Algebraic Bethe Ansatz solutions for the $sl(2|1)^{(2)}$ and $osp(2|1)$ models with boundary terms

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Abstract

This work is concerned with the formulation of the graded quantum inverse scattering method for a class of lattice models with reflecting boundary conditions. The $sl(2|1)^{(2)}$ and $osp(2|1)$ models are considered with their diagonal reflections in BFB grading. This allowed us to derive the eigenvalues and eigenvectors for the corresponding transfer matrices as well as explicit expressions for the Bethe Ansatz equations.

PACS: 05.20.-y; 05.50.+q; 04.20.Jb

Keywords: Algebraic Bethe Ansatz, Open boundary conditions

February 27, 2008

1 Introduction

Integrable quantum systems containing Fermi fields have been attracting increasing interest due to their potential applications in condensed matter physics. The prototypical examples of such systems are the supersymmetric generalizations of the Hubbard and t-J models [1]. They lead to a generalization of the Yang-Baxter (YB) equation [2] associated with the introduction of the a Z_2 grading [3]. In addition to the grading, it is also important to introducing open boundary conditions to study the boundary effects on the bulk system. The most powerful method in the analysis of integrable models is the Bethe Ansatz (BA). The algebraic BA, also known as the quantum inverse scattering method (QISM) [4], is an elegant and important generalization of the coordinate BA [5]

In this work we will study two graded three-state vertex models with reflecting boundary conditions. Their boundary algebraic BA are delineated based in the recent progresses [6, 7], for the formulation of the QISM for the 19-vertex models with boundary condition terms.

Let $V = V_0 \oplus V_1$ be a Z_2-graded vector space where 0 and 1 denote the even and odd parts respectively. Multiplication rules in the graded tensor product space $V \overset{s}{\otimes} V$ differ from the ordinary ones by the appearance of additional signs. The components of a linear operator $A \overset{s}{\otimes} B \in V \overset{s}{\otimes} V$ result in matrix elements of the form

$$(A \overset{s}{\otimes} B)_{\alpha\beta}^{\gamma\delta} = (-)^{p(\beta)(p(\alpha)+p(\gamma))}\ A_{\alpha\gamma} B_{\beta\delta}. \tag{1.1}$$

The action of the graded permutation operator $\mathcal{P}$ on the vector $|\alpha\rangle \overset{s}{\otimes} |\beta\rangle \in V \overset{s}{\otimes} V$ is defined by

$$\mathcal{P}\ \ |\alpha\rangle \overset{s}{\otimes} |\beta\rangle = (-)^{p(\alpha)p(\beta)}\ |\beta\rangle \overset{s}{\otimes} |\alpha\rangle \Longrightarrow (\mathcal{P})_{\alpha\beta}^{\gamma\delta} = (-)^{p(\alpha)p(\beta)} \delta_{\alpha\delta}\ \delta_{\beta\gamma}. \tag{1.2}$$

The graded transposition st and the graded trace str are defined by

$$\left(A^{\rm st}\right)_{\alpha\beta} = (-)^{(p(\alpha)+1)p(\beta)} A_{\beta\alpha}, \quad {\rm str} A = \sum_{\alpha} (-)^{p(\alpha)} A_{\alpha\alpha}. \tag{1.3}$$

where $p(\alpha) = 1$ (0) if $|\alpha\rangle$ is an odd (even) element.

For the graded case the YB equation

$$\mathcal{R}_{12}(u)\mathcal{R}_{13}(u+v)\mathcal{R}_{23}(v) = \mathcal{R}_{23}(v)\mathcal{R}_{13}(u+v)\mathcal{R}_{12}(u) \tag{1.4}$$

and the reflection equation [8, 9]

$$\mathcal{R}_{12}(u-v)K_1^-(u)\mathcal{R}_{21}(u+v)K_2^-(v) = K_2^-(v)\mathcal{R}_{12}(u+v)K_1^-(u)\mathcal{R}_{21}(u-v) \tag{1.5}$$

remain the same as in the non-graded cases and we only need to change the usual tensor product to the graded tensor product.

In general, the dual reflection equation which depends on the unitarity and cross-unitarity relations of the $\mathcal{R}$-matrix takes different forms for different models. For the models considered in this paper, we write the graded dual reflection equation in the following form [10]:

$$\mathcal{R}_{21}^{st_1 st_2}(-u+v)(K_1^{+})^{st_1}(u)M_1^{-1}\mathcal{R}_{12}^{st_1 st_2}(-u-v-2\rho)M_1(K_2^{+})^{st_2}(v)$$

$$=(K_2^{+})^{st_2}(v)M_1\mathcal{R}_{12}^{st_1 st_2}(-u-v-2\rho)M_1^{-1}(K_1^{+})^{st_1}(u)\mathcal{R}_{21}^{st_1 st_2}(-u+v), \tag{1.6}$$

and we will choose a common parity assignment: $p(1)=p(3)=0$ and $p(2)=1$, the BFB grading.

Now, using the relations

$$\mathcal{R}_{12}^{st_1 st_2}(u)=I_1R_{21}(u)I_1, \quad \mathcal{R}_{21}^{st_1 st_2}(u)=I_1R_{12}(u)I_1 \quad \text{and} \quad IK^{+}(u)I=K^{+}(u) \tag{1.7}$$

with $I=\mathrm{diag}(1,-1,1)$ and the property $[M_1M_2,\mathcal{R}(u)]=0$ we can see that the usual isomorphism [11]

$$K^{-}(u) :\to K^{+}(u)=K^{-}(-u-\rho)^{st}M. \tag{1.8}$$

holds with the BFB grading. Here st_i denotes super-transposition in the space i.

A quantum-integrable system is characterized by the monodromy matrix $T(u)$ satisfying the fundamental relation

$$R(u-v)\,[T(u)\otimes T(v)]=[T(v)\otimes T(u)]\,R(u-v) \tag{1.9}$$

where $R(u)$ is given by $R(u)=P\mathcal{R}(u)$.

In the framework of the QISM [4], the simplest monodromies have become known as $\mathcal{L}$ operators, the Lax operators, here defined by $\mathcal{L}_{aq}(u)=\mathcal{R}_{aq}(u)$, where the subscript a represents the auxiliary space, and q represents the quantum space. The monodromy matrix $T(u)$ is defined as the matrix product of N Lax operators on all sites of the lattice,

$$T(u)=\mathcal{L}_{aN}(u)\mathcal{L}_{aN-1}(u)\cdots\mathcal{L}_{a1}(u). \tag{1.10}$$

The main result for open boundaries integrability is: if the boundary equations are satisfied, then the Sklyanin's transfer matrix [8]

$$t(u)=\mathrm{str}_a\left(K^{+}(u)T(u)K^{-}(u)T^{-1}(-u)\right), \tag{1.11}$$

forms a commuting collection of operators in the quantum space

$$[t(u),t(v)]=0, \qquad \forall u,v \tag{1.12}$$

The commutativity of $t(u)$ can be proved by using the unitarity and crossing-unitarity relations, the reflection equation and the dual reflection equation. In particular, it implies the integrability of an open quantum spin chain whose Hamiltonian (with $K^{-}(0) = 1$) is given by [8]

$$H = \sum_{k=1}^{N-1} H_{k,k+1} + \frac{1}{2} \left. \frac{dK_1^-(u)}{du} \right|_{u=0} + \frac{\mathrm{str}_0 K_0^+(0) H_{N,0}}{\mathrm{str} K^+(0)}, \tag{1.13}$$

where the two-site terms are given by

$$H_{k,k+1} = \left. \frac{d}{du} P_{k,k+1} \mathcal{R}_{k,k+1}(u) \right|_{u=0} \tag{1.14}$$

in the standard fashion.

The paper is organized as follows: in Section 2 we define the models to be studied. In section 3, we present our detailed calculations common to both models and in Section 4 the eigenspectra and the corresponding Bethe equations are explicitly presented for each model. Section 5 is reserved for conclusions.

2 The models

The three-state vertex models that we will consider are the $sl(2|1)^{(2)}$ model and the $osp(1|2)$ model. Their $\mathcal{R}$-matrices have a common form

$$\mathcal{R}(u) = \left(\begin{array}{ccc|ccc|ccc}
x_1 & & & & & & & & \\
 & x_2 & & x_5 & & & & & \\
 & & x_3 & & x_6 & & x_7 & & \\
\hline
 & y_5 & & x_2 & & & & & \\
 & & y_6 & & x_4 & & x_6 & & \\
 & & & & & x_2 & & x_5 & \\
\hline
 & & y_7 & & y_6 & & x_3 & & \\
 & & & & & y_5 & & x_2 & \\
 & & & & & & & & x_1
\end{array} \right), \tag{2.1}$$

satisfying the properties

$$\begin{aligned}
\mathrm{regularity} &: \quad \mathcal{R}_{12}(0) = f(0)^{1/2} P_{12}, \\
\mathrm{unitarity} &: \quad \mathcal{R}_{12}(u) \mathcal{R}_{12}^{st_1 st_2}(-u) = f(u), \\
\mathrm{PT-symmetry} &: \quad P_{12} \mathcal{R}_{12}(u) P_{12} = \mathcal{R}_{12}^{st_1 st_2}(u), \\
\mathrm{crossing-symmetry} &: \quad \mathcal{R}_{12}(u) = U_1 \mathcal{R}_{12}^{st_2}(-u-\rho) U_1^{-1},
\end{aligned} \tag{2.2}$$

where $f(u) = x_1(u)x_1(-u)$, $x_1(u)$ being defined for each model bellow. ρ is the crossing parameter and U determines the crossing matrix

$$M = U^t U = M^t. \tag{2.3}$$

Unitarity and crossing-symmetry together imply the useful relation

$$M_1 \mathcal{R}_{12}^{st_2}(-u-\rho) M_1^{-1} \mathcal{R}_{12}^{st_1}(u-\rho) = f^{'}(u). \tag{2.4}$$

2.1 The sl$(2|1)^{(2)}$ model

The solution of the graded YB equation corresponding to $sl(2|1)^{(2)}$ in the fundamental representation has the form (2.1) with non-zero entries [12, 13]:

$$\begin{aligned} x_1(u) &= \sinh(u+2\eta)\cosh(u+\eta), \quad x_2(u) = \sinh u \cosh(u+\eta), \\ x_3(u) &= \sinh u \cosh(u-\eta), \quad x_4(u) = \sinh u \cosh(u+\eta) - \sinh 2\eta \cosh \eta. \\ y_5(u) &= x_5(u) = \sinh 2\eta \cosh(u+\eta), \quad y_6(u) = x_6(u) = \sinh 2\eta \sinh u, \\ y_7(u) &= x_7(u) = \sinh 2\eta \cosh \eta. \end{aligned} \tag{2.5}$$

This $\mathcal{R}$-matrix is regular and unitary, with $f(u) = x_1(u)x_1(-u)$, P- and T-symmetric and crossing-symmetric with $M = 1$ and $\rho = \eta$. The graded version of the crossing-unitarity relation (2.4) is satisfied with $f^{'}(u) = x_1(u + i\frac{\pi}{2})x_1(-u - i\frac{\pi}{2})$.

