Stochastic Geometry and Random Graphs for the Analysis and Design of Wireless Networks

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ICC 2010 Tutorial
Cape Town, South Africa
May 27, 2010

Available at http://www.nd.edu/~mhaenggi/talks/icc10.pdf
Overview

Contents of the tutorial

- Introduction and motivation
- Part I: Single-hop analysis of Poisson networks
- Part II: Random geometric graphs and percolation
- Part III: Multi-hop analysis of Poisson networks
- Part IV: Single-hop analysis of general networks
- Summary
Overview of tutorial parts

- **Part I**: Single-hop analysis
- **Part II**: Random graphs; percolation
- **Part III**: Multi-hop analysis
- **Part IV**: Single-hop analysis

**Poisson networks**

**General networks**
Analysis of Wireless Networks

Comparison of analytical approaches

- **scaling laws**
  - very limited design insight

- analysis of networks with fixed geometry
  - concrete results but no generality

- **stochastic geometry**
  - analysis of the *average* network
  - generality *and* design insight
The Critical Role of the Network Geometry

Wireless transmissions are separated in space, time, or frequency

- Separation in time and frequency not sufficient for wireless networks.
- Need for *spatial reuse*. But separation in space is much more challenging.
Why is spatial reuse hard?

- There is **interference** between concurrent transmissions.
- Transmitter and receiver have a different picture of the situation.
**The cellular solution**

Cellular system with frequency reuse factor 1/7

**A sensible solution: CSMA**

Hidden node

Exposed node

**The simplest solution: ALOHA**

Let nodes transmit independently with probability $p$.

**Performance analysis and protocol design**

We need to analyze first the interference and the outage probabilities and then determine optimum protocols (MAC, routing).
Abstraction

Questions of interest

- What is the interference at $R$? How likely is the transmission $T \rightarrow R$ to succeed?
- How to regulate channel access and perform routing?
Key Result

Setup

- Infinitely many nodes are distributed randomly on the plane, with density $\lambda$, and a fraction $p$ transmits, all at power $P$.
- Channels are Rayleigh fading.
- Path loss is $g(r) = r^{-\alpha}$.
- Noise power is $W$.
- The transmission is successful if the signal-to-interference-plus-noise ratio exceeds a threshold $\theta$.

Result

Probability of successful transmission over distance $R$ [BBM06]:

$$p_s = \mathbb{P}(\text{SINR} > \theta) = \exp \left( -p\lambda\theta^{2/\alpha} C(\alpha) R^2 - \theta WR^{\alpha}/P \right),$$

where $C(\alpha) = \pi \Gamma(1 - 2/\alpha)\Gamma(1 + 2/\alpha) = 2\pi^2/(\alpha \sin(2\pi/\alpha))$. 
Propagation and Physical Layer

Path loss and fading

If a node transmits at power $P$ over a distance $r$, the received power is

$$S = Phg(r),$$

where:

- $g(r)$ is the large-scale (or mean) path loss law, assumed monotonically decreasing. Typically $g(r) = r^{-\alpha}$, where $\alpha$ is the path loss exponent.

- $h$ is the power fading coefficient. We always have $\mathbb{E}h = 1$. We usually assume a block fading model, where $h$ changes from one transmission to the next. Often we consider Rayleigh fading, where $h$ is exponential:

$$F_h(x) = 1 - \exp(-x), \quad x \geq 0.$$ 

The amplitude $\sqrt{h}$ is Rayleigh distributed.
**SINR**

With thermal noise of variance $W$, the signal-to-noise ratio (SNR) is

$$\frac{S}{W} = \frac{Phg(r)}{W}.$$  

The interference $I$ is the cumulative power from all undesired transmitters.

$$I = \sum_{i \in I} P_i h_i g(r_i).$$

This leads to the signal-to-interference-plus-noise ratio (SINR)

$$\text{SINR} = \frac{Phg(r)}{W + I}.$$  

The SINR is our main metric of interest.

