chhsu@math.ncu.edu.tw
when $\alpha = 0$ and $\beta \neq 0$ and
\[
T(u, v) = \left( v, \frac{1}{\beta} (F(v) - z - \alpha u - av) \right),
\]
when $\alpha \neq 0$ and $\beta \neq 0$.

The work only focuses on the complexity of the two-dimensional map $T$ and the results of the one-dimensional map $T_1$ are considered in [Hsu & Lin, 1999]. For these maps, each bounded trajectory will correspond to the outputs of bounded stationary solutions. In practice, if the maps are chaotic, then the stationary solutions of (1) are spatially disordered. However, only stable stationary solutions of (1) should be considered. Therefore, we have to consider the set of all stable bounded orbits of $T$, denoted by $S$. In this paper, for convenience, we also denote $T|_S$ as $T$. After carefully studying the orbits of $T$, we have proved that under certain parameters, the map $T$ has Smale horseshoe structure. Therefore, there exists a hyperbolic invariant subset $\Lambda_{t, r}$ in $\mathbb{R}^2$ (see Sec. 2) such that $T|_{\Lambda_{t, r}}$ is topologically conjugate to a two-sided Bernoulli shift of two symbols. Since the entropy of the Bernoulli shift of two symbols is $\ln 2$, we obtain the chaotic orbits of stable stationary solutions of (1). The main theorem is the following:

**Main Theorem.** Assume (1)–(3) with $\ell, r, a, \alpha$ and $\beta$ being positive real numbers. Define $s = \alpha + a + \beta$ and
\[
\begin{align*}
r^+ & = \frac{-2a + s + 1 + z}{2s - a(s + 1 - z)}, \\
r^- & = \frac{-2a + s + 1 - z}{2s - a(s + 1 - z) - 2sz}, \\
\ell^+ & = \frac{-2a + s + 1 + z}{2s - a(s + 1 + z) + 2sz}, \\
\ell^- & = \frac{-2a + s + 1 - z}{2s - a(s + 1 + z)},
\end{align*}
\]
and the regions $\Sigma_{t, r}^\pm$ in $\mathbb{R}^2$ by
\[
\Sigma_{t, r}^+ = \{(z, a) \in \mathbb{R}^2 | z \geq 0, r < r^+ \text{ and } \ell < \ell^+ \},
\]
\[
\Sigma_{t, r}^- = \{(z, a) \in \mathbb{R}^2 | z \leq 0, r < r^- \text{ and } \ell < \ell^- \}.
\]
If
\[
s - 2a + 1 < z < 2a - 1 - s,
\]
then for any $(z, a)$ lying in $\Sigma_{t, r}^+ \cup \Sigma_{t, r}^-$, there exist $r_0 = r_0(\alpha, a, \beta, z) > 0$ and $\ell_0 = \ell_0(\alpha, a, \beta, z) > 0$ such that for any $r \in (0, r_0)$ and $\ell \in (0, \ell_0), T$ has a hyperbolic invariant subset $\Lambda_{t, r}(z, \alpha, a, \beta)$ in the $(u, v)$ plane such that $T_{\Lambda_{t, r}}$ is topologically conjugate to a two-sided Bernoulli shift of two symbols with entropy equal to $\ln 2$.

The idea of using outputs as dependent variables to study CNN can also be applied to other invertible output functions. Here, we will illustrate the connections between CNN and the other well-known maps. For example, if $f$ satisfies
\[
f = f_H(x) = \begin{cases} \sqrt{x} & \text{if } x \geq 0, \\ -\sqrt{-x} & \text{if } x \leq 0,
\end{cases}
\]
by choosing some suitable template, $T$ is equivalent to the Henon-type map
\[
H(u, v) = (v, \eta(v) - \mu u).
\]
Here, $\eta(v) = \lambda(v - f_H^{-1}(v))$ is a quadratic function of $v$ and $\mu$ is a positive real number. The various results concerning the Henon-type map, e.g. [Afraimovich, 1992] can now be directly applied to the CNN problem with (11).