The most general diagonal solution for $K^-(u)$ has been presented in Ref. [14] and it is given by

$$K^-(u,\beta) = \begin{pmatrix} k_{11}^-(u) & & \\ & 1 & \\ & & k_{33}^-(u) \end{pmatrix}, \tag{2.6}$$

with

$$k_{11}^-(u) = -\frac{\beta \sinh u + 2\cosh u}{\beta \sinh u - 2\cosh u}, \quad k_{33}^-(u) = \frac{\beta \cosh(u+\eta) - 2\sinh(u+\eta)}{\beta \cosh(u-\eta) + 2\sinh(u-\eta)}, \tag{2.7}$$

where β is a free parameter. Due to the automorphism (1.8) the solution for $K^+(u)$ is given by $K^-(-u-\rho, \frac{1}{4}\alpha)$ *i.e.*

$$K^+(u,\alpha) = \begin{pmatrix} k_{11}^+(u) & & \\ & 1 & \\ & & k_{33}^+(u) \end{pmatrix}, \tag{2.8}$$

where

$$k_{11}^+(u) = \frac{\alpha \cosh(u+\eta) - 2\sinh(u+\eta)}{\alpha \cosh(u+\eta) + 2\sinh(u+\eta)}, \quad k_{33}^+(u) = -\frac{\alpha \sinh u + 2\cosh u}{\alpha \sinh(u+2\eta) - 2\cosh(u+2\eta)}, \tag{2.9}$$

and α is another free parameter.

2.2 The osp(2|1) model

The trigonometric solution of the graded YB equation corresponding to $osp(1|2)$ in the fundamental representation has the form (2.1) with non-zero entries [13]:

$$
\begin{aligned}
x_1(u) &= \sinh(u+2\eta)\sinh(u+3\eta), \quad x_2(u) = \sinh u \sinh(u+3\eta) \\
x_3(u) &= \sinh u \sinh(u+\eta), \quad x_4(u) = \sinh u \sinh(u+3\eta) - \sinh 2\eta \sinh 3\eta \\
x_5(u) &= \mathrm{e}^{-u} \sinh 2\eta \sinh(u+3\eta), \quad y_5(u) = \mathrm{e}^{u} \sinh 2\eta \sinh(u+3\eta) \\
x_6(u) &= -\mathrm{e}^{-u-2\eta} \sinh 2\eta \sinh u, \quad y_6(u) = \mathrm{e}^{u+2\eta} \sinh 2\eta \sinh u \\
x_7(u) &= \mathrm{e}^{-u} \sinh 2\eta \left(\sinh(u+3\eta) + \mathrm{e}^{-\eta} \sinh u\right) \\
y_7(u) &= \mathrm{e}^{u} \sinh 2\eta \left(\sinh(u+3\eta) + \mathrm{e}^{\eta} \sinh u\right)
\end{aligned} \tag{2.10}
$$

This $\mathcal{R}$-matrix is regular and unitary, with $f^{'}(u) = f(u) = x_1(u)x_1(-u)$. It is PT-symmetric and crossing-symmetric, with $\rho = 3\eta$ and

$$
M = \begin{pmatrix} \mathrm{e}^{-2\eta} & & \\ & 1 & \\ & & \mathrm{e}^{2\eta} \end{pmatrix}. \tag{2.11}
$$

Diagonal solutions for $K^-(u)$ have been obtained in [15]. It turns out that there are three solutions without free parameters, being $K^-(u) = 1$, $K^-(u) = F^+$ and $K^-(u) = F^-$, with

$$
F^{\pm} = \begin{pmatrix} \mp\mathrm{e}^{-2u} f^{(\pm)}(u) & & \\ & 1 & \\ & & \mp\mathrm{e}^{2u} f^{(\pm)}(u) \end{pmatrix}, \tag{2.12}
$$

where we have defined

$$
f^{(+)}(u) = \frac{\sinh(u+3\eta/2)}{\sinh(u-3\eta/2)}, \quad f^{(-)}(u) = \frac{\cosh(u+3\eta/2)}{\cosh(u-3\eta/2)}. \tag{2.13}
$$

By the automorphism (1.8), three solutions $K^+(u)$ follow as $K^+(u) = M$, $K^+(u) = G^+$ and $K^+(u) = G^-$, with

$$
G^{\pm} = \begin{pmatrix} \mp\mathrm{e}^{2u+4\eta} g^{(\pm)}(u) & & \\ & 1 & \\ & & \mp\mathrm{e}^{-2u-4\eta} g^{(\pm)}(u) \end{pmatrix}, \tag{2.14}
$$

where we have defined

$$
g^{(+)}(u) = \frac{\sinh(u+3\eta/2)}{\sinh(u+9\eta/2)}, \quad g^{(-)}(u) = \frac{\cosh(u+3\eta/2)}{\cosh(u+9\eta/2)}. \tag{2.15}
$$

We have thus nine possibilities for the commuting transfer matrix (1.11). We will only consider three types of boundary solutions, one for each pair $(K^-(u), K^+(u))$ defined by the automorphism (1.8): $(1, M)$, (F^+, G^+) and (F^-, G^-).

3 Algebraic Bethe Ansatz

The monodromy matrix $T(u)$ (1.10) and its reflection $T^{-1}(-u)$ can be written as matrices 3 by 3

$$T(u)=\begin{pmatrix} T_{11}(u) & T_{12}(u) & T_{13}(u) \\ T_{21}(u) & T_{22}(u) & T_{23}(u) \\ T_{31}(u) & T_{32}(u) & T_{33}(u) \end{pmatrix}, \quad T^{-1}(-u)=\begin{pmatrix} T^{-1}_{11}(-u) & T^{-1}_{12}(-u) & T^{-1}_{13}(-u) \\ T^{-1}_{21}(-u) & T^{-1}_{22}(-u) & T^{-1}_{23}(-u) \\ T^{-1}_{31}(-u) & T^{-1}_{32}(-u) & T^{-1}_{33}(-u) \end{pmatrix} \tag{3.1}$$

where

$$T_{ia}(u)=\sum_{k_1,\ldots,k_{N-1}=1}^{3} \mathcal{L}^{(N)}_{ik_1}(u,\eta)\overset{s}{\otimes}\mathcal{L}^{(N-1)}_{k_1k_2}(u,\eta)\overset{s}{\otimes}\cdots\overset{s}{\otimes}\mathcal{L}^{(1)}_{k_{N-1}a}(u,\eta) \tag{3.2}$$

where $\mathcal{L}^{(n)}_{ij}$ are 3×3 matrices acting on the nth site of the lattice, defined by

$$\mathcal{L}^{(n)}_{11}=\begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{pmatrix}, \quad \mathcal{L}^{(n)}_{12}=\begin{pmatrix} 0 & 0 & 0 \\ x_5 & 0 & 0 \\ 0 & x_6 & 0 \end{pmatrix}, \quad \mathcal{L}^{(n)}_{13}=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x_7 & 0 & 0 \end{pmatrix},$$

$$\mathcal{L}^{(n)}_{21}=\begin{pmatrix} 0 & y_5 & 0 \\ 0 & 0 & y_6 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{L}^{(n)}_{22}=\begin{pmatrix} x_2 & 0 & 0 \\ 0 & x_4 & 0 \\ 0 & 0 & x_2 \end{pmatrix}, \quad \mathcal{L}^{(n)}_{23}=\begin{pmatrix} 0 & 0 & 0 \\ x_6 & 0 & 0 \\ 0 & x_5 & 0 \end{pmatrix},$$

$$\mathcal{L}^{(n)}_{31}=\begin{pmatrix} 0 & 0 & y_7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{L}^{(n)}_{32}=\begin{pmatrix} 0 & y_6 & 0 \\ 0 & 0 & y_5 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{L}^{(n)}_{33}=\begin{pmatrix} x_3 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_1 \end{pmatrix}. \tag{3.3}$$

Using the unitary relation in (2.2) we can see that the reflected monodromy $T^{-1}(-u)$ has the following matrix elements

$$T^{-1}_{bj}(-u)=\frac{1}{f(u)^N}\sum_{k_1,\ldots,k_{N-1}=1}^{3} \mathcal{L}^{(1)}_{bk_1}(-u,-\eta)\overset{s}{\otimes}\mathcal{L}^{(2)}_{k_1k_2}(-u,-\eta)\overset{s}{\otimes}\cdots\overset{s}{\otimes}\mathcal{L}^{(N)}_{k_{N-1}j}(-u,-\eta). \tag{3.4}$$

For the vertex models considered in this paper we can choose the highest weight vector of the monodromy matrix in a lattice of N sites as the even (bosonic) completely unoccupied state

$$|0\rangle=\prod_{k=1}^{N}\overset{s}{\otimes}|0\rangle_k, \qquad |0\rangle_k=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \tag{3.5}$$

where $|0\rangle_k$ is the local reference state at the *k-th* lattice site with three components

The action of $T(u)$ and $T^{-1}(-u)$ on this state are

$$T(u)|0\rangle=f^N(u)T^{-1}(-u)|0\rangle=\begin{pmatrix} x_1^N(u)|0\rangle & * & ** \\ 0 & x_2^N(u)|0\rangle & *** \\ 0 & 0 & x_3^N(u)|0\rangle \end{pmatrix}, \tag{3.6}$$

which give us, in the usual BA language, creation and annihilation operators for (3.5). Moreover, we are working out with boundaries and in this case we have a double-row monodromy defined by

$$U(u)=T(u)K^{-}(u)T^{-1}(-u)=\left(\begin{array}{ccc} U_{11}(u) & U_{12}(u) & U_{13}(u) \\ U_{21}(u) & U_{22}(u) & U_{23}(u) \\ U_{31}(u) & U_{32}(u) & U_{33}(u) \end{array}\right) \tag{3.7}$$

where $K^{(-)}(u)$ is a reflection matrix.