**Model for transmission success**

$$p_s \triangleq \mathbb{P}(\text{SINR} > \theta).$$

The rate of transmission is smaller than (but close to) $\log_2(1 + \theta)$. 
Example (Rayleigh block fading with power path loss law)

With $k$ interferers at known distances $r_i$ and path loss law $r^{-\alpha}$:

$$p_s(r) = \mathbb{P}(S > \theta(W + I)) = \exp \left( - \frac{\theta W}{P} r^\alpha \right) \cdot \prod_{i=1}^{k} \frac{1}{1 + \theta \frac{P_i}{P} \left( \frac{r}{r_i} \right)^\alpha}$$

Proof

Let $S = Phr^{-\alpha}$ be the received power, $\tilde{S} = Pr^{-\alpha}$, and $I = \sum_{i=1}^{k} P_i h_i r_i^{-\alpha}$.

$$P_s = \mathbb{P}[S > \theta(W + I)] = \mathbb{E}_I \left\{ \exp \left( - \frac{\theta(I + W)}{\tilde{S}} \right) \right\}$$

$$= \exp \left( - \frac{\theta W}{Pr^{-\alpha}} \right) \cdot \mathbb{E}_I \left\{ \exp \left( - \frac{\theta I}{\tilde{S}} \right) \right\}$$

These are Laplace transforms! $p_s = \mathcal{L}_W(\theta r^\alpha / P) \cdot \mathcal{L}_I(\theta / \tilde{S})$. 
Remarks

- In a wireless network, there is a lot more uncertainty than fading: $k$, $r_i$, perhaps $P_i$. There is a need to model uncertainty in the locations of the nodes.

- Let $I_1$ denote the interference at the receiver. We have

$$\text{SINR}_1 = \frac{Phg(r)}{W + I_1}.$$ 

Now assume all nodes scale their power by a factor $a$. Then $I_a = aI_1$, and

$$\text{SINR}_a = \frac{aPhg(r)}{W + I_a} = \frac{Phg(r)}{W/a + I_1}.$$ 

So, increasing the power improves the SINR, since the noise power $W$ is reduced by $a$.

- The noise term $\exp(-\theta Wr^\alpha/P)$ is less interesting, so we often focus on the SIR only.
The Uncertainty Cube

Three dimensions of uncertainty

- Rayleigh fading
- ALOHA
- Poisson process
- Channel
- Access
- Node positions

The interferer geometry is determined by the point process (node distribution) and the MAC scheme.

- The goal is to characterize the average network, using suitable averaging over the uncertainty.
- We will focus on some of the corner points in Part I and talk about the interior in Part III.
**Node Locations**

**The Poisson point process**

The (homogeneous) Poisson point process (PPP) is the reference model for the distribution of nodes in an ad hoc (or sensor) network.

**Example (PPP of intensity $\lambda$)**

Take a Poisson process $\Phi = \{x_1, x_2, \ldots\}$ of constant intensity $\lambda$ in a square or disk of area $A$. Often, $A \to \infty$ to avoid boundary issues (or use toroidal boundary conditions).
Properties of the PPP

Let $A_i \subset A$ be non-overlapping areas. Then

$$P[\Phi(A_i) = n] = \exp(-\lambda |A_i|) \frac{(\lambda |A_i|)^n}{n!}$$

where $\Phi(\cdot)$ is the counting measure (number of points), $|\cdot|$ the Lebesque measure (area), and $\lambda$ is the density of the (homogeneous) PPP.

1. $\Phi(A_1), \Phi(A_2), \ldots$ are independent.

One of these properties is actually a consequence of the other.

Rényi (1967)

The Poisson distribution implies independence. This is not the case if Property 1 only holds for convex $A_i$. 
Advantages of the PPP model

- Often viewed as worst case (maximum entropy).
- Analytical tractability. Due to the independence property, conditioning on having a point somewhere does not affect the rest of the network. As a consequence, there are many nice results (distances, interference, outage, percolation, connectivity, coverage, ...).

Disadvantages of the PPP model

- Independence property $\iff$ zero interaction between nodes' positions. This is not a good model for many networks.
- Often, the set of transmitters or active nodes is to be modeled. Except for pure ALOHA, the transmitters do not form a PPP even if the underlying process containing all nodes is Poisson.
Simple result for the PPP: Internode distances [Hae05]

The pdf of the distance to the $n$-th nearest neighbor is

$$p_{R_n}(r) = r^{2n-1} (\lambda \pi)^n \frac{2}{\Gamma(n)} \exp(-\lambda \pi r^2), \quad r \geq 0.$$ 

For $n = 1$ (nearest neighbor), this is a Rayleigh distribution. The mean distance is $1/(2\sqrt{\lambda})$. It does not matter whether we measure from an arbitrary point of the plane or from a point of the process.

