On the analytical approximation to the GLAP evolution at small x and moderate Q^{2*}

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Abstract:

Comparing the numerically evaluated solution to the leading order GLAP equations with its analytical small-x approximation we have found that in the domain covered by a large fraction of the HERA data the analytic approximation has to be augmented by the formally non-leading term which has been usually neglected. The corrected formula fits the data much better and provides a natural explanation of some of the deviations from the σ scaling observed in the HERA kinematical range.

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The high-precision data on the deep inelastic structure function $F_2^p(x, Q^2)$ coming from ZEUS [1] and H1 [2] experiments installed in the HERA accelerator have stimulated new interest in the properties of the QCD dynamics in this kinematical domain [3]. The conceptually simplest analysis typically relies on solutions of the GLAP equations which are either evaluated numerically or approximated in the small-x region by a suitable analytical formula. In particular, Forte and Ball [4, 5] pointed out that the HERA small-x data may be interpreted in terms of the so called double asymptotic scaling phenomenon related to the asymptotic behaviour of the GLAP evolution first discovered by DeRujula et al.[6] many years ago. On the other hand, various groups have been able to fit the available data using input characterized by the hard small-x behaviour $x^{-(1+\lambda)}$, $\lambda > 0$ [7].

Notwithstanding the dispute about applicability of the double asymptotic scaling to the HERA data, the problem of an analytical approximation to the GLAP evolution is interesting in itself [8]. A common sense requires that any analytical formula has to follow the full solution with an accuracy at least as good as the magnitude of the experimental error bars, a goal which may not be so easy to achieve given the high precision of the HERA data. Analytical approximations to the GLAP evolution are usually designed to be valid in an extreme kinematics of very small x and/or large Q^2. Starting from this observation, we have considered in the present note the question to which extent the approximation which leads to the double scaling behaviour can be trusted in the domain of moderate Q^2 of the order of ~ 10 GeV^2 which contains a non-negligible fraction of the small-x HERA data, even assuming the most favorable case of absolutely soft initial conditions. As we shall show in the following, the self-consistent approximation to the GLAP evolution in this region indeed seems to require taking into account a formally non-leading term which is normally neglected. Alternatively, one may of course try to avoid the problem by going to higher and higher Q^2, which inevitably results in a lower statistics, especially in the small x region. On the other hand as we show in the following, the improved formula not only provides a much better fit to the small x data, but also the appropriately corrected data points are much more consistent with the asymptotic σ scaling.

Our analysis is performed at leading order, but its generalization to NLO is not particularly difficult. However, before such an extended analysis is performed one has to be sure that there are no other sources of corrections which can be even larger than the difference between the LO and NLO formula [9].

At leading order the solution of GLAP equations for moments of the quark singlet $q_s(x, t)$ parton distribution

$$q_s(n, t) = \int_0^1 dx\, x^n\, q_s(x, t) \tag{1}$$

where $t = \ln(\frac{Q^2}{\Lambda^2})$, has the well known form [10]

$$q_s(n, t) \;=\; ((1 - h_2(n))q_s(n, t_0) - h_1(n)g(n, t_0)) \exp\left[\frac{2}{\beta_0}\lambda_+(n)\zeta\right]$$

$$+ \quad (h_1(n)g(n, t_0) + h_2(n)q_s(n, t_0)) \exp\left[\frac{2}{\beta_0}\lambda_-(n)\zeta\right] \tag{2}$$

where $g(n, t)$ is the corresponding moment of the gluon distribution and $\zeta = \ln\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)$. The coefficients $h_1(n)$ and $h_2(n)$ as well as the eigenvalues of the anomalous dimension matrix $\lambda_\pm$ arise from the diagonalisation of the flavour-singlet evolution equation. To obtain the Bjorken-x distribution from (2) one has to take the inverse Mellin transformation i.e., evaluate the integral

$$xq_s(x, t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dn\, x^{-n} q_s(n, t)\,, \tag{3}$$

where the integration contour runs to the right to all singularities of $q_s(n, t)$. At small-x one expects that the bulk of the integral comes from the part of the contour in the vicinity of the point $n = 0$, which can be quantified e.g., by the saddle point approximation. As the flavour non-singlet part is subleading at sufficiently small-x values, in this region the structure function $F_2(x)$ should be related to a good accuracy to the singlet quark distribution (3) via $F_2(x) = \frac{5}{18} q_s(n, t)$ ($n_f = 4$). In Ref.[4, 5] the HERA data were analyzed with the help of a further analytical approximation to the integral (3), based on the assumption that in the small-x region and for sufficiently large Q^2 the second term in (2), driven by the eigenvalue λ_-, can be neglected with respect to the first one. In this limit the solution of the GLAP equation in the moment space reads

$$q_s(n, t) = q_+(n, t) = ((1 - h_2(n))q_s(n, t_0) - h_1(n)g(n, t_0)) \exp\left[\frac{2}{\beta_0}\lambda_+(n)\zeta\right]\,, \tag{4}$$