For $K^{(-)}(u)=\mathrm{diag}(k_{11}^{-}(u),k_{22}^{-}(u),k_{33}^{-}(u))$, the matrix elements of U have the form

$$U_{ij}(u)=\sum_{a=1}^{3}T_{ia}(u)k_{aa}^{-}(u)T_{aj}^{-1}(-u), \qquad i,j=1,2,3. \tag{3.8}$$

It follows from (3.8) that we will need to know the commutation relations between the operators $T(u)$ and $T^{-1}(-u)$ in order to get the action of $U(u)$ on the reference state (3.5). Using the fundamental relation (1.9) with $u=-v$ we will get the matrix relation

$$T_2^{-1}(-u)R_{12}(2u)T_1(u)=T_1(u)R_{12}(2u)T_2^{-1}(-u) \tag{3.9}$$

Applying both sides of this relation on the reference state, we find the following relations

$$T_{21}(u)T_{12}^{-1}(-u)\left|0\right\rangle = f_1(u)\frac{x_1^{2N}(u)-x_2^{2N}(u)}{f^N(u)}\left|0\right\rangle \tag{3.10}$$

$$T_{31}(u)T_{13}^{-1}(-u)\left|0\right\rangle = \left\{ f_2(u)\frac{x_1^{2N}(u)}{f^N(u)} - f_3(u)f_1(u)\frac{x_2^{2N}(u)}{f^N(u)} - f_4(u)\frac{x_3^{2N}(u)}{f^N(u)}\right\}\left|0\right\rangle \tag{3.11}$$

$$T_{32}(u)T_{23}^{-1}(-u)\left|0\right\rangle = f_2(u)\frac{x_2^{2N}(u)-x_3^{2N}(u)}{f^N(u)}\left|0\right\rangle \tag{3.12}$$

where

$$\begin{aligned} f_1(u) &= \frac{y_5(2u)}{x_1(2u)}, \qquad f_2(u)=\frac{y_7(2u)}{x_1(2u)}, \\ f_3(u) &= -\frac{x_1(2u)y_5(2u)-x_5(2u)y_7(2u)}{x_1(2u)x_4(2u)+x_5(2u)y_5(2u)}, \\ f_4(u) &= \frac{x_4(2u)y_7(2u)+y_5^2(2u)}{x_1(2u)x_4(2u)+x_5(2u)y_5(2u)}. \end{aligned} \tag{3.13}$$

Using these relations we can get the action of each operator U_{ij} on the reference state: for the diagonal entries we have

$$\begin{aligned} U_{11}(u)\left|0\right\rangle &= k_{11}^{-}(u)\frac{x_1^{2N}(u)}{f^N(u)}\left|0\right\rangle \\ U_{22}(u)\left|0\right\rangle &= f_1(u)U_{11}(u)\left|0\right\rangle + \left[k_{22}^{-}(u)-k_{11}^{-}(u)f_1(u)\right]\frac{x_2^{2N}(u)}{f^N(u)}\left|0\right\rangle \\ U_{33}(u)\left|0\right\rangle &= \left[\left(f_2(u)-f_1(u)f_3(u)\right)U_{11}(u)+f_3(u)U_{22}(u)\right]\left|0\right\rangle \\ &\quad + \left[k_{33}^{-}(u)-k_{22}^{-}(u)f_3(u)-k_{11}^{-}(u)f_4(u)\right]\frac{x_3^{2N}(u)}{f^N(u)}\left|0\right\rangle \end{aligned} \tag{3.14}$$

and for the elements out of the diagonal we get

$$U_{ij}(u)\left|0\right\rangle = 0, \quad (i>j), \qquad U_{ij}(u)\left|0\right\rangle \neq \{0, \left|0\right\rangle\}, \quad (i<j) \tag{3.15}$$

Now, we define news operators:

$$\mathcal{D}_1(u) = U_{11}(u), \qquad \mathcal{B}_1(u) = U_{12}(u), \qquad \mathcal{B}_2(u) = U_{13}(u)$$

$$\mathcal{C}_1(u) = U_{21}(u), \qquad \mathcal{D}_2(u) = U_{22}(u) - f_1(u)\mathcal{D}_1(u), \qquad \mathcal{B}_3(u) = U_{23}(u)$$

$$\mathcal{C}_2(u) = U_{31}(u), \qquad \mathcal{C}_3(u) = U_{32}(u), \qquad \mathcal{D}_3(u) = U_{33}(u) - f_2(u)\mathcal{D}_1(u) - f_3(u)\mathcal{D}_2(u) \tag{3.16}$$

to write the double-row monodromy in the form

$$U(u) \to \mathcal{U}(u) = \left(\begin{array}{ccc} \mathcal{D}_1(u) & \mathcal{B}_1(u) & \mathcal{B}_2(u) \\ \mathcal{C}_1(u) & \mathcal{D}_2(u) & \mathcal{B}_3(u) \\ \mathcal{C}_2(u) & \mathcal{C}_3(u) & \mathcal{D}_3(u) \end{array}\right). \tag{3.17}$$

The action of $\mathcal{U}(u)$ on the reference state has the usual BA form

$$\mathcal{U}(u)\left|0\right\rangle = \left(\begin{array}{ccc} \mathcal{X}_1(u)\left|0\right\rangle & * & ** \\ 0 & \mathcal{X}_2(u)\left|0\right\rangle & *** \\ 0 & 0 & \mathcal{X}_3(u)\left|0\right\rangle \end{array}\right) \tag{3.18}$$

where

$$\begin{aligned} \mathcal{X}_1(u) &= k_{11}^-(u)\frac{x_1^{2N}(u)}{f^N(u)} \\ \mathcal{X}_2(u) &= \left[k_{22}^-(u) - k_{11}^-(u)f_1(u)\right]\frac{x_2^{2N}(u)}{f^N(u)} \\ \mathcal{X}_3(u) &= \left[k_{33}^-(u) - k_{22}^-(u)f_3(u) - k_{11}^-(u)f_4(u)\right]\frac{x_3^{2N}(u)}{f^N(u)} \end{aligned} \tag{3.19}$$

The transfer matrix $t(u) = \mathrm{str}(K^+U)$, with diagonal left reflection $K^{(+)} = \mathrm{diag}(k_{11}^+, k_{22}^+ . k_{33}^+)$ and BFB grading, has the form

$$\begin{aligned} t(u) &= k_{11}^+(u)U_{11}(u) - k_{22}^+(u)U_{22}(u) + k_{33}^+(u)U_{33}(u) \\ &= \Omega_1(u)\mathcal{D}_1(u) + \Omega_2(u)\mathcal{D}_2(u) + \Omega_3(u)\mathcal{D}_3(u) \end{aligned} \tag{3.20}$$

where

$$\begin{aligned} \Omega_1(u) &= k_{11}^+(u) - f_1(u)k_{22}^+(u) + f_2(u)k_{33}^+(u) \\ \Omega_2(u) &= -k_{22}^+(u) + f_3(u)k_{33}^+(u) \\ \Omega_3(u) &= k_{33}^+(u) \end{aligned} \tag{3.21}$$

Here we note the sign $(-)$ from the graded trace is absorbed in the definition of $\Omega_2(u)$.

From $\mathcal{U}(u)$ follows the usual algebraic BA structure. Therefore we can look for states created by the operators $\mathcal{B}_i(u)$ from a reference Ψ_0 which will be eigenstates of (3.20). To do this we first recall the magnon number operator

$$M = \sum_{k=1}^{N} M_k, \qquad M_k = \mathrm{diag}(0,1,2) \tag{3.22}$$

This is the analogue of the operator S_T^z used in the coordinate BA construction. The relation $m = N - S_T^z$ ($M\Psi_m = m\Psi_m$) allows us build states Ψ_m such that $t(u)\Psi_m = \Lambda_m\Psi_m$. Therefore, we can start the diagonalization of $t(u)$ by considering all possible values of m in a lattice with N sites.

By the previous construction, Ψ_0 is our reference state $|0>$, which is itself an eigenstate of $t(u)$

$$t(u)\Psi_0 = \Lambda_0(u)\Psi_0 \tag{3.23}$$

with eigenvalue

$$\begin{aligned}
\Lambda_0(u) &= \left[k_{11}^{+}(u) - f_1(u)k_{22}^{+}(u) + f_2(u)k_{33}^{+}(u)\right] k_{11}^{-}(u)\frac{x_1^{2N}(u)}{f^N(u)} \\
&+ \left[-k_{22}^{+}(u) + f_3(u)k_{33}^{+}(u)\right]\left[k_{22}^{-}(u) - k_{11}^{-}(u)f_1(u)\right]\frac{x_2^{2N}(u)}{f^N(u)} \\
&+ k_{33}^{+}(u)\left[k_{33}^{-}(u) - k_{22}^{-}(u)f_3(u) - k_{11}^{-}(u)f_4(u)\right]\frac{x_3^{2N}(u)}{f^N(u)}
\end{aligned} \tag{3.24}$$

It is the only state with $m = 0$.

3.1 The one-particle state

For $m = 1$ we seek a state of the form

$$\Psi_1(u_1) = \mathcal{B}_1(u_1)\left|0\right\rangle. \tag{3.25}$$

The action of $t(u)$ on this state is given by

$$t(u)\Psi_1(u_1) = \Omega_1(u)\mathcal{D}_1(u)\mathcal{B}_1(u_1)\left|0\right\rangle + \Omega_2(u)\mathcal{D}_2(u)\mathcal{B}_1(u_1)\left|0\right\rangle + \Omega_3(u)\mathcal{D}_3(u)\mathcal{B}_1(u_1)\left|0\right\rangle. \tag{3.26}$$

Since we know from (3.18) the action of the operators $\mathcal{D}_i(u)$ on the reference state $|0\rangle$, we need to arrange the operators products

$$\mathcal{D}_1(u)\mathcal{B}_1(u_1), \quad \mathcal{D}_2(u)\mathcal{B}_1(u_1) \quad \text{and} \quad \mathcal{D}_3(u)\mathcal{B}_1(u_1) \tag{3.27}$$

in a normal-ordered form [16]: We anticipate that, in general, the operator-valued function $\Psi_n(u_1, \dots, u_n)$ for a n-particle Bethe state will be composed by a set of normal-ordered monomials. A monomial is said to be in normal order if all elements $\mathcal{B}_i$ are on the left, and all elements $\mathcal{C}_i$ are on the right of the elements $\mathcal{D}_i$.