Proof

The probability that the $n$-th nearest neighbor of a point is further away than $r$ is the probability that there are less than $n$ points in the area $\lambda \pi r^2$.

$$\mathbb{P}[R_n > r] = \exp(-\lambda \pi r^2) \sum_{k=0}^{n-1} \frac{(\lambda \pi r^2)^k}{k!}$$

This is the ccdf, so $p_{R_n}(r) = -\frac{d\mathbb{P}[R_n > r]}{dr}$
Channel Access

**ALOHA**

In ALOHA, each node makes the decision to transmit independently and randomly: In each time slot, each node decides to transmit with probability $p$ and to stay quiet (listen) with probability $1 - p$. This is (slotted) ALOHA.

**CSMA**

With carrier sensing, there is a minimum separation between concurrent transmitters. This usually also imposes a minimum spacing between receivers and interferers.

**Deterministic scheduling**

A better performance can usually be achieved if nodes’ transmissions are scheduled deterministically. This is referred to as time division multiple access (TDMA). But TDMA requires a centralized scheduler.
Part I

Single-Hop Analysis of Poisson Networks
Part I Overview

1. Interference and Outage in PPPs
2. Throughput
3. Other Applications
4. Summary
Section Outline

1. Interference and Outage in PPPs
   - Setup
   - Mapping of a PPP
   - Mean interference
   - Laplace transform
   - Outage for Rayleigh fading
   - Effect of individual interferers
   - Effect of path loss law

2. Throughput

3. Other Applications

4. Summary
Setup

PPP plus a desired transmitter

- PPP \( \{x, x\} \) of intensity \( \lambda \).
- ALOHA with probability \( p \).
  Active nodes (interferers) \( x \) form a PPP of intensity \( \lambda p \).
- \( o \): Receiver under consideration, assumed at origin.
- \( x \): Desired transmitter (not part of the point process), at distance \( R \).

Questions: interference at \( o \)?
Success prob. \( P(\text{SINR} > \theta) \)?
Intensity measure

For a point process $\Phi = \{x_1, x_2, \ldots\}$, the number of nodes in a subset $A \subset \mathbb{R}^2$ is

$$\Phi(A) \triangleq |\Phi \cap A| = \sum_{x \in \Phi} 1(x \in A).$$

$\Phi(A): \mathbb{R}^2 \to \mathbb{N}_0$ is a counting measure for the number of points in $A$. The intensity or mean measure is the expected number of nodes $\mathbb{E}\Phi(A)$:

$$\Lambda(A) \triangleq \mathbb{E}\Phi(A).$$

For a stationary PPP of intensity $\lambda$, $\Lambda(A) = \lambda|A|$. 

M. Haenggi (Univ. of Notre Dame)  Stoch. geometry and random graphs  05/27/2010  24 / 161
Non-homogeneous PPP

A PPP whose intensity depends on the location is a non-homogeneous or non-stationary PPP. In this case,

$$\Lambda(A) = \int_A \lambda(x)dx.$$ 

The number of points in $A$ is Poisson with mean $\Lambda(A)$.

Mapping

Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a mapping function. Then $\Phi^* = \{f(x_1), f(x_2), \ldots\}$ is a one-dimensional point process. By the mapping theorem [Kin93], $\Phi^*$ is a Poisson process with $\Lambda^*(B) = \Lambda(f^{-1}(B))$. 
The one-dimensional PPP of distances

Let \( \Phi \) be stationary with intensity \( \lambda \) and \( f(x) = \|x\| \). For \( B = [0, r] \), \( f^{-1}(B) = b(o, r) \), the ball of radius \( r \) at the origin. We obtain

\[
\Lambda^*(B) = \Lambda(b(o, r)) = \lambda \pi r^2
\]

and

\[
\lambda^*(r) = 2\lambda \pi r, \quad r \geq 0.
\]

So the distances of the points of a PPP that is homogeneous on the plane form a non-homogenous PPP on \( \mathbb{R}^+ \) with linearly increasing density. The squared distances \( \{\|x_1\|^2, \|x_2\|^2, \ldots\} \) form again a homogeneous PPP, with intensity \( \lambda^* = \lambda \pi \).

