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MATHEMATICAL PROBLEMS OF NONLINEARITY

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Optimal Bang-Bang Trajectories in Sub-Finsler Problems on the Engel Group

Yu. Sachkov

The Engel group is the four-dimensional nilpotent Lie group of step 3, with 2 generators. We consider a one-parameter family of left-invariant rank 2 sub-Finsler problems on the Engel group with the set of control parameters given by a square centered at the origin and rotated by an arbitrary angle. We adopt the viewpoint of time-optimal control theory. By Pontryagin's maximum principle, all sub-Finsler length minimizers belong to one of the following types: abnormal, bang-bang, singular, and mixed. Bang-bang controls are piecewise controls with values in the vertices of the set of control parameters.

We describe the phase portrait for bang-bang extremals.

In previous work, it was shown that bang-bang trajectories with low values of the energy integral are optimal for arbitrarily large times. For optimal bang-bang trajectories with high values of the energy integral, a general upper bound on the number of switchings was obtained.

In this paper we improve the bounds on the number of switchings on optimal bang-bang trajectories via a second-order necessary optimality condition due to A. Agrachev and R. Gamkrelidze. This optimality condition provides a quadratic form, whose sign-definiteness is related to optimality of bang-bang trajectories. For each pattern of these trajectories, we compute the maximum number of switchings of optimal control. We show that optimal bang-bang controls may have not more than 9 switchings. For particular patterns of bang-bang controls, we obtain better bounds. In such a way we improve the bounds obtained in previous work.

On the basis of the results of this work we can start to study the cut time along bang-bang trajectories, i.e., the time when these trajectories lose their optimality. This question will be considered in subsequent work.

Keywords: sub-Finsler problem, Engel group, bang-bang extremal, optimality condition

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Yuri Sachkov
yusachkov@gmail.com
Control Processes Research Center
A. K. Ailamazyan Program Systems Institute of RAS
Pereslavl-Zalessky, Russia

1. Introduction

Sub-Finsler geometry is a natural generalization of the sub-Riemannian one. A sub-Riemannian geometry on a smooth manifold M is given by a vector distribution Δ on M and an inner product in Δ. A sub-Finsler structure is defined by a norm in Δ.

In recent years there has been a noticeable interest in sub-Finsler geometry in view of its applications in geometric group theory [1], spaces with length metrics [2], and control theory [3]. An important question of both sub-Finsler and sub-Riemannian geometry is the description of length minimizers and spheres, and the natural simplest cases here are nilpotent structures. The left-invariant sub-Finsler problem on the Heisenberg group was studied in [7, 8]. Nilpotent l_∞ sub-Finsler structures in the Martinet and Grushin cases were studied in [4]. Left-invariant sub-Finsler problems on the Engel and Cartan groups were studied via convex trigonometry techniques in [5]. Moreover, these techniques were applied to generalizations of a series of classical optimization problems to the sub-Finsler case [6].

The next natural case is the Engel group, the 4-dimensional nilpotent Lie group of step 3 and rank 2. A study of a one-parameter family of sub-Finsler structures on the Engel group with the set of control parameters given by a square was started in [9]. The sub-Finsler problems were considered as time-optimal control problems. Pontryagin's maximum principle was applied, and extremal trajectories were described. Some upper bounds on the number of smooth pieces of optimal bang-bang and mixed trajectories were presented.

In this note we continue that work. We describe the phase portrait for bang-bang extremals and present detailed optimality conditions which improve the bounds on the number of smooth pieces of optimal bang-bang trajectories given in [9].

2. Problem statement

The Engel algebra $\mathfrak{g}$ is the 4-dimensional nilpotent Lie algebra with 2 generators, of step 3. In a standard basis of the Engel algebra $\mathfrak{g} = \operatorname{span}(f_1, f_2, f_3, f_4)$ the product table has the form $[f_1, f_2] = f_3$, $[f_1, f_3] = f_4$, $\operatorname{ad} f_4 = 0$. The simply connected Lie group G with the Lie algebra $\mathfrak{g}$ is called the Engel group. In some coordinates $G \cong \mathbb{R}^4_{x,y,z,v}$ the Engel algebra is realized by left-invariant vector fields on G:

$$f_1 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}, \qquad f_2 = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z} + \frac{x^2+y^2}{2}\frac{\partial}{\partial v},$$
$$f_3 = \frac{\partial}{\partial z} + x\frac{\partial}{\partial v}, \qquad f_4 = \frac{\partial}{\partial v}.$$

Define vector fields ($\varphi \in [0, \pi/4]$)

$$X_1 = \cos\varphi f_1 + \sin\varphi f_2, \quad X_2 = -\sin\varphi f_1 + \cos\varphi f_2, \quad X_3 = f_3, \quad X_4 = f_4.$$

Consider the following family of sub-Finsler problems on the Engel group ($\varphi \in [0, \pi/4]$):

$$\dot q = u_1 X_1 + u_2 X_2, \qquad q \in G, \quad u \in U, \tag{2.1}$$
$$U = \{u \in \mathbb{R}^2 \mid \|u\|_\infty = \max(|u_1|, |u_2|) \leqslant 1\}, \tag{2.2}$$
$$q(0) = q_0 = \mathrm{Id}, \qquad q(T) = q_1, \tag{2.3}$$
$$T \to \min. \tag{2.4}$$

The existence of optimal controls follows from the Rashevsky – Chow and Filippov theorems [10].