On the other hand, when $f$ is not invertible, it is difficult to study the complexity of stable stationary solutions as above. For example, if $f = f_{0,0}$ then $f$ is not invertible. The inverse function of $f_{0,0}$ is a set-value function
\[
f_{0,0}^{-1}(v) = \begin{cases} (-\infty, -1) & \text{if } v = -1, \\ (1, \infty) & \text{if } v = 1, \\ \phi & \text{if } |v| > 1.
\end{cases}
\]
In this case, the stationary solutions of (1) with $f = f_{0,0}$ can be considered as the stationary solutions of the discrete Allen–Cahn equations with reaction term $f_{0,0}^{-1}$ as follows
\[
\frac{dx_i}{dt} = -\nu \Delta x_i + f_{0,0}^{-1}(x_i) \text{ for } i \in \mathbb{Z}^1,
\]
here $\Delta$ denotes the discrete Laplace operator. Equation (14) had been studied extensively in [Chow et al., 1996a, 1996b]. Indeed, they obtained the spatial chaos of stationary solutions of (14) within certain parameters. Hence, the CNN also has this phenomenon.
Finally, consider stationary solutions of (1) with perturbation \(\Phi\) defined by
\[
\Phi(y) = \frac{r}{2}y - \frac{1}{r}(|y + 1| - |y - 1|). 
\]
(15)
With some appropriated templates, it can be shown that the stationary solutions of (1) with perturbation \(\Phi\) are equivalent to trajectories of Belykh map, see [Belykh, 1995]. Hence, by the results of [Belykh, 1995], we obtain the existence of a strange attractor of CNN which is analogous to Chua’s attractor [Kevorkian, 1993].

The paper is organized as follows. In Sec. 2, the Smale horseshoe structure of two-dimensional map \(T\) is obtained and the hyperbolic invariant subset \(\Lambda_{\ell,r}\) is constructed such that \(T|_{\Lambda_{\ell,r}}\) is topologically conjugate to a two-sided Bernoulli shift of two symbols. In Sec. 3, we illustrate the variants of CNN closely related to the Henon-type and Belykh maps, and discrete Allen-Cahn equations.

2. Smale Horseshoe Structure

Consider the one-dimensional CNN as (1) but with the output function in (2). It is evident that the stationary solutions of (1) satisfy
\[
x_i = z + \alpha f(x_{i-1}) + af(x_i) + \beta f(x_{i+1}). 
\]
(16)
If \(r > 0\) and \(\ell > 0\), then the output function \(f\) in (2) is invertible with inverse function \(F\) by
\[
F(v) = F_{\ell,r}(v) = \begin{cases} 
\frac{1}{r}v - \frac{1}{r} + 1 & \text{if } v \geq 1, \\
v & \text{if } |v| \leq 1, \\
\frac{1}{\ell}v - \frac{1}{\ell} + 1 & \text{if } v \leq -1. 
\end{cases} 
\]
(17)
It will be proved that the map \(T\) in (6) is chaotic by constructing the Smale horseshoe structure. For convenience, assume that \(z \geq 0\) in the remainder of this paper. The results are similar in the case of \(z \leq 0\). From (6) and (17), the map \(T\) has the following three fixed points,
\[
K_1 = \left(\frac{1 - r + rz}{1 - rs}, \frac{1 - r + rz}{1 - rs}\right), 
\]
(18)
\[
K_0 = \left(\frac{z}{1 - s}, \frac{z}{1 - s}\right), 
\]
(19)
and
\[
K_{-1} = \left(\frac{-1 + \ell + \ell z}{1 - \ell s}, \frac{-1 + \ell + \ell z}{1 - \ell s}\right). 
\]
(20)
The three fixed points correspond to the outputs of the homogeneous steady states of (1). If \(z\) satisfies (10), then
\[
K_1 > 1, \quad |K_0| < 1 \quad \text{and} \quad K_{-1} < -1. 
\]

In the following, by choosing the parameters properly, we can find a region \(\Omega\) in the \((u, v)\) plane such that the map \(T\) acts like the well-known Smale horseshoe map. The method has been used in [Afraimovich & Nekorkin, 1994] to study the set of stationary solutions of discrete bistable reaction–diffusion equations. In [Nekorkin & Chua, 1993], they also used this method to construct Smale horseshoe in a chain of coupled Chua’s circuits and to obtain the spatial disorder of stable stationary solutions.

Now, the region \(\Omega\) can be defined by
\[
\Omega = \{(u, v) \in \mathbb{R}^2 | |u| \leq p \quad \text{and} \quad |v| \leq q\}, 
\]
(20)
where \(p\) and \(q\) are positive real numbers. The following theorem shows how to find appropriated \(p\) and \(q\) such that \(T|_{\Omega}\) has the Smale horseshoe structure.

