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Positive Solutions for p-Laplacian Functional Dynamic Equations on Time Scales*

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Abstract

Using fixed point theorems in cones, we establish some sufficient conditions for the existence of at least single and multiple positive solutions of the boundary value problems for a p-Laplacian functional dynamic equation on time scales. Our results generalize and extend some earlier results in the literature. As application, two examples are also given to illustrate the results.

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1. Introduction

Going back to its founder Stefan Hilger (1988), the study of dynamic equations on time scales is a fairly new area of mathematics. Motivating the subject is the notion that dynamic equations on time scales can build bridges between differential equations and

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difference equations. Now, the study of time scales theory has led to many important applications, for example, in the study of insect population models, neural networks, heat transfer, quantum mechanics, epidemic, crop harvest and stock market [5, 6, 13]. Two examples where time scales models could extend to variable time intervals are Atici's dynamic optimization model in economics [5] and the Thomas–Urena model for West Nile virus [13].

We assume that the reader is familiar with the notion of time scales. Thus note just that $\mathbb{T}$, $\sigma(\rho)$, $\mu(\nu)$, $f^{\Delta}(f^{\nabla})$, and $\int_a^b f(s)\Delta s \left(\int_a^b f(s)\nabla s\right)$ stand for time scale, forward (backward) jump operator, graininess, $\Delta(\nabla)$-derivative of f, and $\Delta(\nabla)$-integral of f from a to b, respectively. A function $f : \mathbb{T} \to \mathbb{R}$ is ld-continuous provided it is continuous at left-dense points in $\mathbb{T}$ and its right sided limit exists (finite) at right-dense points in $\mathbb{T}$. The set of all ld-continuous functions will be denoted by C_{ld}. See [6, 7] by Bohner and Peterson containing a lot of information on time scale calculus and references therein. In this paper, we make the blanket assumption that $-r$, 0, T are points in $\mathbb{T}$. By an interval $(0, T)$ we always mean the intersection of the real interval $(0, T)$ with the given time scale, that is $(0, T) \cap \mathbb{T}$. Other types of intervals are defined similarly.

For convenience, throughout this paper we denote $\varphi_p(u)$ as the p-Laplacian operator, i.e., $\varphi_p(u) = |u|^{p-2}u$ for $p > 1$ with $(\varphi_p)^{-1} = \varphi_q$, where $1/p + 1/q = 1$.

Very recently, there is an increasing attention paid to question of positive solution for second order multi-points boundary value problems on time scales [1–4, 7, 8, 14–17]. But very little work has been done to the existence of positive solutions for functional dynamic equations on time scales [9, 12]. In particular, we would like to mention some results of Kaufmann and Raffoul [9], Song and Xiao [12], Sun, Tang and Wang [16], Sun and Wang [17], which motivate us to consider our problem.

In [9], Kaufman and Raffoul studied the existence of at least one positive solution to the nonlocal eigenvalue problem for a class of nonlinear functional dynamic equations on time scales

$$u^{\Delta\nabla}(t) + \lambda a(t) f(u(t), u(\theta(t))) = 0, \quad t \in (0, T),$$

$$u(s) = \psi(s), \quad s \in [-r, 0], \quad u(0) = 0, \quad \alpha u(\eta) = u(T).$$

In [12], Song and Xiao considered the boundary value problems for a p-Laplacian functional dynamic equation on a time scale

$$\left(\varphi_p\left(x^{\Delta}(t)\right)\right)^{\nabla} + a(t) f(x(t), x(\mu(t))) = 0, \quad t \in (0, T),$$

$$x(s) = \psi(s), \quad s \in [-r, 0], \quad x(0) - B_0(x^{\Delta}(\eta)) = 0, \quad u^{\Delta}(T) = 0.$$

They established the existence result of at least two positive solutions by the twin fixed point theorem.

In [16], the authors considered the eigenvalue problems for p-Laplacian dynamic equations on time scales

$$\left(\varphi_p(u^{\Delta}(t))\right)^{\nabla} + \lambda h(t) f(u(t)) = 0, \quad t \in (0, T),$$

$$u(0) - \beta u^{\Delta}(0) = \gamma u^{\Delta}(\eta), \quad u^{\Delta}(T) = 0.$$

By applying the Krasnosel'skii fixed point theorem [10], they obtained some sufficient conditions for the nonexistence and existence of one or two positive solutions. When $\lambda = 1$, Sun and Wang [17] further established the existence criteria of one or multiple positive solutions of the problem.

Motivated by those works, in this paper we shall discuss the existence of single and multiple positive solutions of the boundary value problems for a p-Laplacian functional dynamic equation on time scales

$$\left(\varphi_p(u^{\Delta}(t))\right)^{\nabla} + h(t) f(u(t), u(\theta(t))) = 0, \quad t \in (0, T), \tag{1.1}$$

$$u(t) = \psi(t), \quad t \in [-r, 0], \quad u(0) - \beta u^{\Delta}(0) = \gamma u^{\Delta}(\eta), \quad u^{\Delta}(T) = 0. \tag{1.2}$$

Some new results are obtained for the existence of at least one, two and three positive solutions for the above problem by using Krasnosel'skii fixed point theorem [10] and Leggett–Williams fixed point theorem [11]. The results are even new for the special cases of differential equations and difference equations, as well as in the general time scale setting.