3. Pontryagin's maximum principle

Introduce Hamiltonians $h_i(\lambda) = \langle\lambda,\ X_i\rangle$, $\lambda \in T^*G$, $i = 1, \ldots, 4$, and the corresponding Hamiltonian vector fields $\vec{h}_i \in \mathrm{Vec}(T^*M)$.

Theorem 1 ([10, 11]). *If a control $u(t)$ and the corresponding trajectory $q(t)$, $t \in [0, T]$, are optimal, then there exist a curve $\lambda_t \in T^*_{q(t)}G$ and a number $\nu \leqslant 0$ for which the following conditions hold:*

$$\begin{aligned} &\dot\lambda_t = u_1(t)\vec{h}_1(\lambda_t) + u_2(t)\vec{h}_2(\lambda_t), \qquad (3.1)\\ &u_1(t)h_1(\lambda_t) + u_2(t)h_2(\lambda_t) = H(\lambda_t) = (|h_1| + |h_2|)(\lambda_t),\\ &\lambda_t \neq 0,\\ &H(\lambda_t) + \nu \equiv 0. \end{aligned}$$

The Hamiltonian system (3.1) has 3 integrals — Casimir functions on the Lie coalgebra $\mathfrak{g}^*$: h_4, $E = h_3^2/2 - (\sin\varphi h_1 + \cos\varphi h_2)h_4$, and the Hamiltonian H.

4. Abnormal trajectories

Let $\nu = 0$.

Then the optimal abnormal controls are $u(t) \equiv \pm(\tan\varphi, 1)$.

5. Classes of normal extremal arcs

Let $-\nu = H(\lambda_t) > 0$. An extremal arc λ_t, $t \in I = (\alpha, \beta) \subset [0, T]$, is called:

- a bang-bang arc if $\mathrm{card}\{t \in I \mid h_1h_2(\lambda_t) = 0\} < \infty$,
- a singular arc if one of the following conditions holds: $h_1(\lambda_t) \equiv 0$ or $h_2(\lambda_t) \equiv 0$,
- a mixed arc if it consists of a finite number of bang-bang and singular arcs.

Remark 1. If $h_i(\lambda_t)|_{(\alpha,\beta)} \neq 0$, then $u_i(t)|_{(\alpha,\beta)} \equiv s_i := \mathrm{sgn}\, h_i(\lambda_t)|_{(\alpha,\beta)}$.

All singular arcs are optimal [9].

6. Bang-bang flow

If $h_1h_2(\lambda_t)|_{(\alpha,\beta)} \neq 0$, then $u(t)|_{(\alpha,\beta)} \equiv (s_1, s_2)$, thus bang-bang extremals satisfy the following Hamiltonian system with the maximized Hamiltonian $H = |h_1| + |h_2|$:

$$\begin{cases} \dot h_1 = -s_2h_3,\\ \dot h_2 = s_1h_3,\\ \dot h_3 = (s_1\cos\varphi - s_2\sin\varphi)h_4,\\ \dot h_4 = 0,\\ \dot q = s_1X_1 + s_2X_2. \end{cases} \qquad (6.1)$$

In view of the symmetry $(\lambda, q) \mapsto (k\lambda, q)$, $k > 0$, we assume in the sequel that $H(\lambda_t) \equiv 1$.

Consider the cylinder

$$C = \mathfrak{g}^* \cap \{H = 1\}.$$

In [9] it was shown that bang-bang trajectories can be represented as images of an exponential mapping: $\{q(t)\} = \mathrm{Exp}(\lambda, t)$, $\lambda \in C$, $t > 0$. The exponential mapping is single-valued for generic $\lambda \in C$, and is multi-valued for certain special subsets of C.