In order to get this normal ordering we recall that the double-row monodromy matrix $\mathcal{U}(u)$ satisfies the fundamental reflection equation

$$\mathcal{R}_{12}(u-v)\mathcal{U}_1(u)\mathcal{R}_{21}(u+v)\mathcal{U}_2(v)=\mathcal{U}_2(v)\mathcal{R}_{12}(u+v)\mathcal{U}_1(u)\mathcal{R}_{21}(u-v), \tag{3.28}$$

where $\mathcal{U}_1(u)=\mathcal{U}(u)\overset{s}{\otimes}1,\ \mathcal{U}_2(u)=1\overset{s}{\otimes}\mathcal{U}(u)$ and $\mathcal{R}_{21}(u)=\mathcal{P}\mathcal{R}_{12}(u)\mathcal{P}$.

In the appendix of [7] it was shown as this equation can be used to recast the non-normal ordered products as the above into a linear combination of normal-ordered ones. However, in this article we will present only the main results. (See below)

For the present case $t(u)\Psi_1(u_1)$ can be computed with the aid of the following normal-ordered (or commutation) relations

$$\begin{aligned}\mathcal{D}_1(u)\mathcal{B}_1(u_1) &= a_{11}(u,u_1)\mathcal{B}_1(u_1)\mathcal{D}_1(u)+a_{12}(u,u_1)\mathcal{B}_1(u)\mathcal{D}_1(u_1)+a_{13}(u,u_1)\mathcal{B}_1(u)\mathcal{D}_2(u_1)\\ &+a_{14}(u,u_1)\mathcal{B}_2(u)\mathcal{C}_1(u_1)+a_{15}(u,u_1)\mathcal{B}_2(u)\mathcal{C}_3(u_1)+a_{16}(u,u_1)\mathcal{B}_2(u_1)\mathcal{C}_1(u)\end{aligned} \tag{3.29}$$

$$\begin{aligned}\mathcal{D}_2(u)\mathcal{B}_1(u_1) &= a_{21}(u,u_1)\mathcal{B}_1(u_1)\mathcal{D}_2(u)+a_{22}(u,u_1)\mathcal{B}_1(u)\mathcal{D}_1(u_1)+a_{23}(u,u_1)\mathcal{B}_1(u)D_2(u_1)\\ &+a_{24}(u,u_1)\mathcal{B}_3(u)\mathcal{D}_1(u_1)+a_{25}(u,u_1)\mathcal{B}_3(u)\mathcal{D}_2(u_1)+a_{26}(u,u_1)\mathcal{B}_2(u)\mathcal{C}_1(u_1)\\ &+a_{27}(u,u_1)\mathcal{B}_2(u)\mathcal{C}_3(u_1)+a_{28}(u,u_1)\mathcal{B}_2(u_1)\mathcal{C}_1(u)+a_{29}(u,u_1)\mathcal{B}_2(u_1)\mathcal{C}_3(u)\end{aligned} \tag{3.30}$$

$$\begin{aligned}\mathcal{D}_3(u)\mathcal{B}_1(u_1) &= a_{31}(u,u_1)\mathcal{B}_1(u_1)\mathcal{D}_3(u)+a_{32}(u,u_1)\mathcal{B}_1(u)\mathcal{D}_1(u_1)+a_{33}(u,u_1)\mathcal{B}_1(u)\mathcal{D}_2(u_1)\\ &+a_{34}(u,u_1)\mathcal{B}_3(u)\mathcal{D}_1(u_1)+a_{35}(u,u_1)\mathcal{B}_3(u)\mathcal{D}_2(u_1)+a_{36}(u,u_1)\mathcal{B}_2(u)\mathcal{C}_1(u_1)\\ &+a_{37}(u,u_1)\mathcal{B}_2(u)\mathcal{C}_3(u_1)+a_{38}(u,u_1)\mathcal{B}_2(u_1)\mathcal{C}_1(u)+a_{39}(u,u_1)\mathcal{B}_2(u_1)\mathcal{C}_3(u)\end{aligned} \tag{3.31}$$

Substituting these relations into (3.26) one gets

$$\begin{aligned}t(u)\Psi_1(u_1) &= \Omega_1(u)\mathcal{D}_1(u)\mathcal{B}_1(u_1)\left|0\right\rangle+\Omega_2(u)\mathcal{D}_2(u)\mathcal{B}_1(u_1)\left|0\right\rangle+\Omega_3(u)\mathcal{D}_3(u)\mathcal{B}_1(u_1)\left|0\right\rangle\\ &= \left[a_{11}(u,u_1)\Omega_1(u)\mathcal{X}_1(u)+a_{21}(u,u_1)\Omega_2(u)\mathcal{X}_2(u)+a_{31}(u,u_1)\Omega_3(u)\mathcal{X}_3(u)\right]\Psi_1(u_1)\\ &+[\mathcal{X}_1(u_1)\sum_{j=1}^{3}\Omega_j(u)a_{j2}(u,u_1)+\mathcal{X}_2(u_1)\sum_{j=1}^{3}\Omega_j(u)a_{j3}(u,u_1)]B_1(u)\left|0\right\rangle\\ &+[\mathcal{X}_1(u_1)\sum_{j=2}^{3}\Omega_j(u)a_{j4}(u,u_1)+\mathcal{X}_2(u_1)\sum_{j=2}^{3}\Omega_j(u)a_{j5}(u,u_1)]B_3(u)\left|0\right\rangle\end{aligned} \tag{3.32}$$

So, $\Psi_1(u_1)$ will be an eigenstate of $t(u)$ with eigenvalue

$$\Lambda_1(u,u_1) = \sum_{j=1}^{3} \Omega_j(u)\mathcal{X}_j(u)a_{j1}(u,u_1) \tag{3.33}$$

provided the following equation is satisfied

$$\frac{\mathcal{X}_1(u_1)}{\mathcal{X}_2(u_1)} = -\frac{\sum_{j=1}^{3}\Omega_j(u)a_{j3}(u,u_1)}{\sum_{j=1}^{3}\Omega_j(u)a_{j2}(u,u_1)} = -\frac{\sum_{j=2}^{3}\Omega_j(u)a_{j5}(u,u_1)}{\sum_{j=2}^{3}\Omega_j(u)a_{j4}(u,u_1)} \equiv \Theta(u_1). \tag{3.34}$$

Explicit calculations of these expressions for the models considered in this paper will be presented in the next section.

3.2 The two-particle state

For $m = 2$ we have to seek eigenstates of $t(u)$ in the form

$$\Psi_2(u_1,u_2) = \mathcal{B}_1(u_1)\mathcal{B}_1(u_2)\left|0\right\rangle + \mathcal{B}_2(u_1)\Gamma(u_1,u_2)\left|0\right\rangle \tag{3.35}$$

where $\Gamma(u_1,u_2)$ is an operator-valued function. Next we will use the condition that $\Psi_2(u_1,u_2)$ must be normal-ordered to find $\Gamma(u_1,u_2)$.

The first term of the right hand side of (3.35) has its normal-ordered form gives by the commutations relations:

$$\begin{aligned}\mathcal{B}_1(u_1)\mathcal{B}_1(u_2) &= \omega(u_1,u_2)\left[\mathcal{B}_1(u_2)\mathcal{B}_1(u_1) + G_{d_1}(u_2,u_1)\mathcal{B}_2(u_2)\mathcal{D}_1(u_1) + G_{d_2}(u_2,u_1)\mathcal{B}_2(u_2)\mathcal{D}_2(u_1)\right] \\ &\quad -G_{d_1}(u_1,u_2)\mathcal{B}_2(u_1)\mathcal{D}_1(u_2) - G_{d_2}(u_1,u_2)\mathcal{B}_2(u_1)\mathcal{D}_2(u_2)\end{aligned} \tag{3.36}$$

where

$$\omega(u_1,u_2) = -\frac{x_3(u_1-u_2)x_4(u_1-u_2) - x_6(u_1-u_2)y_6(u_1-u_2)}{x_1(u_1-u_2)x_3(u_1-u_2)} \tag{3.37}$$

$$G_{d_1}(u_1,u_2) = \frac{x_6(u_1-u_2)}{x_3(u_1-u_2)}\frac{x_2(2u_2)}{x_1(2u_2)} \tag{3.38}$$

$$G_{d_2}(u_1,u_2) = -\frac{x_6(u_1+u_2)}{x_2(u_1+u_2)} \tag{3.39}$$

Here we have used the following identities valid for both models,

$$\frac{y_6(-u)}{x_3(-u)} = \frac{x_3(u)x_6(u) - x_7(u)y_6(u)}{x_3(u)x_4(u) - x_6(u)y_6(u)} \tag{3.40}$$

$$\frac{x_2(2u)}{x_1(2u)} = \frac{y_5(u-v)x_2(u+v) + x_2(u-v)x_5(u+v)f_1(u)}{y_5(u-v)x_1(u+v)} \tag{3.41}$$

Now we can see that (3.35) is normally ordered if it satisfies the condition

$$\Psi_2(u_2, u_1) = \omega(u_2, u_1)\Psi_2(u_1, u_2) \tag{3.42}$$

This condition fixes $\Gamma(u_1, u_2)$ and, by construction, the unique candidate for eigenstate of $t(u)$ in the case $m = 2$ has the form

$$\Psi_2(u_1, u_2) = \mathcal{B}_1(u_1)\mathcal{B}_1(u_2)\left|0\right\rangle + G_{d_1}(u_1, u_2)\mathcal{B}_2(u_1)\mathcal{D}_1(u_2)\left|0\right\rangle + G_{d_2}(u_1, u_2)\mathcal{B}_2(u_1)\mathcal{D}_2(u_2)\left|0\right\rangle \tag{3.43}$$