Analogous: If \( x \) is uniformly randomly distributed on the disk \( b(o, R) \), the radius \( \|x\| \) has probability density \( f_{\|x\|}(r) = 2r/R^2, \ 0 \leq r \leq R \).
Example (Mapping from $x \in \mathbb{R}^2$ to $\|x\| \in \mathbb{R}$)

PPP on $b(o, 3)$ with $\lambda = 1$. $\Lambda(b(o, 3)) = \pi 3^2 \approx 28.2$.

PPP on $[0, 3]$ with $\lambda^*(r) = 2\pi r$. $\Lambda^*([0, 3]) = \Lambda(b(o, r))$. 
Interference

Assume the transmitting nodes form a stationary PPP $\Phi$ of intensity $\lambda$ in $\mathbb{R}^2$. All nodes transmit at unit power, and the path loss $g(r) = r^{-\alpha}$.

Interference at origin:

$$I \triangleq \sum_{x \in \Phi} h_x \|x\|^{-\alpha}.$$ 

Due to the stationarity of the PPP, the distribution of $I$ is the same everywhere.

Equivalently,

$$I = \sum_{r \in \Phi^*} h_r r^{-\alpha},$$

where $\Phi^* = \{\|x_1\|, \|x_2\|, \ldots\}$ is the PPP of the distances.
Mean interference

Since $\mathbb{E} h = 1$,

$$\mathbb{E}(l) = \mathbb{E} \sum_{r \in \Phi^*} h_r r^{-\alpha} = \mathbb{E} \sum_{r \in \Phi^*} r^{-\alpha}.$$  

Campbell’s theorem for PPPs

Let $f$ be non-negative function. Then

$$\mathbb{E} \sum_{x \in \Phi} f(x) = \int_{\mathbb{R}^d} f(x) \Lambda(dx).$$

In our case,

$$\mathbb{E} \sum_{r \in \Phi^*} r^{-\alpha} = \int_{\mathbb{R}^+} r^{-\alpha} 2\pi \lambda r dr = \frac{2\pi \lambda}{2 - \alpha} r^{2-\alpha} \bigg|_{0}^{\infty}, \quad \alpha \neq 2.$$  

This diverges for all $\alpha$! So $\mathbb{E}(l) = \infty$. 
Mean interference

From previous slide: \[ E(I) = \left. \frac{2\pi \lambda}{2 - \alpha} r^{2-\alpha} \right|_0^\infty. \]

If \( \alpha < 2 \), the upper integration bound is the culprit. There is too much interference from all the far nodes.

If \( \alpha > 2 \), the lower integration bound is the culprit. The nodes near the origin make \( E(I) \) diverge, since \( r^{-\alpha} \) grows too quickly as \( r \downarrow 0 \) if \( \alpha > 2 \).

A bounded path loss model would solve the problem for \( \alpha > 2 \). More on that later.

Similarly, if it can be ensured that no node is close to the origin, \( E(I) \) remains finite for \( \alpha > 2 \). Replace the lower integration bound by \( \rho > 0 \) to obtain

\[ E(I) = \frac{2\pi \lambda}{\alpha - 2} \rho^{2-\alpha}. \]

This can be used to model CSMA.
Laplace transform

We would like to find the Laplace transform of $I$ to learn more about the interference.

\[
\mathcal{L}_I(s) = \mathbb{E}(e^{-sl}) = \mathbb{E}_{\Phi^*, h} \left( e^{-s \sum_{r \in \Phi^*} h r^{-\alpha}} \right) \\
= \mathbb{E}_{\Phi^*} \left( \prod_{r \in \Phi^*} \mathbb{E}_h (e^{-s h r^{-\alpha}}) \right).
\]

We need to calculate the expectation of a product over the point process. In the Poisson case, this is easy, thanks to the probability generating functional: For functions $v(x) \in [0, 1],

\[
\mathbb{E} \prod_{x \in \Phi^*} v(x) = G[v].
\]
Probability generating functional (pgfl)

The pgfl of a PPP \( \Phi \) with intensity measure \( \Lambda \) is

\[
G[v] = \mathbb{E} \prod_{x \in \Phi} v(x) = \exp \left( - \int_{\mathbb{R}^2} \left[ 1 - v(x) \right] \Lambda(dx) \right).
\]