Theorem 2.1. If there are a pair of positive numbers \(p\) and \(q\) satisfying the following inequalities,
\[
p > \max\left\{\frac{1 - r + rz}{1 - rs}, \frac{-1 + \ell + \ell z}{1 - \ell s}\right\}, 
\]
(21)
\[
q > \max\left\{\frac{1 - r + rz}{1 - rs}, \frac{-1 + \ell + \ell z}{1 - \ell s}\right\}, 
\]
(22)
\[
q < -\frac{\alpha}{\beta}p + \frac{a}{\beta} - \frac{1}{\beta} - \frac{z}{\beta}, 
\]
(23)
\[
q < -\frac{\alpha}{\beta}p + \frac{a}{\beta} - \frac{1}{\beta} + \frac{z}{\beta}, 
\]
(24)
\[
\frac{\alpha}{\beta}p < -q - \frac{a}{\beta}q + \frac{1}{\beta}\left(\frac{1}{r}q + 1 - \frac{1}{r}\right) + \frac{z}{\beta}, 
\]
(25)
\[
\frac{\alpha}{\beta}p < -q - \frac{a}{\beta}q + \frac{1}{\beta}\left(\frac{1}{r}q - 1 + \frac{1}{r}\right) - \frac{z}{\beta}, 
\]
(26)
\[
p > q, 
\]
(27)
then \(T\) is a Smale horseshoe map on \(\Omega\), i.e. the region \(\Omega\) after one iteration of \(T\) will intersect itself in three regions \(\Omega_{-1}, \Omega_0\) and \(\Omega_1\) as in Fig. 1.
By Theorem 2.1 and Proposition 2.2, we need to find suitable \( p \) and \( q \) satisfying the inequalities in Theorem 2.1 to prove the main theorem. Indeed, we have the following theorem:

**Theorem 2.3.** Consider (1)–(3). If \((z, a)\) lies in the region \( \Sigma_{\ell, r}^- \) defined by (8), then there exist \( \hat{r}_0(z, \alpha, a, \beta) > 0 \) and \( \hat{\ell}_0(z, \alpha, a, \beta) > 0 \) such that for all \( 0 < \ell < \hat{\ell}_0 \) and \( 0 < r < \hat{r}_0 \), there exist \( p \) and \( q \) satisfying the conditions of Theorem 2.1.

Moreover, any compact subset of \( \Omega_{-1} \cup \Omega_1 \) is hyperbolic.

**Proof.** For convenience, define

\[
L^+(p, q) = \frac{\alpha}{\beta} p + q - \frac{a - 1 - z}{\beta},
\]

\[
R^+(p, q) = \frac{\alpha}{\beta} p + \left(1 + \frac{a}{\beta} - \frac{1}{\beta r}\right)q + \frac{1}{\beta} \left(\frac{1}{r} - 1 + z\right),
\]

\[
R^-(p, q) = \frac{\alpha}{\beta} p + \left(1 + \frac{a}{\beta} - \frac{1}{\beta \ell}\right)q + \frac{1}{\beta} \left(\frac{1}{\ell} - 1 - z\right),
\]

then (23)–(26) is equivalent to \( L^+, R^+, R^- < 0 \). Its symbolic dynamics, we know the behavior of \( T \) more precisely. In general, it is difficult to prove the hyperbolicity of an invariant set. However, in [Afraimovich et al., 1993], they gave a set of sufficient conditions to ensure the hyperbolicity of an invariant set. We recall the results as follows:

**Proposition 2.2** [Afraimovich et al., 1993]. Consider a two-dimensional map \( G : U \rightarrow \mathbb{R}^2 \),

\[
G(x, y) = (\overline{x}, \overline{y}),
\]

\[
\overline{x} = g(x, y) \quad \text{and} \quad \overline{y} = k(x, y),
\]

where \( g \) and \( k \) are differentiable real functions defined on \( \mathbb{R}^2 \) and \( k_y \) is invertible. Assume the following conditions hold,