It is also noted that when $\theta(t) = t$, the problem (1.1), (1.2) reduces to the problem in [17], our existence criteria of positive solution generalize and extend the corresponding results in [17]. In the special case of $p = 2$, $\theta(t) = t$ and $\beta = 0$ or $\gamma = 0$, it has been extensively studied by many authors, see [2, 8].

The rest of this paper is organized as follows. In Section 2, we first give three lemmas which are needed later and then state two fixed point theorems due to Krasnosel'skii and Leggett–Williams. Section 3 will develop the existence criteria of at least one positive solution of the problem (1.1), (1.2). In Section 4, we consider the existence of multiple positive solutions of problem (1.1), (1.2). Two examples are also given to illustrate the main results.

For the sake of convenience, we list the following hypotheses:

(H1) $h \in C_{\mathrm{ld}}((0, T), [0, \infty))$ such that $0 < \int_0^T h(s) \nabla s < \infty$, $f \in C([0, \infty) \times [0, \infty), (0, \infty))$;

(H2) β, γ are nonnegative constants, $\eta \in (0, \rho(T))$;

(H3) $\psi \in C([-r, 0], [0, \infty))$, where $r > 0$;

(H4) $\theta \in C([0, T], [-r, T))$ and $\theta(t) \le t$ for all t.

2. Preliminaries

Let the Banach space $B = C_{\mathrm{ld}}[0, T]$ be endowed with the norm $\|u\| = \sup_{t \in [0,T]} |u(t)|$, and choose the cone $P \subset B$ defined by

$$P = \left\{ \begin{array}{c} u \in B : u(t) \ge 0 \text{ for } t \in [0, T] \text{ and} \\ u^{\Delta\nabla}(t) \le 0 \text{ for } t \in (0, T),\ u^{\Delta}(T) = 0 \end{array} \right\}.$$

Clearly, $\|u\| = u(T)$ for $u \in P$.

We notice that u is a solution of the problem (1.1), (1.2) if and only if

$$u(t) = \begin{cases} \displaystyle\int_0^t \varphi_q\left(\int_s^T h(\tau) f(u(\tau), u(\theta(\tau)))\nabla\tau\right)\Delta s \\ \displaystyle +\beta\varphi_q\left(\int_0^T h(s) f(u(s), u(\theta(s)))\nabla s\right) \\ \displaystyle +\gamma\varphi_q\left(\int_\eta^T h(s) f(u(s), u(\theta(s)))\nabla s\right), \quad t \in [0, T], \\ \psi(t), \quad t \in [-r, 0]. \end{cases}$$

For each $u \in B$, we extend u to $[-r, T]$ with $u(t) = \psi(t)$ for $t \in [-r, 0]$, and define an operator $A : P \to B$ by

$$\begin{aligned} Au(t) = &\int_0^t \varphi_q\left(\int_s^T h(\tau) f(u(\tau), u(\theta(\tau)))\nabla\tau\right)\Delta s \\ &+ \beta\varphi_q\left(\int_0^T h(s) f(u(s), u(\theta(s)))\nabla s\right) \\ &+ \gamma\varphi_q\left(\int_\eta^T h(s) f(u(s), u(\theta(s)))\nabla s\right) \\ &\text{for } 0 \le t \le T. \end{aligned}$$

Let u_1 be a fixed point of A in the cone P. Define

$$u(t) = \begin{cases} u_1(t), & t \in [0, T], \\ \psi(t), & t \in [-r, 0]. \end{cases}$$

Then, u is a positive solution of the problem (1.1), (1.2).

Now, we list some lemmas which are needed later.

Lemma 2.1. [15, Lemma 2.3] Assume $g : \mathbb{R} \to \mathbb{R}$ is continuous, $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable on $\mathbb{T}_\kappa$, and $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Then there exists c in the real interval $[\rho(t), t]$ with

$$(f \circ g)^\nabla(t) = f'(g(c))g^\nabla(t).$$

By the definition of the operator A, the monotonicity of $\varphi_q(x)$ and Lemma 2.1, it is easy to see that for each $u \in P$, $Au \in P$ and satisfies (1.2). In addition, $(\varphi_p(Au^\Delta))^\nabla(t) = -h(t)f(u(t), u(\theta(t))) < 0$, and $Au^\Delta(T) = 0$, so, $Au(T)$ is the maximum value of $Au(t)$.

Lemma 2.2. The operator $A : P \to P$ is completely continuous.

The proof of the lemma is similar to that of [16, Lemma 2.2], we omit it here.

Lemma 2.3. [16, Lemma 2.3] If $u \in P$, then $u(t) \geq \frac{t}{T}\|u\|$ for $t \in [0, T]$.

In order to prove our main results, the following fixed point theorems are crucial in our arguments.

Lemma 2.4. [10] Let P be a cone in a Banach space X. Assume Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$. If $A : P \cap (\overline{\Omega}_2 \backslash \Omega_1) \to P$ is a completely continuous operator such that either

(i) $\|Ax\| \leq \|x\|, \forall x \in P \cap \partial\Omega_1$ and $\|Ax\| \geq \|x\|, \forall x \in P \cap \partial\Omega_2$, or

(ii) $\|Ax\| \geq \|x\|, \forall x \in P \cap \partial\Omega_1$ and $\|Ax\| \leq \|x\|, \forall x \in P \cap \partial\Omega_2$,

then A has a fixed point in $P \cap (\overline{\Omega_2} \backslash \Omega_1)$.