Laplace transform

Applied to the interference:

\[
\mathcal{L}_I(s) = \mathcal{L}_I(s) = G[v] = \exp \left( - \int_{\mathbb{R}^+} \left[ 1 - \mathbb{E}_h(e^{-shr^{-\alpha}}) \right] \Lambda^*(dr) \right),
\]

where

\[
\Lambda^*(dr) = \lambda^*(r) dr = 2\pi \lambda r dr.
\]
From previous slide: \( \mathcal{L}_I(s) = \exp \left( -\pi \lambda \int_{\mathbb{R}^+} 2[1 - \mathbb{E}_h(e^{-shr^{-\alpha}})]rdr \right) \)

Swapping expectation and integral and conditioning on \( h \), the integral is:

\[
2 \int_0^\infty \left[ 1 - \exp(-shr^{-\alpha}) \right] rdr \overset{(a)}{=} \int_0^\infty \left[ 1 - \exp(-sh/y) \right] \delta y^{\delta-1} dy \\
\overset{(b)}{=} \int_0^\infty \left[ 1 - \exp(-shx) \right] \delta x^{-\delta-1} dx \\
\overset{(c)}{=} \int_0^\infty x^{-\delta} sh \exp(-shx) dx \\
= (sh)^\delta \Gamma(1 - \delta), \quad 0 < \delta < 1.
\]

(a): \( y \leftarrow r^{1/\alpha}, \delta = 2/\alpha. \)  \hspace{1cm} (b): \( x \leftarrow y^{-1}. \)  \hspace{1cm} (c): Integration by parts.
Laplace transform

Taking the expectation over $h$, we have

$$
\mathcal{L}_I(s) = \exp \left( - \lambda \pi \mathbb{E}(h^\delta) \Gamma(1 - \delta) s^\delta \right), \quad 0 < \delta < 1.
$$

- The interference has a stable distribution with characteristic exponent $\delta$ and dispersion $\lambda \pi \mathbb{E}(h^\delta) \Gamma(1 - \delta)$.
- If $\delta \uparrow 1$ (or $\alpha \downarrow 2$), we have $\mathcal{L}_I(s) \downarrow 0$, for all $s > 0$, so $I \uparrow \infty$ almost surely (a.s.). So we need $\alpha > 2$ for finite interference.
- $I$ does not have any finite moments.
- The interference (power) is very far from Gaussian. The amplitude may be, though.
- For ALOHA with probability $p$, replace $\lambda$ by $\lambda p$. 
Interference distribution

Closed-form expressions for the distribution only exist for $\alpha = 4$. In this case, without fading,

$$\mathcal{L}_I(s) = \exp \left( -\lambda \sqrt{s} \frac{\pi^2}{2} \right).$$

The corresponding distribution is the Lévy distribution

$$f_I(y) = \frac{\pi}{2} \lambda y^{-3/2} e^{-\pi^3 \lambda^2 / 4y}, \quad F_I(y) = 1 - \text{erf} \left( \frac{\pi^{3/2} \lambda}{2 \sqrt{y}} \right)$$

for $y \geq 0$.

It has a heavy tail, as expected from the fact that it does not have a mean.
Laplace transform for Rayleigh fading

Since $\mathbb{E}(h^\delta) = \Gamma(1 + \delta)$:

$$L_i(s) = \exp \left( -\lambda \pi \Gamma(1 + \delta)\Gamma(1 - \delta)s^\delta \right) = \exp \left( -\lambda \pi s^\delta \frac{\pi \delta}{\sin(\pi \delta)} \right).$$

Outage for Rayleigh fading and ALOHA

For a transmission over distance $R$, the received signal power is $S = hR^{-\alpha}$. The success probability is

$$p_s = \mathbb{P}(hR^{-\alpha} > I\theta) = \mathbb{E}(e^{-\theta R^\alpha I}) = \exp \left( -\lambda \pi \Gamma(1 + \delta)\Gamma(1 - \delta)\theta^\delta R^2 \right)$$

with $\delta = d/\alpha$. The Laplace transform gives us the success probability in Rayleigh fading! This is our key result!