The double-scaling solution discussed in Ref.[4, 5] is obtained by assuming soft initial conditions $q_s(n, t_0) \sim g(n, t_0) \sim 1/n$, expanding $\lambda_+(n)$ and $h_i(n)$ (i = 1,2) around the point $n = 0$, retaining only the leading terms, and evaluating the resulting Mellin integral (3). As such simple initial conditions typically overestimate the actual magnitude of $xq(x, Q_0^2)$ one can introduce an additional parameter x_0 which has an interpretation related to the constant term in the Laurent expansion of the initial conditions around their rightmost singularity:

$$q(n, t_0) = A(Q_0^2)\left(\frac{1}{n} + \ln x_0\right) + \ldots = A(Q_0^2)\frac{1}{n}\exp\left(n \ln x_0\right) + \ldots\,. \tag{5}$$

Assuming that $xq(x, Q_0^2) \sim (1 - x)^\beta$ as $x \to 1$, with $\beta \sim 3$ one expects $x_0 \sim e^{-2} \sim 0.1$.

The reasoning which leads to the approximation (4) is based on the series of assumptions which have to be fulfilled by the data under scrutiny. Hence, to be able to justify the small-x and large Q^2 approximations we have selected for the further analysis 83 points in the domain $x \leq 0.01$, $Q^2 \geq 5$ GeV^2 from the recently published H1 data [2]. Although for some data in this domain the value of Q^2 is large, relatively many points have much lower Q^2, of the order of 10 GeV^2. On the other hand, if the perturbative QCD evolution

is to be valid, the starting point of the Q^2 evolution should not be much smaller than, say, 1 GeV^2. Then, as the first consistency check we have plotted the distribution of the saddle point positions $n_0 = (\frac{12}{\beta_0}\frac{\zeta}{\xi})^{1/2}$, $\xi = \ln(x_0/x)$, for $x_0 = 1$ and $x_0 = 0.1$, see full and shadowed histograms on Figure 1 respectively. It is interesting to see that especially in the second case saddle points are not very close to zero, so that one can doubt whether the usual small n approximation to the integral (3) is applicable at all. To quantify this problem, in the next step we have compared the results of the full solution [11] to the evolution equation (2) with the double-scaling approximation (4) at $Q^2 = 10$ GeV^2, assuming the soft initial conditions for quark and gluon distribution functions $q_s(n, t_0) = \frac{A}{n}$, $g(n, t_0) = \frac{B}{n}$ at $Q_0^2 = 1$ GeV^2, taking for simplicity $A = B$. The results are shown on Figure 2. The data points correspond to $Q^2 = 8.5$ GeV^2, and the normalization of initial conditions has been simply adjusted to the data by hand. It turns out that even for such simple initial conditions the double-scaling approximation, represented by the dashed line, approaches the full solution, depicted by the solid line, at values of x much smaller than those experimentally accessible at present. Moreover, in the region covered by the data the difference between the full solution and the double-scaling approximation is significantly larger than the error bars of the data points, making the applicability of the latter to the analysis of the data at least questionable. The situation does not change if we assume more realistic initial conditions, say, $x q_s(x) \approx x g(x, t_0) = (1-x)^3$, see Figure 3 where the appropriate x_0 parameter has been taken into account in the double-scaling approximation. Our simplifying choice $A = B$ cannot be responsible for the discrepancy - a consistent analytical approximation has to be able to accommodate various initial conditions without compromising the accuracy.