Let us parameterize the square $\{(h_1, h_2) \mid H(\lambda) = 1\}$ by an angle coordinate $\theta \in \mathbb{R}/2\pi\mathbb{Z}$:

$$h_1 = \operatorname{sgn}(\cos\theta)\cos^2\theta, \quad h_2 = \operatorname{sgn}(\sin\theta)\sin^2\theta.$$

Then the vertical part of system (6.1) takes the form

$$\begin{cases} \dot\theta = \dfrac{h_3}{|\sin 2\theta|}, \quad \theta \neq \dfrac{\pi n}{2}, \\ \dot h_3 = (s_1\cos\varphi - s_2\sin\varphi)h_4, \\ s_1 = \operatorname{sgn}\cos\theta, \quad s_2 = \operatorname{sgn}\sin\theta. \end{cases} \tag{6.2}$$

System (6.2) is preserved by the group of symmetries $\{\mathrm{Id}, \varepsilon^1\} \cong \mathbb{Z}_2$, where

$$\varepsilon^1\colon\ (h_1, h_2, h_3, h_4) \mapsto (-h_1, -h_2, h_3, -h_4),$$
$$(s_1, s_2) \mapsto (-s_1, -s_2).$$

We factorize by action of this group and reduce system (6.2) to the fundamental domain of this group $\{(h_1, h_2, h_3, h_4) \in \mathbb{R}^4 \mid h_4 \geqslant 0\}$.

7. Phase portrait of system (6.2)

We consider system (6.2) as an oscillator, with the full energy

$$E = \frac{h_3^2}{2} - (\sin\varphi h_1 + \cos\varphi h_2)h_4 = \frac{h_3^2}{2} + U(\theta)$$

and the potential energy

$$U(\theta) = -(\sin\varphi h_1 + \cos\varphi h_2)h_4 = -(s_1\sin\varphi\cos^2\theta + s_2\cos\varphi\sin^2\theta)h_4.$$

The function $U(\theta)$ is C^1-smooth at $\theta = \frac{\pi n}{2}$ and analytic elsewhere.

7.1. Case 1): $h_4 > 0$

7.1.1. Subcase 1a): $\varphi = 0$

The phase portrait of system (6.2) is drawn as a set of curves $h_3 = \pm\sqrt{2(E - U(\theta))}$, see Fig. 1.

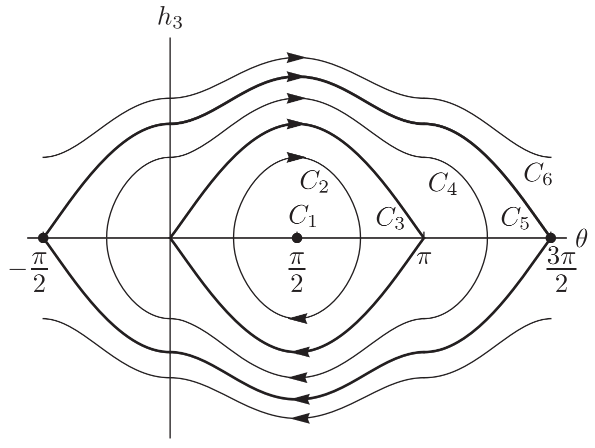

Fig. 1. Phase portrait of system (6.2) in case 1a).

We have a decomposition of a section of the cylinder $C = \mathfrak{g}^* \cap \{H = 1\}$ into domains with qualitatively different trajectories of system (6.2):

$$\{\lambda \in C \mid h_4 > 0\} = \cup_{i=1}^6 C_i,$$
$$C_1 = E^{-1}(-h_4), \quad C_2 = E^{-1}(-h_4, 0), \quad C_3 = E^{-1}(0),$$
$$C_4 = E^{-1}(0, h_4), \quad C_5 = E^{-1}(h_4), \quad C_6 = E^{-1}(h_4, +\infty).$$

7.1.2. Subcase 1b): $\varphi = \pi/4$

The phase portrait of system (6.2) is shown in Fig. 2.

We have a decomposition of a section of the cylinder $C = \mathfrak{g}^* \cap \{H = 1\}$:

$$\{\lambda \in C \mid h_4 > 0\} = \cup_{i=1}^4 C_i,$$
$$C_1 = E^{-1}(-h_4/\sqrt{2}), \qquad C_2 = E^{-1}(-h_4/\sqrt{2}, h_4/\sqrt{2}),$$
$$C_3 = E^{-1}(h_4/\sqrt{2}), \qquad C_4 = E^{-1}(h_4/\sqrt{2}, +\infty).$$

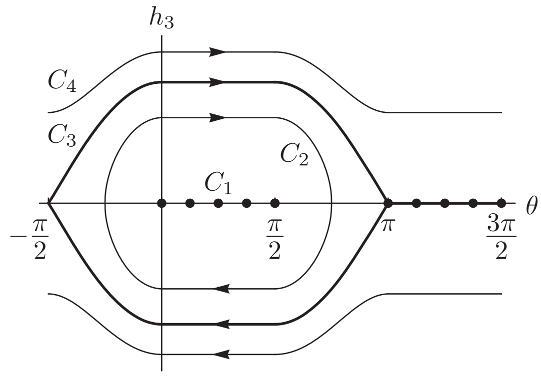

Fig. 2. Phase portrait of system (6.2) in case 1b).