The action of $t(u)$ on this state in its normal ordered form is obtained with the following commutation relations, in addition to those presented in the case $m = 1$ case:

$$\begin{aligned}\mathcal{D}_1(u)\mathcal{B}_2(v) &= b_{11}(u, v)\mathcal{B}_2(v)\mathcal{D}_1(u) + b_{12}(u, v)\mathcal{B}_2(u)\mathcal{D}_1(v) + b_{13}(u, v)\mathcal{B}_2(u)\mathcal{D}_2(v) \\ &\quad + b_{14}(u, v)\mathcal{B}_2(u)\mathcal{D}_3(v) + b_{15}(u, v)\mathcal{B}_1(u)\mathcal{B}_1(v) + b_{16}(u, v)\mathcal{B}_1(u)\mathcal{B}_3(v)\end{aligned} \tag{3.44}$$

$$\begin{aligned}\mathcal{D}_2(u)\mathcal{B}_2(v) &= b_{21}(u, v)\mathcal{B}_2(v)\mathcal{D}_2(u) + b_{22}(u, v)\mathcal{B}_2(u)\mathcal{D}_1(v) + b_{23}(u, v)\mathcal{B}_2(u)\mathcal{D}_2(v) \\ &\quad + b_{24}(u, v)\mathcal{B}_2(u)\mathcal{D}_3(v) + b_{25}(u, v)\mathcal{B}_1(u)\mathcal{B}_1(v) + b_{26}(u, v)\mathcal{B}_1(u)\mathcal{B}_3(v) \\ &\quad + b_{27}(u, v)\mathcal{B}_3(u)\mathcal{B}_1(v) + b_{28}(u, v)\mathcal{B}_3(u)\mathcal{B}_3(v)\end{aligned} \tag{3.45}$$

$$\begin{aligned}\mathcal{D}_3(u)\mathcal{B}_2(v) &= b_{31}(u, v)\mathcal{B}_2(v)\mathcal{D}_3(u) + b_{32}(u, v)\mathcal{B}_2(u)\mathcal{D}_1(v) + b_{33}(u, v)\mathcal{B}_2(u)\mathcal{D}_2(v) \\ &\quad + b_{34}(u, v)\mathcal{B}_2(u)\mathcal{D}_3(v) + b_{35}(u, v)\mathcal{B}_1(u)\mathcal{B}_1(v) + b_{36}(u, v)\mathcal{B}_1(u)\mathcal{B}_3(v) \\ &\quad + b_{37}(u, v)\mathcal{B}_3(u)\mathcal{B}_1(v) + b_{38}(u, v)\mathcal{B}_3(u)\mathcal{B}_3(v)\end{aligned} \tag{3.46}$$

$$\begin{aligned}\mathcal{C}_1(u)\mathcal{B}_1(v) &= c_{11}(u, v)\mathcal{B}_1(v)\mathcal{C}_1(u) + c_{12}(u, v)\mathcal{B}_1(v)\mathcal{C}_3(v) + c_{13}(u, v)\mathcal{B}_1(u)\mathcal{C}_3(v) \\ &\quad + c_{14}(u, v)\mathcal{B}_2(u)\mathcal{C}_3(v) + c_{15}(u, v)\mathcal{B}_2(v)\mathcal{C}_2(u) + c_{16}(u, v)\mathcal{D}_1(v)\mathcal{D}_1(u) \\ &\quad + c_{17}(u, v)\mathcal{D}_1(v)\mathcal{D}_2(u) + c_{18}(u, v)\mathcal{D}_1(u)\mathcal{D}_1(v) + c_{19}(u, v)\mathcal{D}_1(u)\mathcal{D}_2(v) \\ &\quad + c_{110}(u, v)\mathcal{D}_2(u)\mathcal{D}_1(v) + c_{111}(u, v)\mathcal{D}_2(u)\mathcal{D}_2(v)\end{aligned} \tag{3.47}$$

$$\begin{aligned}\mathcal{C}_3(u)\mathcal{B}_1(v) &= c_{21}(u, v)\mathcal{B}_1(v)\mathcal{C}_1(u) + c_{22}(u, v)\mathcal{B}_1(v)\mathcal{C}_3(v) + c_{23}(u, v)\mathcal{B}_1(u)\mathcal{C}_3(v) \\ &\quad + c_{24}(u, v)\mathcal{B}_2(u)\mathcal{C}_3(v) + c_{25}(u, v)\mathcal{B}_2(v)\mathcal{C}_2(u) + c_{26}(u, v)\mathcal{D}_1(v)\mathcal{D}_1(u) \\ &\quad + c_{27}(u, v)\mathcal{D}_1(v)\mathcal{D}_2(u) + c_{28}(u, v)\mathcal{D}_1(u)\mathcal{D}_3(v) + c_{29}(u, v)\mathcal{D}_1(u)\mathcal{D}_2(v) \\ &\quad + c_{210}(u, v)\mathcal{D}_1(u)\mathcal{D}_2(v) + c_{211}(u, v)\mathcal{D}_2(u)\mathcal{D}_1(v) + c_{212}(u, v)\mathcal{D}_2(u)\mathcal{D}_2(v) \\ &\quad + c_{213}(u, v)\mathcal{D}_3(u)\mathcal{D}_1(v) + c_{214}(u, v)\mathcal{D}_3(u)\mathcal{D}_1(v))\end{aligned} \tag{3.48}$$

After a straightforward calculation we obtain

$$t(u)\Psi_2(u_1, u_2) = [\sum_{j=1}^{3} \Omega_j(u)\mathcal{X}_j(u)a_{j1}(u, u_1)a_{j1}(u, u_2)]\Psi_2(u_1, u_2)$$

$$+[a_{11}(u_1,u_2)\mathcal{X}_1(u_1)\sum_{j=1}^{3}\Omega_j(u)a_{j2}(u,u_1)+a_{21}(u_1,u_2)\mathcal{X}_2(u_1)\sum_{j=1}^{3}\Omega_j(u)a_{j3}(u,u_1)]\mathcal{B}_1(u)\mathcal{B}_1(u_2)\left|0\right\rangle$$

$$+[a_{11}(u_1,u_2)\mathcal{X}_1(u_1)\sum_{j=2}^{3}\Omega_j(u)a_{j4}(u,u_1)+a_{21}(u_1,u_2)\mathcal{X}_2(u_1)\sum_{j=2}^{3}\Omega_j(u)a_{j5}(u,u_1)]\mathcal{B}_3(u)\mathcal{B}_1(u_2)\left|0\right\rangle$$

$$+[a_{11}(u_2,u_1)\mathcal{X}_1(u_2)\sum_{j=1}^{3}\Omega_j(u)a_{j2}(u,u_2)+a_{21}(u_2,u_1)\mathcal{X}_2(u_2)\sum_{j=1}^{3}\Omega_j(u)a_{j3}(u,u_2)]\omega(u_1,u_2)\mathcal{B}_1(u)B_1(u_1)\left|0\right\rangle$$

$$+[a_{11}(u_2,u_1)\mathcal{X}_1(u_2)\sum_{j=2}^{3}\Omega_j(u)a_{j4}(u,u_2)+a_{21}(u_2,u_1)\mathcal{X}_2(u_2)\sum_{j=2}^{3}\Omega_j(u)a_{j5}(u,u_2)]\omega(u_1,u_2)\mathcal{B}_3(u)\mathcal{B}_1(u_1)\left|0\right\rangle$$

$$\begin{aligned}&+[\mathcal{X}_1(u_1)\mathcal{X}_1(u_2)\sum_{j=1}^{3}\Omega_j(u)H_{j1}(u_1,u_2)+\mathcal{X}_1(u_1)\mathcal{X}_2(u_2)\sum_{j=1}^{3}\Omega_j(u)H_{j3}(u_1,u_2)\\&+\mathcal{X}_2(u_1)\mathcal{X}_1(u_2)\sum_{j=1}^{3}\Omega_j(u)H_{j2}(u_1,u_2)+\mathcal{X}_2(u_1)\mathcal{X}_2(u_2)\sum_{j=1}^{3}\Omega_j(u)H_{j4}(u_1,u_2)]\mathcal{B}_2(u)\left|0\right\rangle\end{aligned}\qquad(3.49)$$

where

$$\begin{aligned}H_{11}(u_1,u_2)&=a_{14}(u,u_1)\left(c_{16}(u_1,u_2)+c_{18}(u_1,u_2)\right)+a_{15}(u,u_1)\left(c_{26}(u_1,u_2)+c_{29}(u_1,u_2)\right)\\&\quad+b_{12}(u,u_1)G_{d_1}(u_1,u_2)+\omega(u_1,u)a_{11}(u,u_1)a_{12}(u,u_2)G_{d_1}(u,u_1)\\H_{12}(u_1,u_2)&=a_{14}(u,u_1)\left(c_{17}(u_1,u_2)+c_{110}(u_1,u_2)\right)+a_{15}(u,u_1)\left(c_{27}(u_1,u_2)+c_{211}(u_1,u_2)\right)\\&\quad+b_{13}(u,u_1)G_{d_1}(u_1,u_2)+\omega(u_1,u)a_{11}(u,u_1)a_{12}(u,u_2)G_{d_2}(u,u_1)\\H_{13}(u_1,u_2)&=a_{14}(u,u_1)c_{19}(u_1,u_2)+a_{15}(u,u_1)c_{210}(u_1,u_2)+b_{12}(u,u_1)G_{d_2}(u_1,u_2)\\&\quad+\omega(u_1,u)a_{11}(u,u_1)a_{13}(u,u_2)G_{d_1}(u,u_1)\\H_{14}(u_1,u_2)&=a_{14}(u,u_1)c_{111}(u_1,u_2)+a_{15}(u,u_1)c_{212}(u_1,u_2)+b_{13}(u,u_1)G_{d_2}(u_1,u_2)\\&\quad+\omega(u_1,u)a_{11}(u,u_1)a_{13}(u,u_2)G_{d_2}(u,u_1)\end{aligned}\qquad(3.50)$$

and

$$
\begin{aligned}
H_{j1}(u_1,u_2) &= a_{j6}(u,u_1)\left(c_{16}(u_1,u_2)+c_{18}(u_1,u_2)\right)+a_{j7}(u,u_1)\left(c_{26}(u_1,u_2)+c_{29}(u_1,u_2)\right)\\
&+b_{j2}(u,u_1)G_{d_1}(u_1,u_2)+\omega(u_1,u)a_{j1}(u,u_1)a_{j2}(u,u_2)G_{d_1}(u,u_1)\\
&+a_{j1}(u,u_1)a_{j4}(u,u_2)d_{13}(u_1,u)\\
H_{j2}(u_1,u_2) &= a_{j6}(u,u_1)\left(c_{17}(u_1,u_2)+c_{110}(u_1,u_2)\right)+a_{j7}(u,u_1)\left(c_{27}(u_1,u_2)+c_{211}(u_1,u_2)\right)\\
&+b_{j3}(u,u_1)G_{d_1}(u_1,u_2)+\omega(u_1,u)a_{j1}(u,u_1)a_{j2}(u,u_2)G_{d_2}(u,u_1)\\
&+a_{j1}(u,u_1)a_{j4}(u,u_2)d_{14}(u_1,u)\\
H_{j3}(u_1,u_2) &= a_{j6}(u,u_1)c_{19}(u_1,u_2)+a_{j7}(u,u_1)c_{210}(u_1,u_2)+b_{j2}(u,u_1)G_{d_2}(u_1,u_2)\\
&+\omega(u_1,u)a_{j1}(u,u_1)a_{j3}(u,u_2)G_{d_1}(u,u_1)+a_{j1}(u,u_1)a_{j5}(u,u_2)d_{13}(u_1,u)\\
H_{j4}(u_1,u_2) &= a_{j6}(u,u_1)c_{111}(u_1,u_2)+a_{j7}(u,u_1)c_{212}(u_1,u_2)+b_{j3}(u,u_1)G_{d_2}(u_1,u_2)\\
&+\omega(u_1,u)a_{j1}(u,u_1)a_{j3}(u,u_2)G_{d_2}(u,u_1)+a_{j1}(u,u_1)a_{j5}(u,u_2)d_{14}(u_1,u) \qquad (3.51)
\end{aligned}
$$

for $j=2,3$.