Let $0 < a < b$ be given and let α be a nonnegative continuous concave functional on the cone P. Define the convex sets P_a, $P(\alpha, a, b)$ by

$$P_a = \{x \in P : \|x\| < a\},$$
$$P(\alpha, a, b) = \{x \in P : a \leq \alpha(x), \|x\| \leq b\}.$$

Then we state the Leggett–Williams fixed point theorem [11].

Lemma 2.5. Let P be a cone in a real Banach space B, $A : \overline{P}_c \to \overline{P}_c$ be completely continuous and α be a nonnegative continuous concave functional on P with $\alpha(x) \leq \|x\|$ for all $x \in \overline{P}_c$. Suppose there exists $0 < d < a < b \leq c$ such that

(i) $\{x \in P(\alpha, a, b) : \alpha(x) > a\} \neq \emptyset$ and $\alpha(Ax) > a$ for $x \in P(\alpha, a, b)$;

(ii) $\|Ax\| < d$ for $\|x\| \leq d$;

(iii) $\alpha(Ax) > a$ for $x \in P(\alpha, a, c)$ with $\|Ax\| > b$.

Then A has at least three fixed points x_1, x_2, x_3 satisfying

$$\|x_1\| < d,\ a < \alpha(x_2),\ \|x_3\| > d \text{ and } \alpha(x_3) < a.$$

3. Single Positive Solution

In this section, we will apply Lemma 2.4 to establish the existence of at least one positive solution of the problem (1.1), (1.2).

Throughout this paper, we denote $\delta = \eta/T$, define further subsets of $[0, T]$ with respect to the delay θ, set

$$Y_1 = \{t \in [0, T] : \theta(t) \geq 0\}, \quad Y_2 = \{t \in [0, T] : \theta(t) < 0\},$$

and denote

$$M = (T + \beta + \gamma)\varphi_q \left(\int_0^T h(s)\nabla s \right), \tag{3.1}$$

$$m = \delta(\eta + \beta + \gamma)\varphi_q \left(\int_{[\eta,T]\bigcap Y_2} h(s)\nabla s \right). \tag{3.2}$$

For the sake of convenience, we define

$$F_0(s) = \lim_{u\to 0^+} \sup \frac{f(u,\psi(s))}{\varphi_p(u)}, \qquad F_\infty(s) = \lim_{u\to +\infty} \sup \frac{f(u,\psi(s))}{\varphi_p(u)};$$

$$f_0(s) = \lim_{u\to 0^+} \inf \frac{f(u,\psi(s))}{\varphi_p(u)}, \qquad f_\infty(s) = \lim_{u\to +\infty} \inf \frac{f(u,\psi(s))}{\varphi_p(u)}.$$

Theorem 3.1. Suppose (H1)–(H4) hold, and f satisfies the following conditions:

(A1) $F_0(s) \le \varphi_p(1/M)$, uniformly in $s \in [-r, 0]$;

(A2) $\displaystyle\lim_{u_1\to 0^+, u_2\to 0^+} \frac{f(u_1,u_2)}{\max\{\varphi_p(u_1), \varphi(u_2)\}} \le \varphi_p(1/M)$;

(A3) $f_\infty(s) \ge \varphi_p(1/m)$, uniformly in $s \in [-r, 0]$.

Then, the problem (1.1), (1.2) has at least one positive solution.

Proof. By the condition (A1), there exists an $\varepsilon_1 > 0$, such that if $0 < u < \varepsilon_1$, then

$$f(u,\psi(s)) \le \varphi_p(1/M)\,\varphi_p(u) \quad \text{for } s \in [-r, 0]. \tag{3.3}$$

Similarly, by condition (A2), there exists an $\varepsilon_2 > 0$, such that if $0 < u_1, u_2 \le \varepsilon_2$, then

$$f(u_1,u_2) \le \max\{\varphi_p(u_1), \varphi_p(u_2)\}\varphi_p(1/M). \tag{3.4}$$

Let $r_1 = \min\{\varepsilon_1, \varepsilon_2\}$, then, for any $u \in P$ with $\|u\| = r_1$, by (3.1), (3.3) and (3.4), we have

$$\begin{aligned}
\|Au\| = Au(T) &= \int_0^T \varphi_q\left(\int_s^T h(\tau)f(u(\tau), u(\theta(\tau)))\nabla\tau\right)\Delta s \\
&\quad + \beta\varphi_q\left(\int_0^T h(s)f(u(s), u(\theta(s)))\nabla s\right) \\
&\quad + \gamma\varphi_q\left(\int_\eta^T h(s)f(u(s), u(\theta(s)))\nabla s\right) \\
&\le (T+\beta+\gamma)\varphi_q\left(\int_0^T h(s)f(u(s), u(\theta(s)))\nabla s\right) \\
&= (T+\beta+\gamma)\varphi_q\left(\int_{Y_1} h(s)f(u(s), u(\theta(s)))\nabla s\right. \\
&\quad \left. + \int_{Y_2} h(s)f(u(s), \psi(\theta(s)))\nabla s\right) \\
&\le (T+\beta+\gamma)\varphi_q\left(\int_{Y_1} h(s)\max\{\varphi_p(u(s)), \varphi_p(u(\theta(s)))\}\varphi_p\left(1/M\right)\nabla s\right. \\
&\quad \left. + \int_{Y_2} h(s)\varphi_p\left(\frac{u(s)}{M}\right)\nabla s\right) \\
&\le (T+\beta+\gamma)\varphi_q\left(\int_0^T h(s)\nabla s\right)\max_{s\in[0,T]}\{u(s)\}\frac{1}{M} = \|u\|.
\end{aligned}$$