(i) \( \|g_x\| < 1 \) and \( \|k_y^{-1}\| < 1 \),

(ii) \( 1 - \|g_x\| \cdot \|k_y^{-1}\| > 2\sqrt{\|g_y\| \cdot \|k_y^{-1}\|} \cdot \|k_x\| \cdot \|g_y\| \cdot \|k_y^{-1}\| \),

(iii) \( (1 - \|g_x\|)(1 - \|k_y^{-1}\|) > \|g_y \cdot k_y^{-1}\| \cdot \|g_x\| \),

where \( \|\cdot\| = \sup_{(x, y) \in U} |\cdot| \). Then any compact invariant subset \( \Lambda \) in \( U \) is hyperbolic.
Suppose \((q^+, q^-), (p_i, q_i^+)\) and \((p_i, q_i^-)\) satisfy
\[
L^+(q^+, q^-) = 0,
\]
\[
L^+(p_i, q_i^+) = 0, \quad R^+(p_i, q_i^+) = 0,
\]
\[
L^+(p_i, q_i^-) = 0, \quad R^-(p_i, q_i^-) = 0,
\]
then it is obvious that
\[
q^+ = \frac{a - 1 - z}{s - a},
\]
\[
q_i^+ = \frac{2r - ra - 1}{ra - 1} \quad \text{and} \quad q_i^- = \frac{2(\ell - \ell a - 1 + 2\ell z)}{\ell a - 1}.
\]
Therefore, there exist \(p\) and \(q\) that satisfy (23)–(27) if and only if
\[
q_i^+ < q^+ \quad \text{and} \quad q_i^- < q^-.
\] (28)

Since \(z\) satisfies (10), by (7) and a simple computation, we know that
\[
r < r^+ \quad \text{and} \quad \ell < \ell^+
\] (29)
is equivalent to (28) and satisfy (21) and (22). Hence, there exist \(p\) and \(q\) that satisfy the inequalities in Theorem 2.1 if \(r\) and \(\ell\) satisfy (29).

Next, we show that any compact subset of \(\Omega_{-1} \cup \Omega_1\) is hyperbolic. The functions \(g\) and \(k\) in Proposition 2.2 are now defined as follows
\[
g(u, v) = v
\]
and
\[
k(u, v) = \frac{1}{\beta} \left( F(v) - z - \alpha u - av \right).
\]
In the region of \(\Omega_{-1} \cup \Omega_1\), it is clear that \(g_u = 0, g_v = 1, k_u = (-\alpha/\beta)\) and
\[
k_v = \begin{cases} 
\frac{1}{\beta} \left( \frac{1}{r} - a \right), & \text{if } v \geq 1, \\
\frac{1}{\beta} \left( \frac{1}{\ell} - a \right), & \text{if } v \leq -1.
\end{cases}
\]
Hence, if \(r\) and \(\ell\) are positive and small enough, then there exist \(\tilde{r}_0\) and \(\tilde{\ell}_0\) such that if \(0 < r < \tilde{r}_0\) and \(0 < \ell < \tilde{\ell}_0\), the inequalities of Proposition 2.2 hold. Therefore, the results follow by letting \(\tilde{r}_0 \equiv \min\{\tilde{r}_0, r^+\}\) and \(\tilde{\ell}_0 \equiv \min\{\tilde{\ell}_0, \ell^+\}\). The proof is complete. 

From Theorem 2.3, we have constructed the Smale horseshoe map of \(T\) on \(\Omega\). It is still necessary to study the stability of these stationary solutions with respect to the dynamical system (1). The results are as follows:

**Definition 2.4.** Given any stationary solution \(\bar{x} = \{\bar{x}_i\}_{i=\infty}^{\infty}\), the linearized operator of (1) at \(\bar{x}\) is defined by
\[
(L(\bar{x})\xi)_i = -\xi_i + \alpha f'(\bar{x}_{i-1})\xi_{i-1} + \alpha f'(\bar{x}_i)\xi_i + \beta f'(\bar{x}_{i+1})\xi_{i+1} \quad \text{for } \xi \in \ell^2. \tag{30}
\]
\(\bar{x}\) is called stable if all eigenvalues of \(L\) are negative with eigenvectors in \(\ell^2\). \(\bar{x}\) is called unstable if there exists a positive eigenvalue of \(L\) with eigenvectors in \(\ell^2\).