Again, $\Psi_2(u_1,u_2)$ will be an eigenstate of $t(u)$ with eigenvalue

$$
\Lambda_2(u,u_1,u_2)=\sum_{j=1}^{3}\Omega_j(u)\mathcal{X}_j(u)a_{j1}(u,u_1)a_{j1}(u,u_2) \qquad (3.52)
$$

provided the following equations are satisfied

$$
\frac{\mathcal{X}_1(u_1)}{\mathcal{X}_2(u_1)}=\Theta(u,u_1)\frac{a_{21}(u_1,u_2)}{a_{11}(u_1,u_2)},\qquad \frac{\mathcal{X}_1(u_2)}{\mathcal{X}_2(u_2)}=\Theta(u,u_2)\frac{a_{21}(u_2,u_1)}{a_{11}(u_2,u_1)} \qquad (3.53)
$$

where $\Theta(u_i),\ i=1,2$ are given by (3.34).

3.3 The n-particle state

From the previous results we can are seek for operator valued functions with a recurrence relation of the form

$$
\begin{aligned}
\Phi_n(u,\dots,u_n) &= \mathcal{B}_1(u_1)\Phi_{n-1}(u_2,\dots,u_n)\\
&+\mathcal{B}_2(u_1)\sum_{i=2}^{n}F_1^{(i)}(u_1,\dots,u_n)\Phi_{n-2}(u_2,\dots,\overset{\vee}{u}_i,\dots,u_n)\mathcal{D}_1(u_i)\\
&+\mathcal{B}_2(u_1)\sum_{i=2}^{n}F_2^{(i)}(u_1,\dots,u_n)\Phi_{n-2}(u_2,\dots,\overset{\vee}{u}_i,\dots,u_n)\mathcal{D}_2(u_i) \qquad (3.54)
\end{aligned}
$$

It was shown in [6] that operator is normal ordered satisfying $n-1$ exchange conditions

$$
\Phi_n(u_1,\dots,u_i,u_{i+1},\dots,u_n)=\omega(u_i,u_{i+1})\Phi_n(u_1,\dots,u_{i+1},u_i,\dots,u_n) \qquad (3.55)
$$

provided the functions $F_{\alpha}^{(i)}(u_1,\dots,u_n)$ are given by

$$F_{\alpha}^{(i)}(u_1,\dots,u_n)=\prod_{j=2}^{i-1}\omega(u_j,u_i)\prod_{k=2,k\neq i}^{n}a_{\alpha 1}(u_i,u_k)G_{d_\alpha}(u_1,u_i), \qquad (\alpha=1,2) \tag{3.56}$$

Therefore the n-particle state will be given by

$$\Psi_n(u_1,\dots,u_n)=\Phi_n(u,\dots,u_n)\left|0\right\rangle \tag{3.57}$$

and the action of the operators $\mathcal{D}_\alpha(u)$, $\alpha=1,2,3$, on this state will be represented by

$$\begin{aligned}
\mathcal{D}_\alpha(u)\Psi_n(u_1,\dots,u_n) &= \mathcal{X}_\alpha(u)\prod_{i=1}^{n}a_{\alpha 1}(u,u_i)\Psi_n(u_1,\dots,u_n)\\
&+\sum_{i=1}^{n}\prod_{j=1}^{i-1}\omega(u_j,u_i)[\mathcal{X}_1(u)a_{\alpha 2}(u,u_i)\prod_{j\neq i}^{n}a_{11}(u,u_j)+\mathcal{X}_2(u)a_{\alpha 3}(u,u_i)\prod_{j\neq i}^{n}a_{21}(u,u_j)]\mathcal{B}_1(u)\Psi_{n-1}(\overset{\vee}{u_i})\\
&+(1-\delta_{\alpha,1})\sum_{i=1}^{n}\prod_{j=1}^{i-1}\omega(u_j,u_i)[\mathcal{X}_1(u)a_{\alpha 4}(u,u_i)\prod_{j\neq i}^{n}a_{11}(u,u_j)+\mathcal{X}_2(u)a_{\alpha 5}(u,u_i)\prod_{j\neq i}^{n}a_{21}(u,u_j)]\mathcal{B}_3(u)\Psi_{n-1}(\overset{\vee}{u_i})\\
&+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left\{\mathcal{X}_1(u_i)\mathcal{X}_1(u_j)\prod_{k\neq i,j}^{n}a_{11}(u_i,u_k)\prod_{l\neq i,j}^{n}a_{11}(u_j,u_l)H_{\alpha 1}(u_i,u_j)\right.\\
&+\mathcal{X}_2(u_i)\mathcal{X}_1(u_j)\prod_{k\neq i,j}^{n}a_{21}(u_i,u_k)\prod_{l\neq i,j}^{n}a_{11}(u_j,u_l)H_{\alpha 2}(u_i,u_j)\\
&+\mathcal{X}_1(u_i)\mathcal{X}_2(u_j)\prod_{k\neq i,j}^{n}a_{11}(u_i,u_k)\prod_{l\neq i,j}^{n}a_{21}(u_j,u_l)H_{\alpha 3}(u_i,u_j)\\
&\left.+\mathcal{X}_2(u_i)\mathcal{X}_2(u_j)\prod_{k\neq i,j}^{n}a_{21}(u_i,u_k)\prod_{l\neq i,j}^{n}a_{21}(u_j,u_l)H_{\alpha 4}(u_i,u_j)\right\}\\
&\times\prod_{k=1}^{i-1}\omega(u_k,u_i)\prod_{l=1\neq i}^{i-1}\omega(u_l,u_j)\mathcal{B}_2(u)\Psi_{n-2}(\overset{\vee}{u_i},\overset{\vee}{u_j})
\end{aligned} \tag{3.58}$$

Finally, the corresponding n-particle eigenvalue problem will be

$$t(u)\Psi_n(u_1,\dots,u_n)=\left(\sum_{\alpha=1}^{3}\Omega_\alpha(u)\mathcal{X}_\alpha(u)\prod_{i=1}^{n}a_{\alpha 1}(u,u_i)\right)\Psi_n(u_1,\dots,u_n) \tag{3.59}$$

provided that the BA equations are satisfied

$$\frac{\mathcal{X}_1(u_k)}{\mathcal{X}_2(u_k)}=\Theta(u_k)\prod_{j=1,j\neq k}^{n}\frac{a_{21}(u_k,u_j)}{a_{11}(u_k,u_j)}, \qquad (k=1,2,\dots,n) \tag{3.60}$$

4 Explicit solutions

In this section explicit expressions of the eigenvalue problem are presented for both models,. First we recall the fundamental relation (3.28) to get the coefficients $a_{ij}(u,v)$ which appear effectively in the BA expressions (3.59) and (3.60):

$$\begin{aligned}
a_{11}(u,v) &= \frac{x_1(u-v)}{x_2(u-v)}\frac{x_2(u+v)}{x_1(u+v)} \\
a_{21}(u,v) &= -\omega(u,v)[\frac{x_1(u+v)x_4(u+v)+x_5(u+v)y_5(u+v)}{x_1(u+v)x_2(u+v)}] \\
a_{31}(u,v) &= \frac{x_2(u-v)}{x_3(u-v)}[\frac{x_2(u+v)^2+x_6(u+v)y_6(u+v)}{x_2(u+v)x_3(u+v)}]
\end{aligned} \tag{4.1}$$

For the factor with the boundary contributions (3.34) we will consider only the expression

$$\Theta(u_i) = -\frac{\Omega_2(u)a_{25}(u,u_i)+\Omega_3(u)a_{35}(u,u_i)}{\Omega_2(u)a_{24}(u,u_i)+\Omega_3(u)a_{34}(u,u_i)} \tag{4.2}$$

where the $\Omega_j(u)$ are given by (3.21) and

$$\begin{aligned}
a_{24}(u,v) &= -[\frac{x_6(u-v)}{x3(u-v)}\frac{x_3(u+v)}{x_2(u+v)} - f_1(v)\frac{x_6(u+v)}{x_2(u+v)}] \\
a_{25}(u,v) &= \frac{x_6(u+v)}{x_2(u+v)} \\
a_{34}(u,v) &= -f_3(u)[f_1(v)\frac{x_6(u+v)}{x_2(u+v)} - \frac{x_6(u-v)}{x_3(u-v)}\frac{x_3(u+v)}{x_2(u+v)}] \\
&\quad +f_1(v)\frac{y_6(u-v)}{x_3(u-v)}[\frac{x_6(u+v)y_6(u+v)+x_2^2(u+v)}{x_2(u+v)x_3(u+v)}] \\
&\quad -[\frac{x_6(u-v)y_6(u-v)+x_2^2(u-v)}{x_3^2(u-v)}]\frac{y_6(u+v)}{x_2(u+v)} \\
a_{35}(u,v) &= -f_3(u)\frac{x_6(u+v)}{x_2(u+v)} + \frac{y_6(u-v)}{x_3(u-v)}[\frac{x_6(u+v)y_6(u+v)+x_2^2(u+v)}{x_2(u+v)x_3(u+v)}]
\end{aligned} \tag{4.3}$$

with the $f_i(u)$ given by (3.13).