Define $\Omega_1 = \{u \in B : \|u\| < r_1\}$. Then

$$\|Au\| \le \|u\| \text{ for } u \in P \cap \partial\Omega_1. \tag{3.5}$$

By condition (A3), we can find an $R_1 > 2r_1$, such that

$$f(u, \psi(s)) \ge \varphi_p(u/m) \quad \text{for } u \ge R_1, \quad s \in [-r, 0]. \tag{3.6}$$

Pick $u \in P$ such that $\|u\| = R_1$. By Lemma 2.3, we know

$$\min_{t\in[\eta,T]} u(t) \ge \delta\|u\|. \tag{3.7}$$

Now, define $\Omega_2 = \{u \in B : \|u\| < R_1\}$. By (3.2), (3.6) and (3.7), we have

$$\begin{aligned}
\|Au\| = Au(T) &= \int_0^T \varphi_q\left(\int_s^T h(\tau)f(u(\tau), u(\theta(\tau)))\nabla\tau\right)\Delta s \\
&\quad + \beta\varphi_q\left(\int_0^T h(s)f(u(s), u(\theta(s)))\nabla s\right) \\
&\quad + \gamma\varphi_q\left(\int_\eta^T h(s)f(u(s), u(\theta(s)))\nabla s\right) \\
&\geq (\eta+\beta+\gamma)\varphi_q\left(\int_\eta^T h(s)f(u(s), u(\theta(s)))\nabla s\right) \\
&\geq (\eta+\beta+\gamma)\varphi_q\left(\int_{[\eta,T]\bigcap Y_2} h(s)f(u(s), \psi(\theta(s)))\nabla s\right) \\
&\geq (\eta+\beta+\gamma)\varphi_q\left(\int_{[\eta,T]\bigcap Y_2} h(s)\varphi_p\left(\frac{u(s)}{m}\right)\nabla s\right) \\
&\geq \frac{m}{\delta}\cdot\frac{1}{m}\delta\,\|u\| = \|u\|.
\end{aligned}$$

Hence,

$$\|Au\| \geq \|u\| \text{ for } u \in P \cap \partial\Omega_2. \tag{3.8}$$

By (3.5) and (3.8), applying the condition (i) of Lemma 2.4, A has a fixed point $u_1 \in P \cap (\overline{\Omega_2}\backslash\Omega_1)$, and $r_1 \leq \|u_1\| \leq R_2$. It is clear that u is a positive solution of the problem (1.1), (1.2) with the form

$$u(t) = \begin{cases} u_1(t), & t \in [0,T], \\ \psi(t), & t \in [-r,0]. \end{cases}$$

The proof is complete. ∎

Theorem 3.2. Suppose (H1)–(H4) hold, and f satisfies the following conditions:

(B1) $f_0(s) \geq \varphi_p(1/m)$, uniformly in $s \in [-r,0]$;

(B2) $F_\infty(s) \leq \varphi_p(1/M)$, uniformly in $s \in [-r,0]$;

(B3) $\displaystyle\lim_{u_1\to\infty, u_2\to\infty} \frac{f(u_1,u_2)}{\max\{\varphi_p(u_1), \varphi_p(u_2)\}} \leq \varphi_p(1/M)$.

Then, the problem (1.1), (1.2) has at least one positive solution.

Proof. By condition (B1), there exists an $r_2 > 0$ such that

$$f(u, \psi(s)) \geq \varphi_p(u/m), \quad 0 < u \leq r_2, \quad s \in [-r,0]. \tag{3.9}$$

Let $u \in P$ with $\|u\| = r_2$. From (3.7) and (3.9), we have

$$\begin{aligned}\|Au\| = Au(T) &\geq (\eta+\beta+\gamma)\varphi_q\left(\int_\eta^T h(s)f(u(s), u(\theta(s)))\nabla s\right)\\ &\geq (\eta+\beta+\gamma)\varphi_q\left(\int_{[\eta,T]\bigcap Y_2} h(s)f(u(s), \psi(\theta(s)))\nabla s\right)\\ &\geq (\eta+\beta+\gamma)\varphi_q\left(\int_{[\eta,T]\bigcap Y_2} h(s)\varphi_p\left(\frac{u(s)}{m}\right)\nabla s\right) \geq \frac{m}{\delta}\cdot\frac{1}{m}\delta\,\|u\| = \|u\|\,.\end{aligned}$$

Set $\Omega_1 = \{u \in B : \|u\| < r_2\}$. Then

$$\|Au\| \geq \|u\| \quad \text{for } u \in P \cap \partial\Omega_1. \tag{3.10}$$

To construct Ω_2, we need to consider two cases: f is bounded and f is unbounded.