Since the function \(f = f_{t,r}\) is not differentiable at the transition state \(|\bar{x}_i| = 1\), (30) may not be well-defined. However, if we consider the trajectory \(\bar{x} = \{\bar{x}_i\}_{i=\infty}^{\infty}\) of \(T\) be such that \(|\bar{x}_i| \neq 1\), then we have the stability results as follows.

**Proposition 2.5.** Let \(\bar{x} = \{\bar{x}_i\}_{i=\infty}^{\infty}\) be a stationary solution of (1) and (2). If \(a > 1\) then there exists \(\bar{r}_0 > 0\) and \(\bar{\ell}_0 > 0\) such that

(i) if there exists \(i\) such that
\[
(0 < r < \bar{r}_0, f_{t,r}(\bar{x}_i)) \in \Omega_0 \cap \omega_0
\]
for some \(i \in \mathbb{Z}^1, \)
then \(\bar{x}\) is unstable,

(ii) if \(0 < r < \bar{r}_0\) and \(0 < \ell < \bar{\ell}_0\),
\[
(0 < \ell < \bar{\ell}_0, f_{t,r}(\bar{x}_i)) \in (\Omega_{-1} \cap \omega_{-1}) \cup (\Omega_1 \cap \omega_1)
\]
for all \(i \in \mathbb{Z}^1, \)
then \(\bar{x}\) is stable.

Proof. Let \(E\) be a subset of \(\mathbb{Z}^1\) and assume that \(\bar{x}\) satisfying
\[
|\bar{x}_i| < 1 \quad \text{for } i \in E \subset \mathbb{Z}^1
\]
and
\[
|\bar{x}_i| > 1 \quad \text{for } i \notin E.
\]
By (30), if \(E = \phi, rs < 1\) and \(\ell s < 1\), define \(\bar{r}_0 = \bar{\ell}_0 = (1/s)\), then \(-L(\bar{x})\) is a positive operator. Hence, the eigenvalues of \(L(\bar{x})\) are negative.
and \( \pi \) is stable. If \( E \neq \phi \), let \( k \in E \) and define \( e = \{e_i\} \in \ell^2 \) by

\[
e_k = 1 \quad \text{and} \quad e_i = 0 \quad \text{for} \quad i \neq k,
\]
then \( e \) is an eigenvector with positive eigenvalue. Hence, \( \pi \) is unstable. Since \( E \neq \phi \) and \( E = \phi \) are equivalent to the conditions (i) and (ii) respectively. The proof is complete. \( \blacksquare \)

By Proposition 2.5, we only have to consider the trajectories of \( T \) in the subset of \( (\Omega_{-1} \cap \omega_{-1}) \cup (\Omega_1 \cap \omega_1) \).

**Proof of the Main Theorem.** Combine the results of Theorems 2.3 and Proposition 2.5, the main theorem is proved by taking \( r_0 = \min\{\bar{r}_0, \bar{r}_0\} \) and \( \ell_0 = \min\{\bar{t}_0, \bar{t}_0\} \). The proof is complete. \( \blacksquare \)

Next, it is interesting to investigate the relationship of stable stationary solutions of (1) between different output functions \( f_{0,0} \) and \( f_{\ell, r} \). Now, consider the stable stationary solutions of (1) with \( A = [\alpha, a, \beta] \) and output function \( f(x) = f_{0,0}(x) = (1/2)(|x + 1| - |x - 1|) \). By the same method used in [Juang & Lin, 2000], we can partition the parameters space \( \mathbb{R}^2 = \{(z, a - 1) : z \in \mathbb{R}^1 \text{ and } a \in \mathbb{R}^1\} \) into finite regions \([m, n], 0 \leq m, n \leq 4\), see Fig. 2, such that the entropy \( h \) of a stable stationary solution is

\[
h = \begin{cases} 
\ln 2 & \text{if } (z, a) \in [4, 4], \\
\ln \lambda & \text{if } (z, a) \in [4, 3] \text{ or } [3, 4], \\
\frac{1 + \sqrt{5}}{2} & \text{if } (z, a) \in [3, 3], \\
0 & \text{otherwise}. 
\end{cases}
\]

(31)

Here \( \lambda \) is the maximal root of \( \lambda^3 - 2\lambda \lambda^2 + \lambda - 1 = 0 \) and region \([4, 4]\) is defined by

\[
[4, 4] = \{(z, a - 1)|\alpha + \beta -(a - 1) < z < (a - 1) - \alpha - \beta\}.
\]

By (8) and (9), it is easy to verify that \( \Sigma_{T, r}^+ \cup \Sigma_{T, r}^- \subset [4, 4] \) as in Fig. 2, and

\[
\lim_{\ell, r \to 0} (\Sigma_{T, r}^+ \cup \Sigma_{T, r}^-) = [4, 4].
\]

Consequently, the results of (31) with \((z, a) \in [4, 4]\) can be regarded as the limiting case of (1) but with \( f = f_{\ell, r} \) when \( \ell, r \to 0^+ \).