4.1 $\mathbf{sl(2|1)^{(2)}}$ model

Substituting the matrix elements of the $\mathcal{R}$ matrix (2.5) and the matrix elements of the K matrices (2.7) and (2.9) for this model we get the following expressions in our algebraic BA: first the coefficients $a_{j1}(u,u_i)$ for the eigenvalue $\Lambda_n(u,\{u_i\})$

$$a_{11}(u,u_i) = \frac{\sinh(u+u_i)}{\sinh(u+u_i+2\eta)}\frac{\sinh(u-u_i-2\eta)}{\sinh(u-u_i)}, \tag{4.4}$$

$$a_{21}(u,u_i) = \frac{\sinh(u+u_i)}{\sinh(u+u_i+2\eta)}\frac{\sinh(u-u_i-2\eta)}{\sinh(u-u_i)}\frac{\cosh(u+u_i+3\eta)}{\cosh(u+u_i+\eta)}\frac{\cosh(u-u_i+\eta)}{\cosh(u-u_i-\eta)}, \tag{4.5}$$

$$a_{31}(u,u_i)=\frac{\cosh(u+u_i+3\eta)}{\cosh(u+u_i+\eta)}\frac{\cosh(u-u_i+\eta)}{\cosh(u-u_i-\eta)},$$

Second, the factors of $\Lambda_n(u,\{u_i\})$ and of the BA equations with boundary contributions

$$\Omega_1(u)=\frac{\cosh(2u+3\eta)}{\cosh(2u+\eta)}\frac{\alpha\sinh u-2\cosh u}{\alpha\sinh(u+2\eta)-2\cosh(u+2\eta)}\frac{\alpha\cosh(u+\eta)-2\sinh(u+\eta)}{\alpha\cosh(u+\eta)+2\sinh(u+\eta)}, \tag{4.6}$$

$$\Omega_2(u)=-\frac{\sinh(2u+2\eta)}{\sinh(2u)}\frac{\alpha\sinh u-2\cosh u}{\alpha\sinh(u+2\eta)-2\cosh(u+2\eta)}, \tag{4.7}$$

$$\Omega_3(u)=-\frac{\alpha\sinh u-2\cosh u}{\alpha\sinh(u+2\eta)-2\cosh(u+2\eta)}, \tag{4.8}$$

$$\mathcal{X}_1(u)=-\frac{\beta\sinh u+2\cosh u}{\beta\sinh u-2\cosh u}\frac{x_1^{2N}(u)}{f^N(u)}, \tag{4.9}$$

$$\mathcal{X}_2(u)=\frac{\sinh(2u)}{\sinh(2u+2\eta)}\frac{\beta\sinh(u+2\eta)-2\cosh(u+2\eta)}{\beta\sinh u-2\cosh u}\frac{x_2^{2N}(u)}{f^N(u)}, \tag{4.10}$$

$$\mathcal{X}_3(u)=\frac{\cosh(2u-\eta)}{\cosh(2u+\eta)}\frac{\beta\sinh(u+2\eta)-2\cosh(u+2\eta)}{\beta\sinh u-2\cosh u}\frac{\beta\cosh(u+\eta)-2\sinh(u+\eta)}{\beta\cosh(u-\eta)+2\sinh(u-\eta)}\frac{x_3^{2N}(u)}{f^N(u)}, \tag{4.11}$$

and

$$\Theta(u_i)=\frac{\sinh(2u_i+2\eta)}{\sinh(2u_i)}\frac{\alpha\cosh(u_i+\eta)+2\sinh(u_i+\eta)}{\alpha\cosh(u_i+\eta)-2\sinh(u_i+\eta)}. \tag{4.12}$$

From these data one can see that the n-particle state $\Psi_n(\{u_i\})$ is an eigenfuction of the transfer matrix (3.20) for the $sl(2|1)^{(2)}$ vertex model with eigenvalue

$$\begin{aligned}\Lambda_n(u,\{u_i\}) &= \Omega_1(u)\mathcal{X}_1(u)\prod_{i=1}^{n}\frac{\sinh(u+u_i-\eta)}{\sinh(u+u_i+\eta)}\frac{\sinh(u-u_i-\eta)}{\sinh(u-u_i+\eta)}\\ &-\Omega_2(u)\mathcal{X}_2(u)\prod_{i=1}^{n}\frac{\sinh(u+u_i-\eta)}{\sinh(u+u_i+\eta)}\frac{\sinh(u-u_i-\eta)}{\sinh(u-u_i+\eta)}\\ &\times\frac{\cosh(u+u_i+2\eta)}{\cosh(u+u_i)}\frac{\cosh(u-u_i+2\eta)}{\cosh(u-u_i)}\\ &+\Omega_3(u)\mathcal{X}_3(u)\prod_{i=1}^{n}\frac{\cosh(u+u_i+2\eta)}{\cosh(u+u_i)}\frac{\cosh(u-u_i+2\eta)}{\cosh(u-u_i)}\end{aligned} \tag{4.13}$$

provided that the parameters $\{u_i\}$ satisfy the BA equations

$$\begin{aligned}\left(\frac{\sinh(u_i+\eta}{\sinh(u_i-\eta)}\right)^{2N} &= -\frac{\alpha\cosh u_i+2\sinh u_i}{\alpha\cosh u_i-2\sinh u_i}\frac{\beta\sinh(u_i+\eta)-2\cosh(u_i+\eta)}{\beta\sinh(u_i-\eta)-2\cosh(u_i-\eta)}\\ &\times\prod_{\{j\neq i\}=1}^{n}\frac{\cosh(u_i+u_j+\eta)}{\cosh(u_i+u_j-\eta)}\frac{\cosh(u_i-u_j+\eta)}{\cosh(u_i-u_j-\eta)}\\ i &= 1,2,...,n\end{aligned} \tag{4.14}$$

where we have used the shifts $u_i \rightarrow u_i = u_i - \eta$ to bring these expressions into a symmetric form in η.

The formulation of this model in terms of the QISM presented here is new. However, one can verify that our results give the energy eigenspectrum previously obtained in the framework of coordinate BA by Fireman *et al.* [17].

4.2 osp(2|1) model

For this model the K matrices have no free parameters but we have to consider three cases. For both cases the coefficients $a_{j1}(u, u_i)$ are given by

$$
\begin{aligned}
a_{11}(u,u_i) &= \frac{\sin(u+u_i)}{\sinh(u+u_i+2\eta)} \frac{\sinh(u-u_i-2\eta)}{\sinh(u-u_i)}, \\
a_{21}(u,u_i) &= \frac{\sin(u+u_i+4\eta)}{\sinh(u+u_i+3\eta)} \frac{\sinh(u+u_i+\eta)}{\sinh(u+u_i+2\eta)} \frac{\sin(u-u_i+2\eta)}{\sinh(u-u_i)} \frac{\sinh(u-u_i-\eta)}{\sinh(u-u_i+\eta)}, \\
a_{31}(u,u_i) &= \frac{\sinh(u+u_i+5\eta)}{\sinh(u+u_i+3\eta)} \frac{\sinh(u-u_i+3\eta)}{\sinh(u-u_i+\eta)}.
\end{aligned} \tag{4.15}
$$

Though, the boundary contributions are different for each cases:

4.2.1 The (1,M) case

In this case we have

$$
\begin{aligned}
\Omega_1(u) &= \frac{\sinh(2u+\eta)}{\sinh(2u+2\eta)} \frac{\sinh(2u+6\eta)}{\sinh(2u+3\eta)} \\
\Omega_2(u) &= -\mathrm{e}^{2\eta} \frac{\sinh(2u+6\eta)}{\sinh(2u+4\eta)} \\
\Omega_3(u) &= \mathrm{e}^{2\eta}
\end{aligned} \tag{4.16}
$$

$$
\begin{aligned}
\mathcal{X}_1(u) &= \frac{x_1^{2N}(u)}{f^N(u)} \\
\mathcal{X}_2(u) &= \mathrm{e}^{-2\eta} \frac{\sinh(2u)}{\sinh(2u+2\eta)} \frac{x_2^{2N}(u)}{f^N(u)} \\
\mathcal{X}_3(u) &= \mathrm{e}^{-2\eta} \frac{\sinh(2u)}{\sinh(2u+4\eta)} \frac{\sinh(2u+5\eta)}{\sinh(2u+3\eta)} \frac{x_3^{2N}(u)}{f^N(u)}
\end{aligned} \tag{4.17}
$$

and

$$
\Theta(u_i) = \mathrm{e}^{2\eta} \frac{\sinh(2u_i+2\eta)}{\sinh(2u_i)}, \qquad i = 1, ..., n \tag{4.18}
$$

Therefore, the n-particle state $\Psi_n(\{u_i\})$ is an eigenfuction of the transfer matrix (3.20) for the

$osp(2|1)$ vertex model with boundaries $(1, M)$. The corresponding eigenvalue is given by

$$
\begin{aligned}
\Lambda_n(u,\{u_i\}) &= \frac{\sinh(2u+\eta)}{\sinh(2u+2\eta)}\frac{\sinh(2u+6\eta)}{\sinh(2u+3\eta)}\frac{x_1^{2N}(u)}{f^N(u)}\prod_{i=1}^{n}\frac{\sin(u+u_i)}{\sinh(u+u_i+2\eta)}\frac{\sinh(u-u_i-2\eta)}{\sinh(u-u_i)} \\
&\quad -\frac{\sinh(2u+6\eta)}{\sinh(2u+4\eta)}\frac{\sinh(2u)}{\sinh(2u+2\eta)}\frac{x_2^{2N}(u)}{f^N(u)}\prod_{i=1}^{n}\frac{\sin(u+u_i+4\eta)}{\sinh(u+u_i+3\eta)}\frac{\sinh(u+u_i+\eta)}{\sinh(u+u_i+2\eta)} \\
&\quad \times\frac{\sin(u-u_i+2\eta)}{\sinh(u-u_i)}\frac{\sinh(u-u_i-\eta)}{\sinh(u-u_i+\eta)} \\
&\quad +\frac{\sinh(2u)}{\sinh(2u+4\eta)}\frac{\sinh(2u+5\eta)}{\sinh(2u+3\eta)}\frac{x_3^{2N}(u)}{f^N(u)}\prod_{i=1}^{n}\frac{\sinh(u+u_i+5\eta)}{\sinh(u+u_i+3\eta)}\frac{\sinh(u-u_i+3\eta)}{\sinh(u-u_i+\eta)} \qquad (4.19)
\end{aligned}
$$

provided that its parameters $\{u_i\}$are solutions of the BA equations

$$
\begin{aligned}
\left(\frac{\sinh(u_i+2\eta)}{\sinh u_i}\right)^{2N} &= \prod_{\{j\neq i\}=1}^{n}\frac{\sin(u_i+u_j+4\eta)}{\sinh(u_i+u_j+3\eta)}\frac{\sinh(u_i+u_j+\eta)}{\sin(u_i+u_j)}\frac{\sin(u_i-u_j+2\eta)}{\sinh(u_i-u_j-2\eta)}\frac{\sinh(u_i-u_j-\eta)}{\sinh(u_i-u_j+\eta)} \\
i &= 1,..,n \qquad (4.20)
\end{aligned}
$$