7.1.3. Subcase 1c): $\varphi \in (0, \pi/4)$

The phase portrait of system (6.2) is shown in Fig. 3.

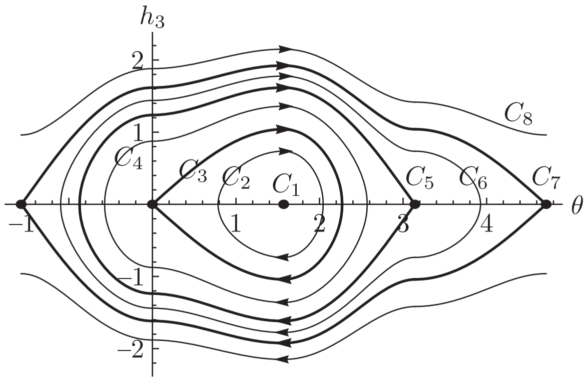

Fig. 3. Phase portrait of system (6.2) in case 1c).

We have a decomposition of a section of the cylinder $C = \mathfrak{g}^* \cap \{H = 1\}$:

$$\{\lambda \in C \mid h_4 > 0\} = \cup_{i=1}^{8} C_8,$$
$$C_1 = E^{-1}(-h_4 \cos \varphi), \qquad C_2 = E^{-1}(-h_4 \cos \varphi, -h_4 \sin \varphi),$$
$$C_3 = E^{-1}(-h_4 \sin \varphi), \qquad C_4 = E^{-1}(-h_4 \sin \varphi, h_4 \sin \varphi),$$
$$C_5 = E^{-1}(h_4 \sin \varphi), \qquad C_6 = E^{-1}(h_4 \sin \varphi, h_4 \cos \varphi),$$
$$C_7 = E^{-1}(h_4 \cos \varphi), \qquad C_8 = E^{-1}(h_4 \cos \varphi, +\infty).$$

7.2. Case 2): $h_4 = 0$

In this case the phase portrait of (6.2) is shown in Fig. 4.

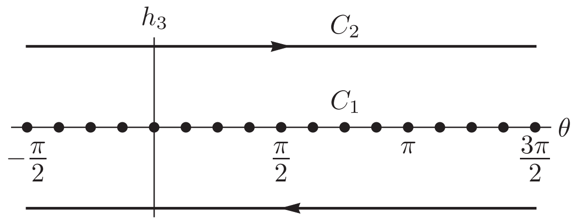

Fig. 4. Phase portrait of system (6.2) in case 2).

The critical level line $C_1 = E^{-1}(0)$ consists of fixed points, and the domain of regular values of energy is $C_2 = E^{-1}(0, +\infty)$. We have

$$\{\lambda \in C \mid h_4 = 0\} = C_1 \cup C_2.$$

8. Optimality of bang-bang trajectories

8.1. Bang-bang trajectories with low energy E

In [9] the following optimality result was obtained for bang-bang trajectories with low energy E.

Theorem 2 ([9]). *If a bang-bang extremal $\lambda_t, t \in [0, +\infty)$, satisfies the conditions*

$$\varphi \in [0, \pi/4), \qquad -|h_4| \cos \varphi < E \leqslant -|h_4| \sin \varphi,$$

then it is optimal.

8.2. Bang-bang trajectories with high energy E

Further, in [9] the following optimality result was obtained for bang-bang trajectories with high energy E.

Theorem 3 ([9]). *If $(\varphi \in [0, \pi/4)$ and $-|h_4| \sin \varphi < E)$ or $\varphi = \pi/4$, then optimal trajectories have not more than* 10 *switchings.*

The main goal of this paper is to obtain detailed optimality results for each pattern of bang-bang trajectory and to improve Theorem 3.

9. The Agrachev – Gamkrelidze theorem

We obtain an upper bound on the number of switchings on optimal bang-bang trajectories via the following theorem due to A. Agrachev and R. Gamkrelidze.