Comparing the situation for various values of Q^2 we have found that the situation improves considerably when Q^2 increases, or in other words in the above discussion we have considered, for the sake of clarity of the argument, a really extreme example. By the same token one can suspect that for a reasonable description of relatively low Q^2 data the term driven by the eigenvalue λ_- should not be neglected. We have tested this hypothesis by comparing again the full solution with the approximation in which both the first and the second terms in (2) are expanded in n to the leading order, i.e.

$$
\begin{aligned}
q_s(n, t) &= (\,0.198\, n\, q_s(n, t_0) + 0.444\, n\, g(n, t_0)\,) \exp\left[\frac{2}{\beta_0}(6/n - 5.648)\,\zeta\right] \\
&+ q_s(n, t_0) \exp\left[-\frac{2}{\beta_0} 1.185\,\zeta\right].
\end{aligned} \tag{6}
$$

All real coefficients above result from expansion of corresponding coefficients in (2) in n and keeping only the first terms. After transformation into x-space the final expression for the structure function $F_2(x)$ to be fitted to the H1-data in the following reads:

$$
F_2(x, t) = \frac{5}{18}\,(\,0.198\, A + 0.444\, B\,) \exp\left[-\frac{2}{\beta_0} 5.648\,\zeta\right] \left(\frac{\gamma^2\,\zeta}{\xi}\right)^{\frac{1}{2}} I_1\left(2\,\gamma\sqrt{\zeta\xi}\right)
$$

$$+ \quad \frac{5}{18} A \exp\left[-\frac{2}{\beta_0} 1.185\,\zeta\right] , \tag{7}$$

where $\gamma = \sqrt{12/\beta_0}$. As it can be seen from Figure 4, the formula (7) considerably improves the quality of the approximation, which in our opinion makes its applicability much better justified.

Motivated by these considerations we have compared fits to the small-x data set based on formulae (4) and (7) respectively. For that purpose we have used the minimization package MINUIT from the CERN library [12]. The results are summarized in Tables 1 and 2. We have kept $n_f = 4$, used $\Lambda_{QCD} = 250$ MeV and a fixed input scale $Q_0^2 = 1\text{GeV}^2$ such that pQCD should be still meaningful. The improved approximation (7) results in χ^2 which is smaller approximately by a factor 3. We note that if we had not restricted Q_0^2 to a fixed value of $Q_0^2 = 1$ GeV^2, the fit would drive Q_0^2 to much lower values of about 0.5 GeV^2, simultaneously pushing x_0 to values larger than 1, which is certainly disturbing as far as the interpretation of x_0 is concerned. On the other hand χ^2 hardly changes by such a shift, and hence we are sure that the values presented in Table 1 represent a sensible set of fit parameters. In addition, if the data points in the Q^2 bins around 5 GeV^2 and 6.5 GeV^2 are removed from the data set, the fit improves once more considerably, see corresponding entry in Table 2.

data range	$0.198\,A + 0.444\,B$	x_0	$\chi^2/d.o.f.$
$Q^2 \geq 5$ GeV^2, $x \leq 0.01$	1.60 ± 0.06	0.14 ± 0.01	1.57
$Q^2 \geq 8.5$ GeV^2, $x \leq 0.01$	1.41 ± 0.06	0.19 ± 0.02	1.35

Table 1: LO fit with double-scaling solution

data range	A	B	x_0	$\chi^2/d.o.f.$
$Q^2 \geq 5$ GeV^2, $x \leq 0.01$	0.83 ± 0.09	2.47 ± 0.18	0.15 ± 0.02	0.49
$Q^2 \geq 8.5$ GeV^2, $x \leq 0.01$	0.80 ± 0.10	2.30 ± 0.18	0.18 ± 0.03	0.35

Table 2: LO fit with improved solution

What is the influence of the correction taken into account in (7) on the double asymptotic scaling considered in [4, 5]? To see this we have compared the LO double asymptotic scaling formula which results from (4) with the data in the domain $x \leq x_0$, $Q^2 \geq 8.5$ GeV^2 before and after the correction due to the extra term in (7) is taken into account. The so called σ-scaling predicts [4] that after rescaling the F_2-data by the factor

$$R_F = N_F\,\rho\,\sqrt{\sigma}\exp(-2\,\gamma\,\sigma + \delta_+\,\sigma/\rho) \tag{8}$$

where $\delta_+ = (11 + \frac{2}{27}n_f)/\beta_0$, one should observe constant behaviour at large ρ, where $\rho = \sqrt{\xi/\zeta}$, $\sigma = \sqrt{\xi\zeta}$, $\gamma = \sqrt{12/\beta_0}$) and N_F is an arbitrary normalization constant. The

results are presented on Figure 5. The upper plot corresponds to the original double-scaling formula (8). For the lower plot our fit result for the non-scaling contribution arising from the formally non-leading term in (7) has been subtracted from each data point before the rescaling. For $\rho \geq 1.5$, points on the lower plot clearly have much smaller spread around the horizontal line. The ρ scaling predicts that on the plot $\ln(R_{F'} F_2)$ versus σ the data points, with the rescaling factor given by