Case 1. Suppose f is bounded. Then there exists some $N > 0$ such that

$$f(u_1, u_2) \leq \varphi_p(N),\ u_1, u_2 \in (0, \infty)\,. \tag{3.11}$$

Set $R_2 = \max\{2r_2, NM\}$. Then, for each $u \in P$ with $\|u\| = R_2$, we have

$$\begin{aligned}\|Au\| = Au(T) &\leq (T+\beta+\gamma)\varphi_q\left(\int_0^T h(s)f(u(s), u(\theta(s)))\nabla s\right)\\ &\leq N(T+\beta+\gamma)\varphi_q\left(\int_0^T h(s)\nabla s\right) \leq R_2.\end{aligned}$$

Case 2. Suppose f is unbounded. In view of the condition (B2), we know there exists a $\rho_1 > r_2$ such that if $u > \rho_1$, then

$$f(u, \psi(s)) \leq \varphi_p(u/M) \quad \text{for } s \in [-r, 0]. \tag{3.12}$$

Similarly by condition (B3), there exists a $\rho_2 > r_2$, if $u_1 \geq \rho_2$, $u_2 \geq \rho_2$, we have

$$f(u_1, u_2) \leq \max\{\varphi_p(u_1), \varphi_p(u_2)\}\varphi_p\,(1/M)\,. \tag{3.13}$$

Let $R_2 = \max\{\rho_1, \rho_2\}$. For any $u \in P$ with $\|u\| = R_2$, by (3.1), (3.12) and (3.13), we have

$$\begin{aligned}
\|Au\| &\le (T+\beta+\gamma)\varphi_q\left(\int_0^T h(s)f(u(s),u(\theta(s)))\nabla s\right)\\
&= (T+\beta+\gamma)\varphi_q\left(\int_{Y_1} h(s)f(u(s),u(\theta(s)))\nabla s\right.\\
&\quad \left.+\int_{Y_2} h(s)f(u(s),\psi(\theta(s)))\nabla s\right)\\
&\le (T+\beta+\gamma)\varphi_q\left(\int_{Y_1} h(s)\max\{\varphi_p(u(s)),\varphi_p(u(\theta(s)))\}\varphi_p\left(1/M\right)\nabla s\right.\\
&\quad \left.+\int_{Y_2} h(s)\varphi_p\left(\frac{u(s)}{M}\right)\nabla s\right)\\
&\le (T+\beta+\gamma)\varphi_q\left(\int_0^T h(s)\nabla s\right)\max_{t\in[0,T]}\{u(t)\}\frac{1}{M} = R_2.
\end{aligned}$$

Hence, in both Case 1 and Case 2, if we set $\Omega_2 = \{u \in B : \|u\| < R_2\}$, then

$$\|Au\| \le \|u\| \quad \text{for } u \in P\cap\partial\Omega_2. \tag{3.14}$$

By (3.10), (3.14) and Lemma 2.4 (ii), we know A has a fixed point $u_1 \in P\cap(\overline{\Omega_2}\backslash\Omega_1)$ with $r_2 \le \|u_1\| \le R_2$. So u is a positive solution of (1.1), (1.2) with the form

$$u(t) = \begin{cases} u_1(t), & t\in[0,T],\\ \psi(t), & t\in[-r,0].\end{cases}$$

The proof is complete. ■

4. Multiple Positive Solutions

This section we devote to applying Lemma 2.4 and Lemma 2.5 to establish the existence of at least two and three positive solutions to the problem (1.1), (1.2) respectively.

Theorem 4.1. Assume (H1)–(H4) hold, and f satisfies the following conditions:

(C1) $f_0(s) \ge \varphi_p(1/m)$, uniformly in $s \in [-r,0]$;

(C2) $f_\infty(s) \ge \varphi_p(1/m)$, uniformly in $s \in [-r,0]$;

(C3) there exists a $q_1 > 0$, such that if $0 < u \le q_1$, then $f(u,\psi(s)) \le \varphi_p(q_1/M)$, $s\in[-r,0]$, and $f(u_1,u_2) \le \varphi_p(q_1/M)$ for $0 < u_1, u_2 \le q_1$.

Then (1.1), (1.2) has at least two positive solutions.

Proof. Similar to the proof of Theorem 3.2, in view of (C1), there exists an r_1, $0 < r_1 < q_1$ such that

$$f(u,\psi(s)) \ge \varphi_p(u/m), \quad 0\le u\le r_1, \quad s\in[-r,0]. \tag{4.1}$$

Let $u \in P$ with $\|u\| = r_1$. From (3.2), (3.7) and (4.1) we have

$$\|Au\| \geq (\eta + \beta + \gamma)\varphi_q \left(\int_{[\eta,T]\bigcap Y_2} h(s) f(u(s), \psi((\theta(s)))\nabla s \right)$$
$$\geq (\eta + \beta + \gamma)\varphi_q \left(\int_{[\eta,T]\bigcap Y_2} h(s)\varphi_p \left(\frac{u(s)}{m} \right) \nabla s \right) \geq \|u\| .$$