4.2.2 The ($\mathbf{F}^+$,$\mathbf{G}^+$) case

Here we have

$$
\begin{aligned}
\Omega_1(u) &= -\mathrm{e}^{2u}\frac{\sinh(2u+6\eta)}{\sinh(2u+2\eta)}\frac{\sinh(u+\frac{5}{2}\eta)}{\sinh(u+\frac{9}{2}\eta)}\frac{\cosh(u+\frac{1}{2}\eta)}{\cosh(u+\frac{3}{2}\eta)} \\
\Omega_2(u) &= -\frac{\sinh(2u+6\eta)}{\sinh(2u+4\eta)}\frac{\sinh(u+\frac{5}{2}\eta)}{\sinh(u+\frac{9}{2}\eta)} \\
\Omega_3(u) &= -\mathrm{e}^{-2u-4\eta}\frac{\sinh(u+\frac{3}{2}\eta)}{\sinh(u+\frac{9}{2}\eta)} \qquad (4.21)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{X}_1(u) &= -\mathrm{e}^{-2u}\frac{\sinh(u+\frac{3}{2}\eta)}{\sinh(u-\frac{3}{2}\eta)}\frac{x_1^{2N}(u)}{f^N(u)} \\
\mathcal{X}_2(u) &= \frac{\sinh(2u)}{\sinh(2u+2\eta)}\frac{\sinh(u+\frac{1}{2}\eta)}{\sinh(u-\frac{3}{2}\eta)}\frac{x_2^{2N}(u)}{f^N(u)} \\
\mathcal{X}_3(u) &= -\mathrm{e}^{2u+4\eta}\frac{\sinh(2u)}{\sinh(2u+4\eta)}\frac{\sinh(u+\frac{1}{2}\eta)}{\sinh(u-\frac{3}{2}\eta)}\frac{\cosh(u+\frac{5}{2}\eta)}{\cosh(u+\frac{3}{2}\eta)}\frac{x_3^{2N}(u)}{f^N(u)} \qquad (4.22)
\end{aligned}
$$

and

$$
\Theta(u_i) = -\mathrm{e}^{-2u_i}\frac{\sinh(2u_i+2\eta)}{\sinh(2u_i)}\frac{\sinh(u_i+\frac{1}{2}\eta)}{\sinh(u_i+\frac{3}{2}\eta)} \qquad (4.23)
$$

Therefore, the n-particle state $\Psi_n(\{u_i\})$ is an eigenfuction of the transfer matrix (3.20) for the $osp(2|1)$ vertex model with boundaries (F^+, G^+). The corresponding eigenvalue is given by

$$\begin{aligned}\Lambda_n(u,\{u_i\}) &= \Omega_1(u)\mathcal{X}_1(u)\prod_{i=1}^{n}\frac{\sin(u+u_i-\eta)}{\sinh(u+u_i+\eta)}\frac{\sinh(u-u_i-\eta)}{\sinh(u-u_i+\eta)} \\ &+\Omega_2(u)\mathcal{X}_2(u)\prod_{i=1}^{n}\frac{\sin(u+u_i+3\eta)}{\sinh(u+u_i+2\eta)}\frac{\sin(u-u_i+3\eta)}{\sinh(u-u_i+\eta)} \\ &\times\frac{\sinh(u+u_i)}{\sinh(u+u_i+\eta)}\frac{\sinh(u-u_i)}{\sinh(u-u_i+2\eta)} \\ &+\Omega_3(u)\mathcal{X}_3(u)\prod_{i=1}^{n}\frac{\sinh(u+u_i+4\eta)}{\sinh(u+u_i+2\eta)}\frac{\sinh(u-u_i+4\eta)}{\sinh(u-u_i+2\eta)}\end{aligned} \tag{4.24}$$

with the BA equations

$$\begin{aligned}\left(\frac{\sinh(u_i+\eta)}{\sinh(u_i-\eta)}\right)^{2N} &= \left(\frac{\sinh(u_i-\frac{1}{2}\eta)}{\sinh(u_i+\frac{1}{2}\eta)}\right)^{2}\prod_{\{j\neq i\}=1}^{n}\frac{\sin(u_i+u_j+2\eta)}{\sinh(u_i+u_j+\eta)}\frac{\sin(u_i-u_j+2\eta)}{\sinh(u_i-u_j-2\eta)} \\ &\times\frac{\sinh(u_i+u_j-\eta)}{\sin(u_i+u_j+\eta)}\frac{\sinh(u_i-u_j-\eta)}{\sinh(u_i-u_j+\eta)} \\ i &= 1,...,n\end{aligned} \tag{4.25}$$

Again, $\Lambda_n(u,\{u_i\})$ and the Bethe equations have been written in their symmetric form.

4.2.3 The $(\mathbf{F}^-,\mathbf{G}^-)$ case

Here we have

$$\begin{aligned}\Omega_1(u) &= \mathrm{e}^{2u}\frac{\sinh(2u+6\eta)}{\sinh(2u+2\eta)}\frac{\cosh(u+\frac{5}{2}\eta)}{\cosh(u+\frac{9}{2}\eta)}\frac{\sinh(u+\frac{1}{2}\eta)}{\sinh(u+\frac{3}{2}\eta)} \\ \Omega_2(u) &= -\frac{\sinh(2u+6\eta)}{\sinh(2u+4\eta)}\frac{\cosh(u+\frac{5}{2}\eta)}{\cosh(u+\frac{9}{2}\eta)} \\ \Omega_3(u) &= \mathrm{e}^{-2u-4\eta}\frac{\cosh(u+\frac{3}{2}\eta)}{\cosh(u+\frac{9}{2}\eta)}\end{aligned} \tag{4.26}$$

$$\begin{aligned}\mathcal{X}_1(u) &= \mathrm{e}^{-2u}\frac{\cosh(u+\frac{3}{2}\eta)}{\cosh(u-\frac{3}{2}\eta)}\frac{x_1^{2N}(u)}{f^N(u)} \\ \mathcal{X}_2(u) &= \frac{\sinh(2u)}{\sinh(2u+2\eta)}\frac{\cosh(u+\frac{1}{2}\eta)}{\cosh(u-\frac{3}{2}\eta)}\frac{x_2^{2N}(u)}{f^N(u)} \\ \mathcal{X}_3(u) &= \mathrm{e}^{2u+4\eta}\frac{\sinh(2u)}{\sinh(2u+4\eta)}\frac{\cosh(u+\frac{1}{2}\eta)}{\cosh(u-\frac{3}{2}\eta)}\frac{\sinh(u+\frac{5}{2}\eta)}{\sinh(u+\frac{3}{2}\eta)}\frac{x_3^{2N}(u)}{f^N(u)}\end{aligned} \tag{4.27}$$

$$\Theta(u_i) = \mathrm{e}^{-2u_i}\frac{\sinh(2u_i+2\eta)}{\sinh(2u_i)}\frac{\cosh(u_i+\frac{1}{2}\eta)}{\cosh(u_i+\frac{3}{2}\eta)} \tag{4.28}$$

Therefore, the n-particle state $\Psi_n(\{u_i\})$ is an eigenfuction of the transfer matrix (3.20) for the $osp(2|1)$ vertex model with boundaries (F^-, G^-). The corresponding eigenvalue is given by

$$\begin{aligned}
\Lambda_n(u,\{u_i\}) \;=\; & \Omega_1(u)\mathcal{X}_1(u)\prod_{i=1}^{n}\frac{\sin(u+u_i-\eta)}{\sinh(u+u_i+\eta)}\frac{\sinh(u-u_i-\eta)}{\sinh(u-u_i+\eta)} \\
& +\Omega_2(u)\mathcal{X}_2(u)\prod_{i=1}^{n}\frac{\sin(u+u_i+3\eta)}{\sinh(u+u_i+2\eta)}\frac{\sin(u-u_i+3\eta)}{\sinh(u-u_i+\eta)} \\
& \times\frac{\sinh(u+u_i)}{\sinh(u+u_i+\eta)}\frac{\sinh(u-u_i)}{\sinh(u-u_i+2\eta)} \\
& +\Omega_3(u)\mathcal{X}_3(u)\prod_{i=1}^{n}\frac{\sinh(u+u_i+4\eta)}{\sinh(u+u_i+2\eta)}\frac{\sinh(u-u_i+4\eta)}{\sinh(u-u_i+2\eta)}
\end{aligned} \tag{4.29}$$

and the BA equations are now given by

$$\begin{aligned}
\left(\frac{\sinh(u_i+\eta)}{\sinh(u_i-\eta)}\right)^{2N} \;=\; & \left(\frac{\cosh(u_i-\frac{1}{2}\eta)}{\cosh(u_i+\frac{1}{2}\eta)}\right)^{2}\prod_{\{j\neq i\}=1}^{n}\frac{\sin(u_i+u_j+2\eta)}{\sinh(u_i+u_j+\eta)}\frac{\sin(u_i-u_j+2\eta)}{\sinh(u_i-u_j-2\eta)} \\
& \times\frac{\sinh(u_i+u_j-\eta)}{\sin(u_i+u_j-2\eta)}\frac{\sinh(u_i-u_j-\eta)}{\sinh(u_i-u_j+\eta)}
\end{aligned} \tag{4.30}$$

Here, both $\Lambda_n(u,\{u_i\})$ and the BA equation were written with $u_i \to u_i - \eta$.

These three cases were also considered in [17] via the coordinate BA.

5 Conclusion

Here, with the aid of previous works [6, 7], two of the three-state graded 19-vertex models have their boundary algebraic BA derived using a generalization of the Tarasov's approach [16]. From our results for the transfer matrix one can in principle derive the free-energy thermodynamics, the quasi-particle excitations behavior as well as the classes of universality governing the criticality of gapless regimes with integrable boundary conditions. Moreover, the rather universal formula we obtained for the eigenvectors could be useful in future computations of off-shell properties such as form factors and correlation functions with boundary conditions of relevant operators.

The algebraic BA for n-state models with periodic boundary conditions was developed by Martins in [18]. In a recent paper Galleas and Martins [19] have presented the algebraic BA for the vertex models based on superalgebras. Therefore we believe that the Martins's approach can be generalized to include the diagonal open boundary conditions.

Finally we observe that the vertex models discussed in this paper share a common algebraic structure with the non-graded 19-vertex models, the Zamolodchikov-Fateev [20] and the Izergin-Korepin [21] models. Thus, we expected that a fusion procedure for the $sl(2|1)^{(2)}$ model as well as the analytical BA formulation based in the quantum group invariance of the $osp(2|1)$ model can reproduce our results.

Acknowledgment: This work was supported in part by Fundação de Amparo à Pesquisa do Estado de São Paulo-FAPESP-Brasil and by Conselho Nacional de Desenvolvimento-CNPq-Brasil.

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