$$R_{F'} = N_{F'} \rho \sqrt{\sigma} \exp(\delta_+ \sigma/\rho) \tag{9}$$

should lie on the straight line with the slope equal to 2γ. To see the effect of (7) on the ρ scaling hypothesis we have displayed on Figure 6 the scaling behaviour before (upper plot) and after (lower plot) the non-leading correction has been subtracted from the data points. Although both plots clearly differ in the domain of $\sigma \leq 1$, the scaling behaviour is hardly influenced for $\sigma \geq 1.5$. The dashed line represents the expected behaviour according to the first, scaling term in (7) i.e.,

$$R_{F'} F_2 = N \sqrt{\sigma} I_1(2\gamma\sigma) \tag{10}$$

with the constant N adjusted to the normalization of the large σ data points.

For large σ the data plotted in Fig.6 should follow a straight line with slope 2γ [13]. Restricting our data range to $Q^2 \geq 8.5$ GeV^2, $x \leq 0.01$ we have determined, using MINUIT as well, the slope before and after the non-leading correction has been subtracted from the data points. Our results, summarized in Table 3, clearly show that with the non-leading correction taken into account we get better agreement with the value expected from the theory, $2\gamma_{th} = 2.4$

	data range	2γ
double scaling solution	$\sigma \geq 1.0$	2.15 ± 0.04
improved solution	$\sigma \geq 1.0$	2.47 ± 0.05

Table 3: Slope 2γ obtained from a fit to F_2-data

To summarize, starting from the observation that the experimental errors set the magnitude for the minimal sensible accuracy of an analytical approximation to the GLAP evolution, we have found that the description in the region of $Q^2 \sim 10$ GeV^2 improves considerably after a formally non-leading, and thus usually neglected term is included. The improved formula provides a better description of the available small x HERA data and it improves the observed σ scaling.

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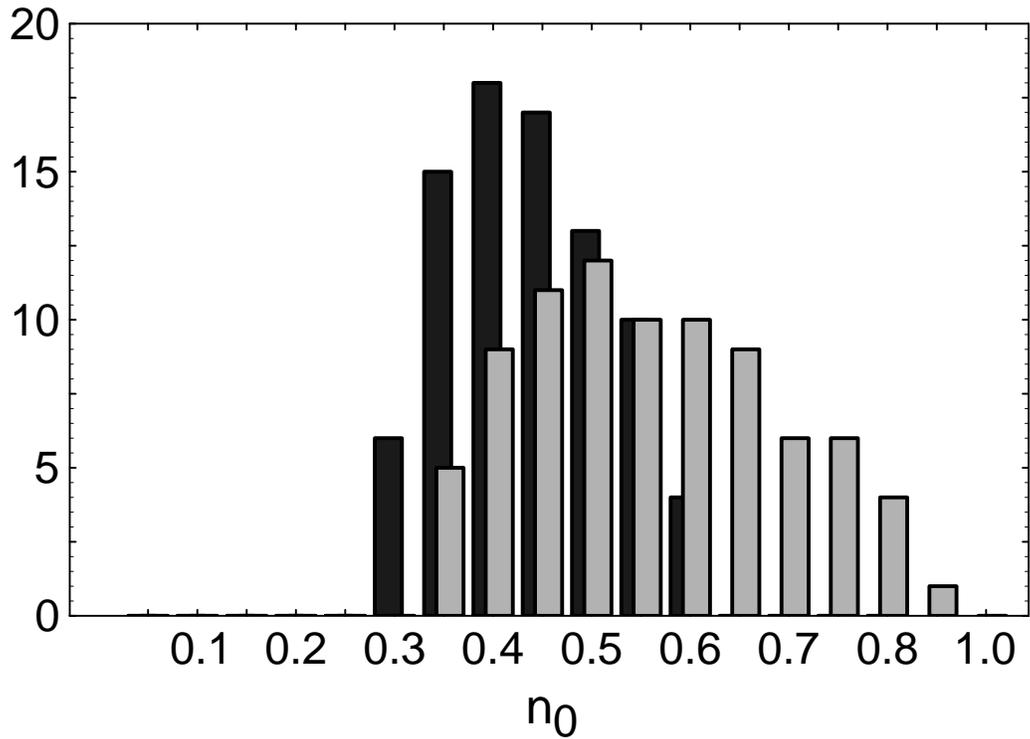[13] R. Ball and S. Forte, Phys. Lett. **B336**, 77 (1994).