Theorem 4 ([4, 12]). *Let $(q(\cdot), u(\cdot))$ be an extremal pair for problem (2.1)–(2.4) and let $\lambda.$ be an extremal lift of $q(\cdot)$. Assume that $\lambda.$ is the unique extremal lift of $q(\cdot)$, up to multiplication by a positive scalar. Assume that there exist $0 = t_0 < t_1 < t_2 < \ldots < t_k < \tau_{k+1} = T$ and $u^0, \ldots, u^k \in U$ such that $u(\cdot)$ is constantly equal to u^j on (τ_j, τ_{j+1}) for $j = 0, \ldots, k$.*

Fix $j = 1, \ldots, k$. For $i = 0, \ldots, k$ let $Y_i = u_1^i X_1 + u_2^i X_2$ and define recursively the operators

$$P_j = P_{j-1} = \underset{\mathrm{Vec}(M)}{\mathrm{Id}},$$

$$P_i = P_{i-1} \circ e^{(t_i - t_{i-1}) \operatorname{ad} Y_{i-1}}, \qquad i = j+1, \ldots, k,$$

$$P_i = P_{i+1} \circ e^{-(t_{i+2} - t_{i+1}) \operatorname{ad} Y_{i+1}}, \qquad i = 0, \ldots, j-2.$$

Define the vector fields

$$Z_i = P_i(Y_i), \quad i = 0, \ldots, k.$$

Let Q be the quadratic form

$$Q(\alpha) = \sum_{0 \leqslant i < l \leqslant k} \alpha_i \alpha_l \langle \lambda_{t_j}, [Z_i, Z_l](q(t_j)) \rangle,$$

defined on the space

$$W = \left\{ \alpha = (\alpha_0, \ldots, \alpha_k) \in \mathbb{R}^{k+1} \mid \sum_{i=0}^{k} \alpha_i = 0, \quad \sum_{i=0}^{k} \alpha_i Z_i(q(t_j)) = 0 \right\}.$$

If Q is not negative-semidefinite, then $q(\cdot)$ is not optimal.

We will check the sign of the quadratic form $Q|_W$ via the following test.

Consider a quadratic form

$$A(x) = \sum_{i,j=1}^{n} a_{ij} x_i x_j, \qquad a_{ij} = a_{ji}, \quad x_i \in \mathbb{R}.$$

Denote a minor

$$A \begin{pmatrix} i_1 & i_2 & \ldots & i_p \\ i_1 & i_2 & \ldots & i_p \end{pmatrix} = \begin{vmatrix} a_{i_1 i_1} & a_{i_1 i_2} & \ldots & a_{i_1 i_p} \\ a_{i_2 i_1} & a_{i_2 i_2} & \ldots & a_{i_2 i_p} \\ \ldots & \ldots & \ldots & \ldots \\ a_{i_p i_1} & a_{i_p i_2} & \ldots & a_{i_p i_p} \end{vmatrix}.$$

Theorem 5 ([13]). *A quadratic form $A(x)$ is negative-semidefinite iff the following inequalities hold:*

$$(-1)^p A \begin{pmatrix} i_1 & i_2 & \ldots & i_p \\ i_1 & i_2 & \ldots & i_p \end{pmatrix} \geqslant 0, \qquad 1 \leqslant i_1 < i_2 < \ldots < i_p \leqslant n, \quad p = 1, 2, \ldots, n.$$

10. Bounds on the number of switchings on optimal bang-bang trajectories

We apply necessary optimality conditions for bang-bang trajectories of A. A. Agrachev and R. V. Gamkrelidze given by Theorem 4 and improve the bound of Theorem 3.

10.1. Case 1a): $h_4 > 0$, $\varphi = 0$

Theorem 6. *Let $h_4 > 0$, $\varphi = 0$, and $\lambda \in \cup_{i=1}^{3} C_i$. Then the bang-bang trajectory* $\mathrm{Exp}(\lambda, t)$ *is optimal.*

Proof. Apply Theorem 2. □

Theorem 7. *Let $h_4 > 0$, $\varphi = 0$, and $\lambda \in \cup_{i=4}^{6} C_i$. Then the bang-bang trajectory* $\mathrm{Exp}(\lambda, t)$ *with k switchings is not optimal, where k is given by the following tables:*

- $\lambda \in C_4 \Rightarrow$ *Table* 1,
- $\lambda \in C_5 \Rightarrow$ *Table* 2,
- $\lambda \in C_6 \Rightarrow$ *Table* 3.

Table 1. $\lambda \in C_4$

Start	$(+,+)_+$	$(-,+)_+$	$(-,-)$	$(-,+)_-$	$(+,+)_-$	$(+,-)$
k	8	9	7	7	9	7

Table 2. $\lambda \in C_5$

Start	$(-,+)_+$	$(+,+)_+$	$(+,-)_+$	$(-,-)_+$	$(-,-)_-$	$(-,+)_-$	$(+,+)_-$	$(+,-)_-$
$- -$	7	8	7	7	6	5	8	7
$- +$	8	8	7	7	6	5	8	8
$+ -$	8	5	6	8	7	8	8	8
$+ +$	8	5	6	7	7	8	8	7

Table 3. $\lambda \in C_6$

Start	$(+,-)$	$(+,+)$	$(-,+)$	$(-,-)$
k	6	5	6	5

Remark 2. We explain now how Tables 1–3 should be read. Consider Table 1. The first line — Start — gives the values of $(u_1(0), u_2(0)) = (\operatorname{sgn} h_1(0), \operatorname{sgn} h_2(0))$. For example, the first column of Table 1 corresponds to

$$(u_1(0), u_2(0)) = (\operatorname{sgn} h_1(0), \operatorname{sgn} h_2(0)) = (+1, +1).$$

The second column of Table 1 corresponds to the initial values $(u_1(0), u_2(0)) = (\operatorname{sgn} h_1(0), \operatorname{sgn} h_2(0)) = (-1, +1)$. The lower index $\pm$ near $(\pm, \pm)$ indicates the value of $\operatorname{sgn} h_3(0)$.