3. Relations of CNN to Other Maps

In this section, we will study the stationary solutions of (1) but with different output functions from the piecewise-linear one. By choosing appropriated output functions, we can find that CNN has a closed relation with other dynamical systems, e.g. the Henon-type map, Belykh map, and discrete Allen–Cahn equation. Through the study of these models, we can have a better understanding of the structure of the set of all stationary solutions of CNN.
3.1. Henón-type map

We first consider (1) with the square-root output function as follows:

\[
f = f_H(x) = \begin{cases} 
\sqrt{x} & \text{if } x \geq 0, \\
-\sqrt{-x} & \text{if } x \leq 0.
\end{cases}
\] (32)

By (4), we obtain the two-dimensional map \(H\) by

\[
H(u, v) = \left( v, -\frac{\alpha}{\beta} u - \frac{a}{\beta} v + \frac{1}{\beta} Q(v) \right), \tag{33}
\]

where \(Q\) is a quadratic function by

\[
Q(v) = \begin{cases} 
v^2 & \text{if } |v| \geq 0, \\
-v^2 & \text{if } |v| \leq 0.
\end{cases}
\] (34)

Assume that \(z = 0\) and the \(A\)-template given by

\[
A \equiv [\alpha, a, \beta] = \left[ -\frac{\mu}{\lambda}, 1, -\frac{1}{\lambda} \right], \tag{35}
\]

where \(\lambda\) and \(\mu\) are positive numbers. From (33) and (35), \(H\) can be rewritten as

\[
H(u, v) = (v, -\mu u + \lambda v - \lambda Q(v)) = (v, \eta(v) - \mu u), \tag{36}
\]

where \(\eta(v) = \lambda v - Q(v)\).

From (36), we know that \(H\) is the well-known Henón-type map. If we denote the rectangles \(E, E_0\) and \(E_1\) by

\[
E = \{(u, v)|0 \leq u \leq 1, 0 \leq v \leq 1\},
\]

\[
E_0 = \{(u, v)|0 \leq u \leq 1, u \leq v \leq v_1 + \xi\},
\]

\[
E_1 = \{(u, v)|0 \leq u \leq 1, v_2 - \xi \leq v \leq 1\},
\]

where

\[
v_1 = \frac{1}{2} \left( 1 - \sqrt{1 - \frac{4}{\lambda}} \right),
\]

\[
v_2 = \frac{1}{2} \left( 1 + \sqrt{1 - \frac{4}{\lambda}} \right) \quad \text{and} \quad \xi < \frac{1}{2} \sqrt{1 - \frac{4}{\lambda}},
\]

then we have \(\eta(v) = \lambda v(1 - v)\) in \(E\). Now, the results of [Afraimovich, 1992] can be applied. In particular we recall

**Theorem 3.1** [Afraimovich, 1992]

(i) If \(\lambda > 4\) and \(0 \leq \mu \leq \lambda \xi \sqrt{1 - (4/\lambda)}\), then there exists a Smale horseshoe map of \(H\).

(ii) If \(\lambda \geq 2 + \sqrt{5}\) and \(\xi\) is small enough, then the Smale horseshoe map in (i) satisfies the hyperbolicity.

From Theorem 3.1, we know that there exists a hyperbolic invariant subset \(\Lambda_H\) in \(E\) such that \(H|_{\Lambda_H}\) is topologically conjugate to two-sided Bernoulli shift of two symbols. Hence, we obtain the spatial disorder of stationary solutions of (1) with the square-root output function (32).