Fig. 1 Distribution of saddle points $n_0 = (\frac{12}{\beta_0} \frac{\zeta}{\xi})^{1/2}$ over the analyzed HERA data points [2]. The shadowed and filled histograms correspond to $x_0 = 0.1$ and $x_0 = 1$, respectively.
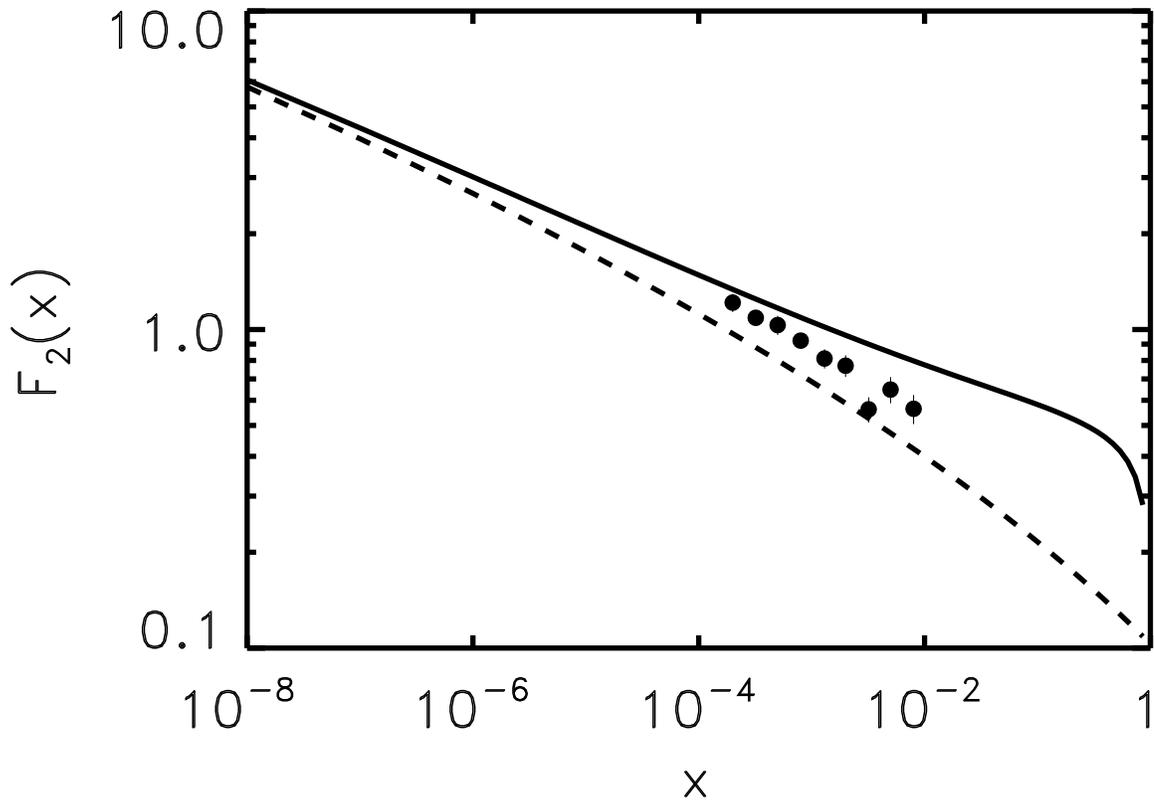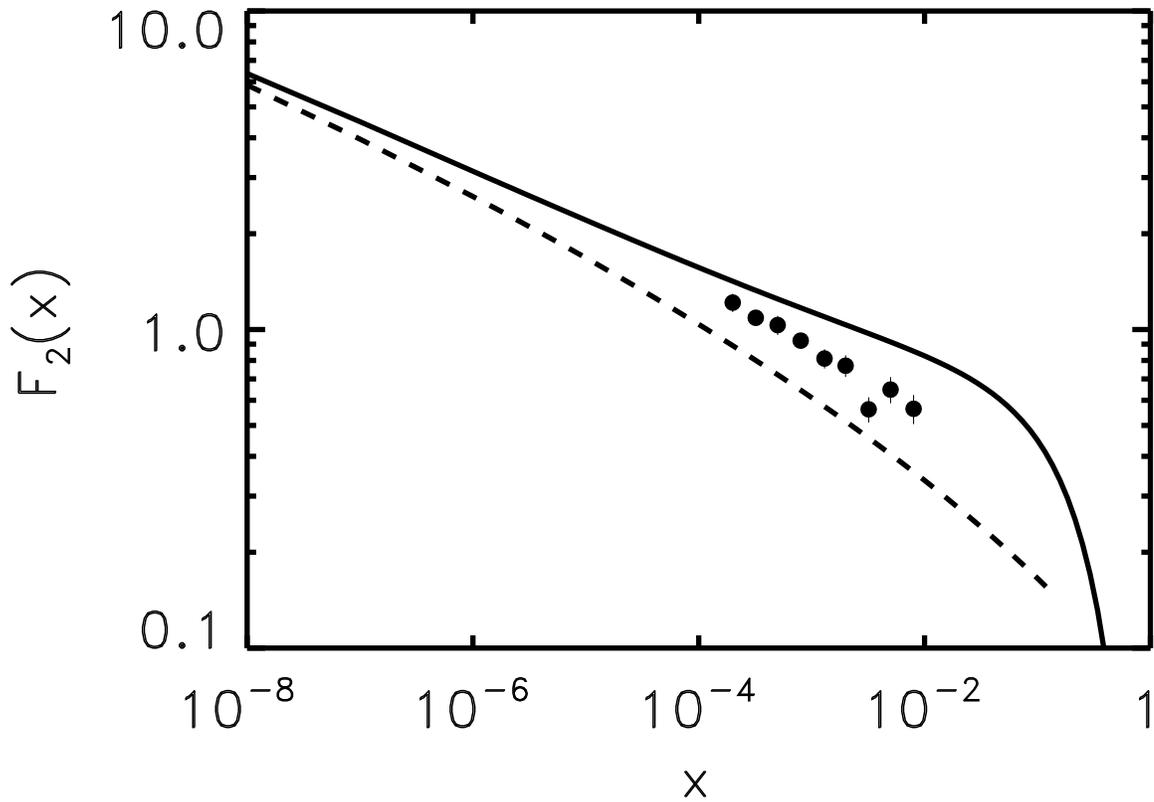
Fig. 2 Comparison of the full evolution (solid line), equation (2) with the approximate double-scaling solution (4) for the data around $Q^2 = 10 \text{ GeV}^2$ [2], and the soft initial conditions $q(n, t_0) = g(n, t_0) = 0.45/n$ at $Q_0^2 = 1 \text{ GeV}^2$.