If we define $\Omega_1 = \{u \in B : \|u\| < r_1\}$, then

$$\|Au\| \geq \|u\| \ \text{ for } u \in P \cap \partial\Omega_1. \tag{4.2}$$

Now, we consider $u \in P$ with $\|u\| = q_1$. By the condition (C3), we get

$$\|Au\| \leq (T + \beta + \gamma)\varphi_q \left(\int_0^T h(s) f(u(s), u(\theta(s)))\nabla s \right)$$
$$= (T + \beta + \gamma)\varphi_q \left(\int_{Y_1} h(s) f(u(s), u(\theta(s)))\nabla s \right.$$
$$\left. + \int_{Y_2} h(s) f(u(s), \psi(\theta(s)))\nabla s \right)$$
$$\leq (T + \beta + \gamma)\varphi_q \left(\int_0^T h(s)\nabla s \right) \frac{q_1}{M} = q_1.$$

Let $\Omega_2 = \{u \in B : \|u\| < q_1\}$. From the above inequality, we have

$$\|Au\| \leq \|u\| \ \text{ for } u \in P \cap \partial\Omega_2. \tag{4.3}$$

An application of Lemma 2.4 (ii) yields a fixed point u_1 of A, $u_1 \in P \cap (\overline{\Omega_2}\backslash\Omega_1)$ with $r_1 \leq \|u_1\| \leq q_1$.

By condition (C2), we know there exists an $R_1 > q_1$ such that

$$f(u, \psi(s)) \geq \varphi_p(u/m) \quad \text{for } u \geq R_1, s \in [-r, 0]. \tag{4.4}$$

We pick $u \in P$ such that $\|u\| = R_1$. Furthermore, define $\Omega_3 = \{u \in B : \|u\| < R_1\}$. By using (3.2), (3.7) and (4.4), similar to the second part of the proof of Theorem 3.1 we get

$$\|Au\| \geq \|u\| \ \text{ for } u \in P \cap \partial\Omega_3. \tag{4.5}$$

By (4.2), (4.5) and Lemma 2.4 (i), there is a fixed point u_2 of A with $q_1 \leq \|u_2\| \leq R_1$.

Then, there exist two positive solutions of (1.1), (1.2) with the form

$$u(t) = \begin{cases} u_i(t), & t \in [0, T], \quad i = 1, 2 \\ \psi(t), & t \in [-r, 0]. \end{cases}$$

The proof is complete. ∎

Theorem 4.2. Assume (H1)–(H4) hold, and f satisfies the following conditions:

(D1) $F_0(s) \leq \varphi_p(1/M)$, uniformly in $s \in [-r, 0]$;

(D2) $\lim\limits_{u_1 \to 0^+, u_2 \to 0^+} \dfrac{f(u_1, u_2)}{\max\{\varphi_p(u_1), \varphi_p(u_2)\}} \leq \varphi_p(1/M)$;

(D3) $F_\infty(s) \leq \varphi_p(1/M)$, uniformly in $s \in [-r, 0]$;

(D4) $\lim\limits_{u_1 \to \infty, u_2 \to \infty} \dfrac{f(u_1, u_2)}{\max\{\varphi_p(u_1), \varphi_p(u_2)\}} \leq \varphi_p(1/M)$;

(D5) there is $q_2 > 0$, such that if $0 < u < q_2$, then $f(u, \psi(s)) \geq \varphi_p(u/m)$, uniformly in $s \in [-r, 0]$.

Then, the problem (1.1), (1.2) has at least two positive solutions.

Proof. Firstly, since the conditions (D1) and (D2) are similar to the conditions (A1) and (A2) in Theorem 3.1, with a similar proof process, we can find an Ω_1 with respect to $u \in B$ such that

$$\|Au\| \leq \|u\| \text{ for } u \in P \cap \partial\Omega_1. \tag{4.6}$$

By the condition (D5), pick $u \in P$ with $\|u\| = q_2$, and set $\Omega_2 = \{u \in B : \|u\| < q_2\}$. We have that

$$\begin{aligned} \|Au\| &\geq (\eta + \beta + \gamma)\varphi_q \left(\int_{[\eta,T] \bigcap Y_2} h(s)\varphi_p\left(\frac{u(s)}{m}\right) \nabla s \right) \\ &= \frac{m}{\delta} \cdot \frac{\delta \|u\|}{m} = q_2. \end{aligned}$$

So we get another inequality

$$\|Au\| \geq \|u\| \text{ for } u \in P \cap \partial\Omega_2. \tag{4.7}$$

Hence, the inequalities (4.6), (4.7) together with the condition (i) of Lemma 2.4 imply that A has a fixed point u_1 of A which satisfies $0 < \|u_1\| \leq q_2$.