So while we do not have a distribution of $I$ in general, we have a distribution of the SIR: $p_s = \mathbb{P}(\text{SIR} > \theta)$ is its ccdf.
Interference from the nearest node

\[ \mathbb{P}(I_1 \leq x) = \mathbb{P}(R^{-\alpha} \leq x) = \mathbb{P}(R > x^{-1/\alpha}) = \exp(-\lambda \pi x^{-\delta}) , \]

where \( \delta = 2/\alpha \). We get

\[ \mathbb{E}I_1 = \pi^{1/\delta} \Gamma(1 - 1/\delta) , \quad \delta > 1 . \]

If \( \delta < 1 \) then \( \mathbb{E}(I_1) \) does not exist. Generally, \( \mathbb{E}(I_1^p) \) exists for \( p < \delta \) since

\[ \mathbb{P}(I_1 > x) \sim \lambda \pi x^{-\delta} \quad x \to \infty . \]

Interference from \( n \)-nearest node

\[ \mathbb{P}(I_n > x) \sim \frac{1}{n!}(\lambda \pi)^n x^{-n\delta} . \]

This means that \( \mathbb{E}(I_n^p) \) exists for \( p < n\delta \). For example, we would need to cancel \( k > \alpha \) interferers to have a finite second moment.

On the other hand, canceling all nodes within distance \( \epsilon > 0 \) or using a bounded path loss law would ensure finite moments.
Effect of Path Loss Law

Interference with bounded path loss

Consider a path loss law \( g(r) = \min\{1, r^{-\alpha}\} \), and let the diameter of the network be \( D > 1 \). So we consider a PPP of intensity \( \lambda \) on \( b(o, D) \). In this case, from Campbell’s theorem,

\[
\mathbb{E}(I_D) = \int_{\mathbb{R}^+} g(r) 2\lambda \pi r dr = \int_0^1 2\lambda \pi r dr + \int_1^D r^{-\alpha} 2\lambda \pi r dr
\]

\[
= \lambda \left( \pi + \frac{2\pi}{\alpha - 2} (1 - D^{2-\alpha}) \right) .
\]

The Laplace transform can also be calculated; it involves incomplete gamma functions. Moments can be obtained by

\[
\mathbb{E}(I^m) = (-1)^m \left. \frac{d^m}{ds^m} \log(L_I(s)) \right|_{s=0} .
\]
Section Outline

1. Interference and Outage in PPPs

2. Throughput
   - Probabilistic throughput
   - Shannon throughput
   - Spatial Shannon throughput
   - Transmission capacity

3. Other Applications

4. Summary
Throughput

Unconditional success probability

Previously we assumed that the desired transmitter transmits and the receiver listens. The *unconditioned success probability* is

\[
\text{ALOHA: } p_T \triangleq p(1 - p) p_s(p).
\]

Optimum ALOHA transmit probability

Let \( p_s(p) = e^{-p\gamma} \). For the PPP with Rayleigh fading, e.g.,

\[
\gamma = \lambda \pi R^2 \theta^\delta \frac{\pi \delta}{\sin(\pi \delta)}.
\]

We find

\[
p_{\text{opt}}(\gamma) = \frac{1}{\gamma} - \frac{1}{2} \left( \sqrt{1 + \frac{4}{\gamma^2}} - 1 \right) \quad \text{(half-duplex penalty)}.
\]
Remark on optimum transmit probability $p$

Let

$$p_{s}^{\text{opt}}(\gamma) = p_{s}(p_{\text{opt}}(\gamma))$$

be the optimum transmit probability for a given $\gamma$.

The resulting $p_{s}^{\text{opt}}(\gamma)$ decays quickly with $\gamma$. For example, $p_{s}^{\text{opt}}(\gamma) < 1/2$ for

$$\gamma > \frac{\log 2(2 - \log 2)}{1 - \log 2} \approx 2.9.$$ 

Although throughput-optimum, such low success probabilities are not acceptable for many applications, and they waste energy. We will get back to this later.
Definition (Probabilistic throughput)

If the transmission rate is \( \log(1 + \theta) \) (nats/s/Hz), there is an outage if SIR < \( \theta \). So for fixed-rate transmission, it is natural to set the rate to \( \log(1 + \theta) \). The probabilistic throughput is

\[
T(\theta) \triangleq p_T(\theta) \log(1 + \theta)
\]

Optimum SIR threshold for full-duplex operation [Hae09]

The probabilistic throughput \( T^f = p \exp(-p\gamma) \log(1 + \theta) \) is maximized at the rate (spectral efficiency)

\[
R_T^{\text{opt}} = \log(1 + \theta) = \mathcal{W} \left( -\frac{1}{\delta} e^{-1/\delta} \right) + \frac{1}{\delta} \quad \text{(nats/s/Hz)},
\]

where \( \mathcal{W} \) is the principal branch of the Lambert W function, i.e.,

\[
\mathcal{W}(x)e^{\mathcal{W}(x)} = x.
\]