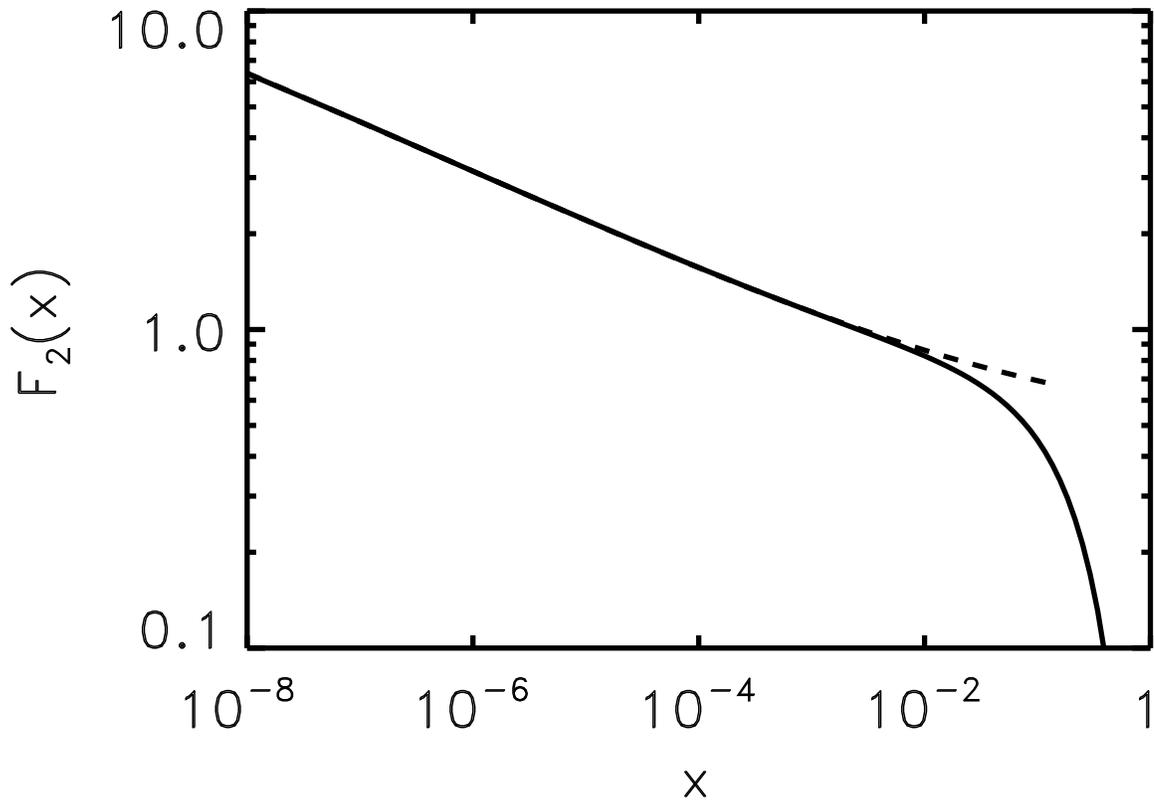**Fig. 3** Comparison of the full evolution (solid line), equation (2) with the approximate double-scaling solution (4) for the data around $Q^2 = 10$ GeV^2 [2], and the soft initial conditions $xq(x, t_0) = xg(x, t_0) = 0.6 \, (1-x)^3$ at $Q_0^2 = 1$ GeV^2.