The same agreement on reading similar tables is used in subsequent subsections.

We prove Theorem 7.

Proof. Let $\lambda \in C_4$, the cases $\lambda \in C_5$ and $\lambda \in C_6$ are considered similarly. Then system (6.2) has the phase portrait shown in Fig. 1.

Consider the first column of Table 1 — a control starting from $(1,1)_+$ and having $k = 8$ switchings (controls starting from other values are considered similarly). We apply Theorem 4 and show that such control is not optimal. We have $0 = t_0 < t_1 < \ldots < t_9 = T$, where

$$t_1 \in (0, \tau_1], \quad t_2 - t_1 = t_4 - t_3 = t_5 - t_4 = t_7 - t_6 = t_8 - t_7 = \tau_1,$$
$$t_3 - t_2 = t_6 - t_5 = \tau_2, \quad t_9 - t_8 \in (0, \tau_2],$$

and

$$\tau_1 = \frac{\sqrt{2(E + h_4)} - \sqrt{2E}}{h_4} = \frac{2}{\sqrt{2(E + h_4)} + \sqrt{2E}}, \quad \tau_2 = \frac{2\sqrt{2E}}{h_4}.$$

Further, we have

$$u|_{(t_0,t_1)} = u|_{(t_4,t_5)} = u|_{(t_6,t_7)} = (1,1),$$
$$u|_{(t_1,t_2)} = u|_{(t_3,t_4)} = u|_{(t_7,t_8)} = (-1,1),$$
$$u|_{(t_2,t_3)} = u|_{(t_8,t_9)} = (-1,-1), \qquad u|_{(t_5,t_6)} = (1,-1),$$

see Fig. 1. We apply Theorem 4 in the case $k = 8$, $j = 4$. We use the basis (X_+, X_-, X_3, X_4) in the Lie algebra $\mathfrak{g}$, where $X_+ = X_1 + X_2$, $X_- = X_1 - X_2$. Then

$$Y_0 = -Y_2 = Y_4 = Y_6 = -Y_8 = X_+,$$
$$Y_1 = Y_3 = -Y_5 = Y_7 = -X_-.$$

Further,

$$P_4 = P_3 = \mathrm{Id}, \qquad P_5 = e^{\tau_1 \operatorname{ad} X_+},$$
$$P_6 = P_5 \circ e^{\tau_2 \operatorname{ad} X_-}, \qquad P_7 = P_6 \circ e^{\tau_1 \operatorname{ad} X_+},$$
$$P_2 = e^{\tau_1 \operatorname{ad} X_-}, \qquad P_1 = P_2 \circ e^{\tau_2 \operatorname{ad} X_+},$$
$$P_0 = P_1 \circ e^{\tau_1 \operatorname{ad} X_-}, \qquad P_8 = P_7 \circ e^{-\tau_1 \operatorname{ad} X_-}.$$

Thus,

$$Z_0 = X_+ + 4\tau_1 X_3 + (4\tau_1^2 + 2\tau_1\tau_2)X_4,$$
$$Z_1 = -X_- + 2\tau_2 X_3 + (\tau_2^2 + 2\tau_1\tau_2)X_4,$$
$$Z_2 = -X_+ - 2\tau_2 X_3 - \tau_2^2 X_4,$$
$$Z_3 = -X_-,$$
$$Z_4 = X_+,$$
$$Z_5 = X_- - 2\tau_1 X_3 - \tau_1^2 X_4,$$
$$Z_6 = X_+ + 2\tau_2 X_3 + (\tau_2^2 + 2\tau_1\tau_2)X_4,$$
$$Z_7 = -X_- + 4\tau_1 X_3 + (4\tau_1^2 + 2\tau_1\tau_2)X_4,$$
$$Z_8 = -X_+ + (2\tau_1 - 2\tau_2)X_3 + (3\tau_1^2 - \tau_2^2)X_4.$$