3.2. Belykh map

Now, consider the one-dimensional CNN with perturbation term as follows

\[
\frac{dx_i}{dt} = -x_i + z + \alpha f(x_{i-1}) + af(x_i) + \beta f(x_{i+1}) + \Phi(f(x_i)) + \xi. \tag{37}
\]

Here \(f = f_0, 0\) and \(\Phi\) is the perturbation function satisfying

\[
\Phi(y) = \frac{1}{2} y - \frac{1}{\lambda} \left( y^2 - 1 \right). \tag{38}
\]

It is clear that the stationary solutions of (37) satisfy

\[
x_i = z + \alpha f(x_{i-1}) + af(x_i) + \beta f(x_{i+1}) + \Phi(f(x_i)). \tag{39}
\]

Now, denote \(v_i = f(x_i)\) and function \(\phi(v)\) by

\[
\phi(v) \equiv \frac{F(v) - \Phi(v)}{\beta} \tag{40}
\]

\[
= \frac{-1}{\beta v^2} + \frac{1}{2} \left( 1 + \frac{1}{\beta v^2} \right) \left( |v| - 1 \right).
\]

(41)

then (39) can be written as

\[
\phi(v_i) = \frac{1}{\beta} (z + \alpha v_{i-1} + av_i + \beta v_{i+1}). \tag{42}
\]

For simplicity, assume that \(z = 0\) and that the \(A\)-template is in the form

\[
A = [\alpha, a, \beta] = [\beta \rho, (-1 - \rho) \beta, \beta], \tag{43}
\]

where \(\rho\) is positive in \(\mathbb{R}^1\). Equation (42) can be written as

\[
\phi(v_i) = v_{i+1} - v_i - \rho(v_i - v_{i-1}). \tag{44}
\]
Furthermore, denote
\[ u_i = \rho(v_i - v_{i-1}), \tag{45} \]
we can rewrite (44) as
\[
\begin{cases}
  u_{i+1} = \rho u_i + \rho \phi(v_i), \\
  v_{i+1} = u_i + v_i + \phi(v_i).
\end{cases} \tag{46}
\]
Hence, the two-dimensional map (46) is called a “Belykh map”. Therefore, the results of [Belykh, 1995] can be applied. In particular, we recall the following result:

**Theorem 3.2** [Belykh, 1995]. If \(0 < \rho \leq 1, \beta > 0\) and \((1/\beta r) > 2(1 + \rho)\), then there exists a strange attractor of (46) whose orbits give stationary solutions of (37).

### 3.3. Allen–Cahn equation

Finally, we consider the spatially discrete versions of the Allen–Cahn equation as follows
\[
\frac{dx_i}{dt} = -\nu \Delta x_i - g(x_i) \quad \text{for } i \in \mathbb{Z}, \tag{47}
\]
here \(\Delta\) denotes the discrete Laplace operator given by
\[
\Delta x_i = x_{i+1} + x_{i-1} - 2x_i. \tag{48}
\]
The stationary solutions of (47) and (48) satisfy
\[
x_{i+1} = -x_{i-1} + 2x_i - \frac{1}{\nu} g(x_i). \tag{49}
\]
By using the same transformation of (4), we may consider (49) as a two-dimensional dynamical system \(T_A\) by
\[
T_A(u, v) = \left(v, -\frac{1}{\nu} g(v) + 2v - u\right). \tag{50}
\]
It is clear that (50) is equivalent to (6) with
\[
A = [-\nu, 2\nu, -\nu], \quad z = 0, \quad \text{and } g = F.
\]
If we consider \(g = F = f_{0.5}^{1}\), then \(g\) will be a set-valued function defined as follows
\[
g(v) = \begin{cases} 
  (-\infty, -1] & \text{if } v = -1, \\
  v & \text{if } |v| < 1, \\
  [1, \infty) & \text{if } v = 1, \\
  \phi & \text{if } |v| > 1.
\end{cases} \tag{51}
\]
Hence, the complexity of outputs of stationary solutions of CY–CNN is equivalent to the stationary solutions of (47). In [Chow et al., 1996b], they considered the stationary solutions of (47) with reaction function \(g\) as (51) and proved that there is a phase transition from pattern formation to spatial chaos occurring at the value of \(\nu = (1/3)\). Hence, the patterns of CNN display spatial chaos, too.

From the previous discussion, we find that CNN is analogous to many models in dynamical systems. Through the study of these models, we can better understand the complexity of stationary solutions of CNN, and this is important in the applications of CNN.

**References**