Fig. 4 Comparison of the full evolution (solid line), equation (2) with the approximation which includes besides the double-scaling solution also the leading contribution from the term driven by the non-leading eigenvalue $\lambda_-(n)$, equation (7), for the soft initial conditions $xq(x, t_0) = xg(x, t_0) = 0.6 \ (1 - x)^3$ at $Q_0^2 = 1 \ \text{GeV}^2$.
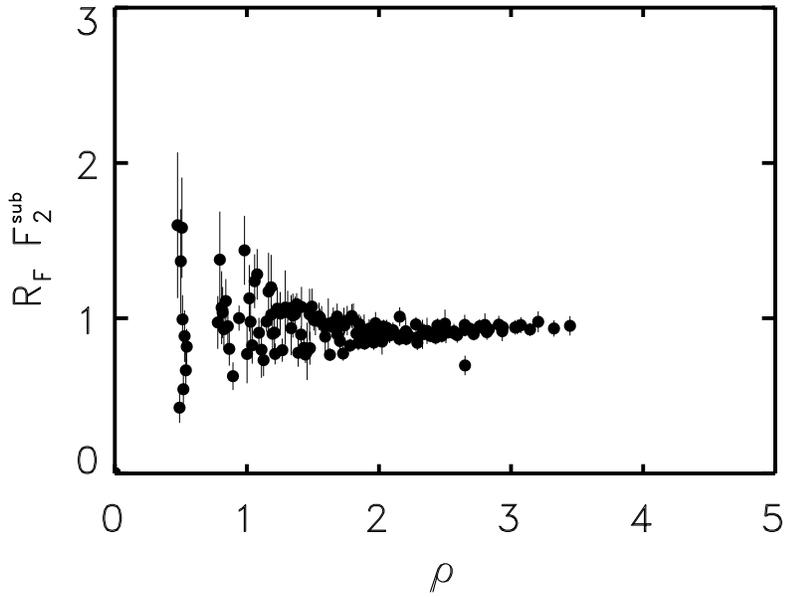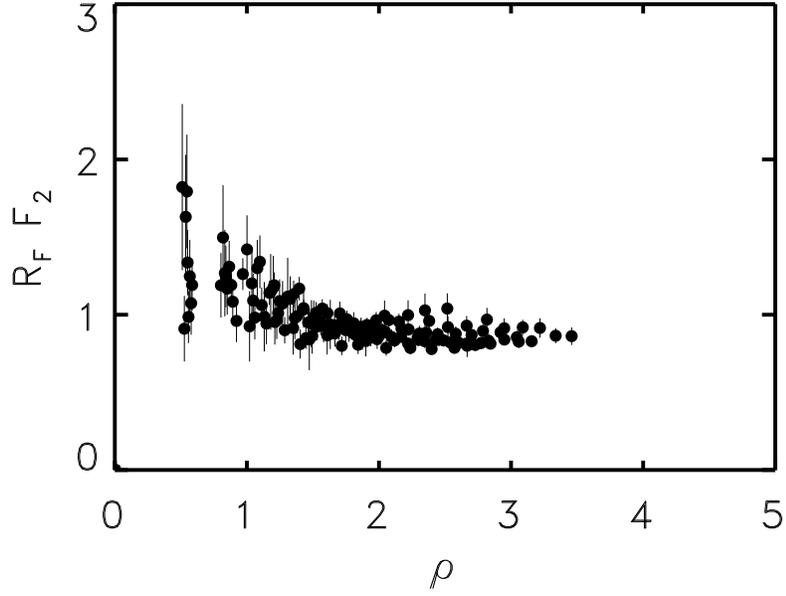
Fig. 5 The σ scaling formula from Ref.