Then $Q(\alpha) = \sum_{0 \leqslant i < l \leqslant 8} \sigma_{il} \alpha_i \alpha_l$, where

$$\begin{aligned}
&\sigma_{01} = h_3 + (2\tau_1 + \tau_2)h_4, && \sigma_{24} = \tau_1 h_4,\\
&\sigma_{02} = \tau_1 h_4, && \sigma_{25} = h_3 + 2\tau_1 h_4,\\
&\sigma_{03} = h_3 + 2\tau_1 h_4, && \sigma_{26} = (\tau_1 - \tau_2)h_4,\\
&\sigma_{04} = -2\tau_1 h_4, && \sigma_{27} = -h_3 - 3\tau_1 h_4,\\
&\sigma_{05} = -h_3 - 3\tau_1 h_4, && \sigma_{28} = (\tau_2 - 2\tau_1)h_4,\\
&\sigma_{06} = (\tau_2 - 2\tau_1)h_4, && \sigma_{34} = -h_3,\\
&\sigma_{07} = h_3 + 4\tau_1 h_4, && \sigma_{35} = \tau_1 h_4,\\
&\sigma_{08} = (3\tau_1 - \tau_2)h_4, && \sigma_{36} = -h_3 - \tau_2 h_4,\\
&\sigma_{12} = h_3 + (\tau_1 + \tau_2)h_4, && \sigma_{37} = -2\tau_1 h_4,\\
&\sigma_{13} = \tau_2 h_4, && \sigma_{38} = h_3 + (\tau_2 - \tau_1)h_4,\\
&\sigma_{14} = -h_3 - \tau_2 h_4, && \sigma_{45} = -h_3 - \tau_1 h_4,\\
&\sigma_{15} = (\tau_1 - \tau_2)h_4, && \sigma_{46} = \tau_2 h_4,\\
&\sigma_{16} = -h_3 - 2\tau_2 h_4, && \sigma_{47} = h_3 + 2\tau_1 h_4,\\
&\sigma_{17} = (-2\tau_1 + \tau_2)h_4, && \sigma_{48} = (\tau_1 - \tau_2)h_4,\\
&\sigma_{18} = h_3 + (2\tau_2 - \tau_1)h_4, && \sigma_{56} = h_3 + (\tau_1 + \tau_2)h_4,\\
&\sigma_{23} = -h_3 - \tau_1 h_4, && \sigma_{57} = \tau_1 h_4,\\
&\sigma_{58} = -h_3 - \tau_2 h_4, && \sigma_{67} = h_3 + (2\tau_1 + \tau_2)h_4,\\
&\sigma_{68} = \tau_1 h_4, && \sigma_{78} = h_3 + (\tau_1 + \tau_2)h_4.
\end{aligned}$$

Further,

$$\begin{aligned}
W &= \left\{(\alpha_0, \ldots, \alpha_8) \in \mathbb{R}^9 \mid \sum_{i=0}^{8} \alpha_i = 0, \quad \sum_{i=0}^{8} \alpha_i Z_i(q(t_1)) = 0\right\}\\
&= \{(\alpha_0, \ldots, \alpha_8) \in \mathbb{R}^8 \quad | \quad \alpha_1 = 2\gamma\alpha_0 - \alpha_6 - 2\gamma\alpha_7 + (1 - 2\gamma)\alpha_8,\\
&\qquad \alpha_3 = 2\gamma\alpha_0 - \alpha_2 + \alpha_6 + (2\gamma - 1)\alpha_7 + (2\gamma - 2)\alpha_8,\\
&\qquad \alpha_4 = -\alpha_0 + \alpha_2 - \alpha_6 + \alpha_8, \qquad \alpha_5 = -\alpha_2 - \alpha_8\},\\
\gamma &= \tau_1/\tau_2,\\
Q|_W &= -4\sqrt{2}/a(f_1 + a f_2 - \sqrt{a(a+1)} f_3),\\
a &= E/h_4 \in (0, 1),\\
f_1 &= (\alpha_0 + \alpha_7 + \alpha_8)^2,\\
f_2 &= \alpha_0^2 + \alpha_2^2 - 2\alpha_2\alpha_6 + 2\alpha_6^2 + \alpha_7(2\alpha_2 - 4\alpha_6 + 3\alpha_7)\\
&\qquad + (4\alpha_2 - 7\alpha_6 + 9\alpha_7)\alpha_8 + 8\alpha_8^2 + 2\alpha_0(\alpha_7 + \alpha_8),\\
f_3 &= \alpha_0^2 + \alpha_0(\alpha_2 - 2\alpha_6 + 3\alpha_7 + 5\alpha_8 + (\alpha_7 + \alpha_8)(\alpha_2 - 2\alpha_6 + 3\alpha_7 + 5\alpha_8).
\end{aligned}$$

Then

$$\Delta_{27} = \begin{vmatrix} a_{32} & a_{25} \\ a_{52} & a_{55} \end{vmatrix} = 8\sqrt{1+a}/\sqrt{a}\left(11\sqrt{a(1+a)} - 4 - 12a\right) < 0.$$