By the same reason, because the conditions (D3) and (D4) are completely parallel with the conditions (B2) and (B3) of Theorem 3.2, the second part of the proof of Theorem 3.2 carries over verbatim. We can construct an Ω_3 with respect to $u \in B$ naturally such that the inequality

$$\|Au\| \leq \|u\|, \quad u \in P \cap \partial\Omega_3 \tag{4.8}$$

holds. Now, we have obtained (4.7) and (4.8), together with the condition (ii) of Lemma 2.4, A has another fixed point u_2, $u_2 \in P \cap (\overline{\Omega_3} \backslash \Omega_2)$, and $q_2 \leq \|u_2\| \leq R_2$, clearly u is a positive solution of (1.1), (1.2) with the form

$$u(t) = \begin{cases} u_i(t), & t \in [0, T], \quad i = 1, 2 \\ \psi(t), & t \in [-r, 0]. \end{cases}$$

The proof is complete. ■

In order to establish existence criteria of at least three positive solutions of the problem (1.1), (1.2), we define a nonnegative continuous concave functional $\alpha : P \to [0, \infty)$ by

$$\alpha(u) = \min_{t\in[\eta,T]} u(t) = u(\eta).$$

Theorem 4.3. Assume (H1)–(H4) hold, there exist $a, b, d = b/\delta$ and c, with $0 < a < b < d \le c$, and f satisfies the following conditions:

(F1) $f(u, \psi(s)) < \varphi_p(a/M)$ for $0 \le u \le a$, uniformly in $s \in [-r, 0]$, and $f(u_1, u_2) < \varphi_p(a/M)$ for $0 \le u_1, u_2 \le a$;

(F2) $f(u, \psi(s)) > \varphi_p(b/m)$ for $b \le u \le d$, uniformly in $s \in [-r, 0]$;

(F3) $f(u, \psi(s)) \le \varphi_p(c/M)$ for $0 \le u \le c$, uniformly in $s \in [-r, 0]$, and $f(u_1, u_2) \le \varphi_p(c/M)$ for $0 \le u_1, u_2 \le c$.

Then the problem (1.1), (1.2) has at least three positive solutions u_1, u_2 and u_3, with

$$\|u_1\| \le a \le \|u_2\| \text{ and } \min_{t\in[\eta,T]} u_3(t) < b < \min_{t\in[\eta,T]} u_2(t).$$

Proof. By the definition of the operator A and its properties, it suffices to show that the conditions of Lemma 2.5 hold with respect to A.

First, we show that if (F3) and (F1) hold, then $A\overline{P_c} \subset \overline{P_c}$ and $A\overline{P_a} \subset P_a$. Obviously, Lemma 2.2 guarantees $A\overline{P_c} \subset P$. For any $u \in P_c$, we have $0 \le u(t) \le c$, $t \in [0, T]$. By condition (F3), we have

$$\begin{aligned}
\|Au\| = Au(T) &= \int_0^T \varphi_q\left(\int_s^T h(\tau)f(u(\tau), u(\theta(\tau)))\Delta\tau\right)\nabla s \\
&\quad + \beta\varphi_q\left(\int_0^T h(s)f(u(s), u(\theta(s)))\nabla ds\right) \\
&\quad + \gamma\varphi_q\left(\int_\eta^T h(s)f(u(s), u(\theta(s)))\nabla s\right) \\
&\le (T + \beta + \gamma)\varphi_q\left(\int_{Y1} h(s)f(u(s), u(\theta(s)))\nabla s\right. \\
&\quad \left. + \int_{Y2} h(s)f(u(s), \psi(\theta(s)))\nabla s\right) \\
&\le (T + \beta + \gamma)\varphi_q\left(\int_0^T h(s)\nabla s\right)\frac{c}{M} = c.
\end{aligned}$$

Therefore, $\|Au\| \le c$ and $A\overline{P_c} \subset \overline{P_c}$. Similarly, by (F1), we obtain $Au \in P_a$ for all $u \in \overline{P_a}$.

Next, we verify $\{u \in P(\alpha, b, d) : \alpha(u) > b\} \neq \emptyset$ and $\alpha(Au) > b$ for $u \in P(\alpha, b, d)$. In fact, $u = \dfrac{b+d}{2} \in \{P(\alpha, b, d) : \alpha(u) > b\}$. For $u \in P(\alpha, b, d)$, we have $b \leq \min_{t\in[\eta,T]} \leq u(t) \leq d$, $t \in [\eta, T]$. In view of condition (F2), we have that

$$\begin{aligned}
\alpha(Au) = Au(\eta) &= \int_0^\eta \varphi_q\left(\int_s^T h(\tau)f(u(\tau), u(\theta(\tau)))\Delta\tau\right)\nabla s \\
&\quad + \beta\varphi_q\left(\int_0^T h(s)f(u(s), u(\theta(s)))\nabla ds\right) \\
&\quad + \gamma\varphi_q\left(\int_\eta^T h(s)f(u(s), u(\theta(s)))\nabla s\right) \\
&\geq (\eta+\beta+\gamma)\varphi_q\left(\int_{[\eta,T]\bigcap Y_2} h(s)f(u(s), \psi(s))\nabla s\right) > \frac{m}{\delta}\cdot\frac{\delta b}{m} = b.
\end{aligned}$$

Finally, we assert that $\alpha(Au) > b$ hold for all $u \in P(\alpha, b, c)$ and $\|Au\| \geq d$. If $u \in P(\alpha, b, c)$ and $\|Au\| \geq d$, by Lemma 2.3, we have

$$\alpha(Au) = Au(\eta) \geq \delta\,\|Au\| > \delta d/\delta = d > b. \tag{4.9}$$