[4, 5] versus the small x HERA data [2]. For the lower plot the formally non-leading contribution taken into account in (7) has been subtracted from each data point before rescaling of the data [2] according to (8).
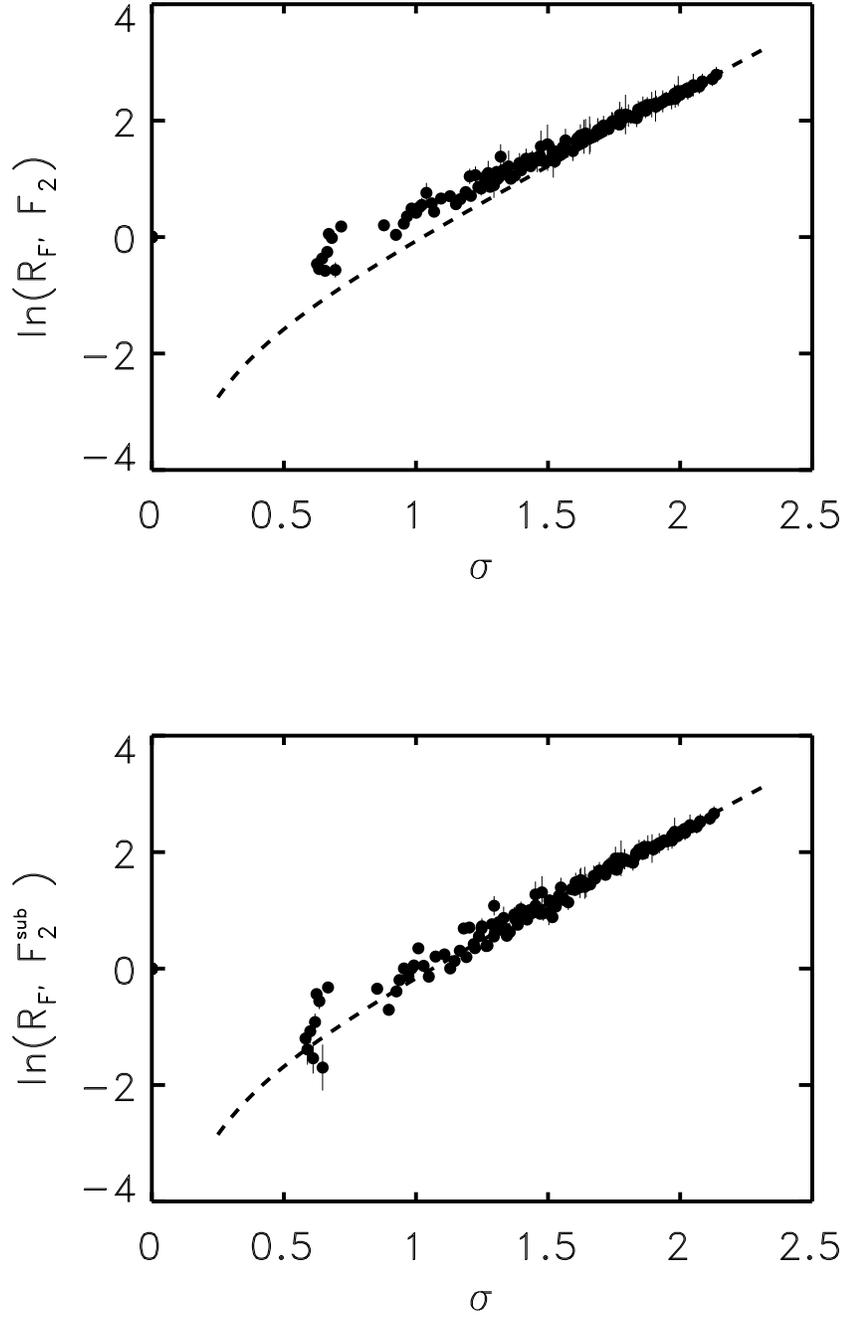
Fig. 6 The ρ scaling formula from Ref.[4, 5] versus the small x HERA data [2]. For the lower plot the formally non-leading contribution taken into account in (7) has been subtracted from each data point before rescaling of the data [2] according to (9). The dashed line represents the expected behaviour according to the first, scaling term in (7).