By Theorem 5, the quadratic form $Q|_W$ is not negative semidefinite. By Theorem 4, the control u is not optimal. □

10.2. Case 1b): $h_4 > 0$, $\varphi = \pi/4$

Theorem 8. *Let* $h_4 > 0$, $\varphi = \pi/4$, *and* $\lambda \in C_1$. *Then the bang-bang trajectory* $\operatorname{Exp}(\lambda, t)$ *is optimal.*

Proof. Apply Theorem 2. □

Theorem 9. *Let* $h_4 > 0$, $\varphi = 0$, *and* $\lambda \in \cup_{i=2}^{4} C_i$. *Then the bang-bang trajectory* $\operatorname{Exp}(\lambda, t)$ *with* k *switchings is not optimal, where* k *is given by the following tables:*

- $\lambda \in C_2 \Rightarrow$ *Table* 4,
- $\lambda \in C_3 \Rightarrow$ *Table* 5,
- $\lambda \in C_4 \Rightarrow$ *Table* 6.

Proof. Similarly to the proof of Theorem 7. □

Table 4. $\lambda \in C_2$

Start	$(+,+)_+$	$(-,+)$	$(+,+)_-$	$(+,-)$
k	7	6	5	4

Table 5. $\lambda \in C_3$

Start	$(+,-)$	$(+,+)_+$
k	4	5

Table 6. $\lambda \in C_4$

Start	$(+,-)$	$(+,+)$	$(-,+)$	$(-,-)$
k	6	6	5	7

10.3. Case 1c): $h_4 > 0$, $\varphi \in (0, \pi/4)$

Theorem 10. *Let* $h_4 > 0$, $\varphi \in (0, \pi/4)$, *and* $\lambda \in \cup_{i=1}^{3} C_i$. *Then the bang-bang trajectory* $\operatorname{Exp}(\lambda, t)$ *is optimal.*

Proof. Apply Theorem 2. □

Theorem 11. *Let* $h_4 > 0$, $\varphi \in (0, \pi/4)$, *and* $\lambda \in \cup_{i=4}^{8} C_i$. *Then the bang-bang trajectory* $\operatorname{Exp}(\lambda, t)$ *with* k *switchings is not optimal, where* k *is given by the following tables:*

- $\lambda \in C_4 \Rightarrow$ *Table* 7,
- $\lambda \in C_5 \Rightarrow$ *Table* 8,
- $\lambda \in C_6 \Rightarrow$ *Table* 9,
- $\lambda \in C_7 \Rightarrow$ *Table* 10,
- $\lambda \in C_8 \Rightarrow$ *Table* 11.

Proof. Similarly to the proof of Theorem 7. □

Table 7. $\lambda \in C_4$

Start	$(+,+)_+$	$(-,+)$	$(+,+)_-$	$(+,-)$
k	7	8	8	7

Table 8. $\lambda \in C_5$

Start	$(+,+)_+$	$(-,+)$	$(+,+)_-$	$(+,-)$
k	7	8	8	7

Table 9. $\lambda \in C_6$

Start	$(+,+)_+$	$(-,+)_+$	$(-,-)$	$(-,+)_-$	$(+,+)_-$	$(+,-)$
k	8	9	8	8	9	8

Table 10. $\lambda \in C_7$

Start	$(+,+)_+$	$(-,+)_+$	$(-,-)$	$(-,+)_-$	$(+,+)_-$	$(+,-)$
$--$	10	9	8	7	8	7
$-+$	8	9	7	10	9	8
$+-$	10	9	8	10	9	8
$++$	8	8	7	10	9	8

Table 11. $\lambda \in C_8$

Start	$(+,+)$	$(-,+)$	$(-,-)$	$(+,-)$
k	8	8	7	7

10.4. Case 2): $h_4 = 0$

Theorem 12. *Let* $h_4 = 0$, *and* $\lambda \in C_2$. *Then the bang-bang trajectory* $\mathrm{Exp}(\lambda, t)$ *with* 7 *switchings is not optimal.*

Proof. Similarly to the proof of Theorem 7. □

Now Theorems 7–12 imply the following statement.

Corollary 1. *If* ($\varphi \in [0, \pi/4)$ *and* $-|h_4| \sin\varphi < E$) *or* $\varphi = \pi/4$, *then optimal bang-bang trajectories have not more than* 9 *switchings.*

11. Conclusion

Many interesting questions on the sub-Finsler problem on the Engel group considered in this paper remain open:

- precise description of the cut time along extremal trajectories,
- optimal synthesis,
- sub-Finsler sphere and distance.

We hope to address these questions in the forthcoming works.

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