Tight bound: \( R_T^{\text{opt}}(\alpha) \lesssim \alpha - 2 \).
Optimum threshold and throughput for PPP ALOHA network with Rayleigh fading

- Transmissions at relatively low rate are optimum.
- The throughput increases linearly with $\alpha$.
- There is a fixed small penalty factor for half-duplex operation.
Shannon Throughput

**Shannon throughput**

If the transmitter has full knowledge of $S$ and $I$, it can adjust its rate of transmission accordingly. Alternatively, if there is enough time diversity in $S$ and $I$, e.g., through fast FH, the transmitter can signal at $E \log(1 + \text{SIR})$.

Either way, the resulting throughput is the **Shannon throughput**

$$E \log(1 + \text{SIR})$$

**Definition (Shannon throughput $C$)**

$$C \triangleq E \log(1 + \text{SIR}) = \int_{0}^{\infty} -\log(1 + \theta)dp_s(\theta),$$

where $p_s(\theta)$ is the ccdf of the SIR.
Alternate expression for Shannon throughput

Let $p_s(\theta)$ be the success probability as a function of the SINR threshold. The Shannon throughput can also be calculated as follows:

$$
\mathbb{E} \log(1 + \text{SIR}) = \int \mathbb{P}(\log(1 + \text{SIR}) > \theta) d\theta
$$

$$
= \int \mathbb{P}(\text{SIR} > e^\theta - 1) d\theta
$$

$$
= \int p_s(e^\theta - 1) d\theta.
$$

This is sometimes easier to evaluate or bound.
Example (PPP ALOHA network with Rayleigh fading)

For $\alpha = 4$ and $R = 1$:

$$C = 2\Re\{q\} \cos(p\lambda \pi^2/2) - 2\Im\{q\} \sin(p\lambda \pi^2/2), \quad q \triangleq \text{Ei}(1, jp\lambda \pi^2/2),$$

where $\text{Ei}(1, z) = \int_1^\infty \exp(-xz)x^{-1}dx$ is the exponential integral.

For general $\alpha$:

$$C > \int_1^\infty -\log(\theta)dp_s(\theta) = \frac{\alpha}{2} \text{Ei}(1, p\lambda C(\alpha)).$$

Spatial Shannon throughput

Since $C(p) \to \infty$ as $p \to 0$, the Shannon throughput itself does not give insight on how to choose $p$. Instead, as before, use the spatial Shannon throughput or throughput density:

$$p(1 - p)C(p)$$
Spatial Shannon throughput for PPP ALOHA network

The SIR-based capacity diverges as $p \to 0$, so a better metric is the spatial capacity $p(1 - p)C$ (half-duplex) and $pC$ (full-duplex).

- The optimum $p$ is independent of $\alpha$. For half-duplex operation, $p_{\text{opt}} \approx 1/9$.
- The maxima are about 2.5 times higher than for the probabilistic throughput. So rate adaptation can result in a significant gain.
Transmission Capacity

Definition (Transmission capacity)

The maximum density of successful transmissions subject to an outage constraint $\epsilon$, multiplied by $1 - \epsilon$ [WYAdV05]:

$$ \text{TC}(\epsilon) = (1 - \epsilon) \sup \{ \lambda : p_s(\lambda) > 1 - \epsilon \} . $$

Or, if $p_s$ is viewed as just a function of $\lambda$ and the inverse is $p_s^{-1}$,

$$ \text{TC}(\epsilon) = (1 - \epsilon)p_s^{-1}(1 - \epsilon) . $$

It is available in closed-form whenever $p_s$ is known and invertible.

Advantage over throughput as a metric

The transmission capacity metric imposes a maximum outage probability and is thus very useful in applications that cannot tolerate high packet loss rates.
Example (Transmission capacity for PPP with Rayleigh fading)

In this case, the success probability is invertible, and we have

\[ \text{TC}(\epsilon) = (1 - \epsilon) \frac{- \log(1 - \epsilon)}{R^2 \theta^2/\alpha C(\alpha)} = \frac{\epsilon}{R^2 \theta^2/\alpha C(\alpha)} + \Theta(\epsilon^2), \quad \epsilon \to 0. \]

This is \textit{linear} in \( \epsilon \) and inversely proportional to the area of the disk \( b(o, R) \).
### Transmission capacity in more general settings [WAJ10]