Hence, with an application of Lemma 2.5, we know that there exist three positive solutions of (1.1), (1.2), with the form

$$u = \begin{cases} u_i(t) & t \in [0, T], \quad i = 1, 2, 3 \\ \psi(t) & t \in [-r, 0]. \end{cases}$$

The proof is complete. ∎

Example 4.4. Let $\mathbb{T} = \left[-\frac{1}{3}, 0\right] \cup \left\{\left(\frac{1}{2}\right)^{\mathbb{N}_0}\right\} \cup \left\{\frac{3}{8}, \frac{3}{4}\right\}$, where $\mathbb{N}_0$ denotes the set of nonnegative integers. Suppose A and B are positive constant numbers. We consider the boundary value problem

$$\begin{cases} u^{\Delta\nabla}(t) + \dfrac{Bu(t)}{u^2(t) + u^2(t-\frac{1}{4}) + A} = 0, \\ \psi(t) = 0, \quad t \in \left[-\dfrac{1}{4}, 0\right], \quad u(0) - \dfrac{1}{2}u^\Delta(0) = \dfrac{1}{2}u^\Delta\left(\dfrac{3}{8}\right), \quad u^\Delta(1) = 0, \end{cases} \tag{4.10}$$

where $T = 1$, $\eta = \frac{3}{8}$, $p = 2$, $\beta = \gamma = \frac{1}{2}$, $h(s) \equiv 1$, $\theta : [0, 1] \to \left[-\frac{1}{4}, \frac{3}{4}\right]$, $\theta(t) = t - \frac{1}{4}$, and

$$f(u_1, u_2) = \frac{Bu_1}{u_1{}^2 + u_2{}^2 + A}.$$

Then we get $\delta = \frac{3}{8}$, $Y_1 = \left[\frac{1}{4}, 1\right]$, $Y_2 = \left[0, \frac{1}{4}\right]$ and $[\eta, T] \cap Y_2 = \left[\frac{1}{8}, 1\right] \cap \left[0, \frac{1}{4}\right] = \left\{\frac{1}{8}, \frac{1}{4}\right\}$. By (3.1) and (3.2), we have that

$$M = (T + \beta + \gamma)\varphi_q \left(\int_0^T h(s)\nabla s\right) = 2,$$

$$m = \delta(\eta + \beta + \gamma)\varphi_q \left(\int_{[\eta,T]\bigcap Y_2} h(s)\nabla s\right) = \frac{33}{512}.$$

Clearly, the conditions (H1)–(H4) hold.

If we choose the positive numbers A and B such that $B > 16A$, then

$$f_0(s) = \lim_{u\to 0^+} \frac{B}{u^2 + A} = \frac{B}{A} > \frac{1}{m},$$

$$F_\infty = 0 < \frac{1}{M}, \qquad \lim_{u_1\to\infty, u_2\to\infty} \frac{f(u_1, u_2)}{\max\{u_1, u_2\}} = 0 < \frac{1}{M}.$$

Thus, by Theorem 3.2, the problem (4.10) has at least one positive solution.

Example 4.5. Let $f(u_1, u_2) = \dfrac{32u_1^2}{u_1^2 + u_2^2 + 1}$, while the other parameters are the same as those in Example 4.4. Obviously, f is increasing with respect to u_1. If we take $a = 0.01$, $b = 1$, $c = 70$, then

$$a < b < d = b/\delta = 8/3 < c.$$

Now, we check that the conditions in Theorem 4.3 are satisfied. Observe that

$$f(u, \psi(s)) \le f(a, 0) \approx 0.003 < 0.005 = a/M,\ 0 \le u \le 0.01,$$

$$f(u_1, u_2) \le f(a, 0) \approx 0.003 < 0.005 = a/M,\ 0 \le u_1, u_2 \le 0.01,$$

so (F1) in Theorem 4.3 is satisfied. To verify (F2), note that $f(b, 0) = f(1, 0) = 16$, then,

$$f(u, \psi(s)) \ge f(b, 0) = 16 > \frac{512}{33} = \frac{b}{m},\ 1 \le u \le \frac{8}{3}.$$

Finally, in view of $f(c, 0) = f(70, 0) \approx 31.99$, we have

$$f(u, \psi(s)) \le f(c, 0) \approx 31.99 < 35 = c/M,\ 0 \le u \le 70,$$

$$f(u_1, u_2) \le f(c, 0) \approx 31.99 < 35 = c/M,\ 0 \le u_1, u_2 \le 70,$$

and thus, (F3) holds too. By Theorem 4.3, the boundary value problem

$$\begin{cases} u^{\Delta\nabla}(t) + \dfrac{32u^2(t)}{u^2(t) + u^2(t - \frac{1}{4}) + 1} = 0, \\ \psi(t) = 0, \quad t \in \left[-\frac{1}{4}, 0\right], \quad u(0) - \frac{1}{2}u^\Delta(0) = \frac{1}{2}u^\Delta\left(\frac{3}{8}\right), \quad u^\Delta(1) = 0, \end{cases} \tag{4.11}$$

has at least three positive solutions u_1, u_2, u_3 satisfying

$$\|u_1\| \leq 0.01 \leq \|u_2\| \text{ and } \min_{t\in[3/8,1]} u_3(t) < 1 < \min_{t\in[3/8,1]} u_2(t).$$

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