- Approximations for general fading are possible.
- If there is no fading and $\alpha = 4$, the TC follows from the Lévy distribution.
- Considering only the nearest interferer yields an upper bound on the TC. The bound is tight when $\alpha$ is not too close to 2.
- MIMO: With $N_t$ transmit and $N_r$ receive antennas used for diversity,

$$\Omega(\max\{N_t, N_r\}^\delta) = TC(\epsilon) = O((N_t N_r)^\delta), \quad N_t, N_r \to \infty.$$  

- SIC with $N_t = 1$ and $N_r > 1$: Canceling the strongest $N_r - 1$ interferers yields $TC(\epsilon) = \Theta(N_r^{1-\delta})$. This is much better for larger $\alpha$ — but requires knowledge of the interferers’ channels.
Section Outline

1. Interference and Outage in PPPs

2. Throughput

3. Other Applications
   - CSMA
   - Spread-spectrum techniques
   - Opportunistic ALOHA
   - Power control
   - Bandwidth partitioning

4. Summary
Other Applications

CSMA emulation using guard zone

With a path loss law $g(r) = r^{-\alpha}1(r > \rho)$, the success probability with CSMA can be calculated as a function of the guard zone radius $\rho$.

PPP with $\lambda = 1/10$ and $\alpha = 4$.

The exclusion radii are chosen such that 1, 2, and 3 interferers are muted on average.
Direct-sequence vs. frequency hopping spread spectrum [AWH07]

Consider spreading signals by a factor $M$, either using DS-SS or FH-SS. With DS-SS, the density of interferers stays the same, but the interference is reduced by a factor $M$.

$$p_{s}^{\text{DS}}(\theta, M) = \mathbb{E}(e^{-\theta I/M}) = p_{s}(\theta / M).$$

DH-SS: $\log p_{s}(\theta) \propto \theta^{\delta} \implies \frac{\log p_{s}(\theta / M)}{\log p_{s}(\theta)} = M^{-\delta}$.

With FH-SS, the density of interferers is reduced by a factor $M$:

$$\text{FH-SS: } \log p_{s}(\theta) \propto \lambda \implies \frac{\log p_{s}^{\text{FH}}(\theta)}{\log p_{s}(\theta)} = M^{-1}.$$

Since $\delta < 1$, the benefit of FH-SS is larger; the difference is more drastic for small $\delta$, i.e., for large $\alpha$. 
Opportunistic ALOHA

Exploiting CSI

If the transmitter knows the channel, it can transmit whenever the channel state is good instead of transmitting "blindly" with probability $p$ as in regular ALOHA.

For a threshold $\nu$, let each node $x$ transmit when $h_x > \nu$. For all other nodes, this is the same as ALOHA with $p = \mathbb{P}(h > \nu)$. We have

$$p_s(\nu) = \mathbb{P}(S > l\theta \mid h > \nu).$$

For Rayleigh fading, we needed $\mathbb{E}(h^{-\delta}) = \Gamma(1 - \delta)$. With conditioning,

$$\mathbb{E}(h^{-\delta} \mid h > \nu) = \Gamma(1 - \delta, \nu),$$

where $\Gamma(\cdot, \cdot)$ is the upper incomplete gamma function

$$\Gamma(a, z) = \int_z^\infty t^{a-1} \exp(-t) dt.$$
Other Applications  

Opportunistic ALOHA in Rayleigh fading

Upper bound [WAJ06]: 

\[ p_s < \exp \left( -p \lambda \pi \Gamma(1 + \delta) \Gamma(1 - \delta, \nu) \theta^\delta R^2 \right), \]

where \( p = \mathbb{P}(h > \nu) = \exp(-\nu). \)

Comparison of \( p_s(\theta) \) for \( \lambda = 1/2, R = 1, \theta = 5, \) and \( \alpha = 4 \) (\( \delta = 1/2 \)).
Power Control

Channel inversion without fading

Assume each transmitter talks to its nearest neighbor at distance $R$. Since $R$ is Rayleigh with mean $1/(2\sqrt{\lambda})$, the transmitter power is Weibull distributed:

$$\mathbb{P}(P \leq x) = 1 - \exp(-\lambda \pi x^\delta)$$

Power control at the transmitters acts like fading. Since $\mathbb{E}(P^\delta) = 1/(\pi \lambda)$,

$$\mathcal{L}_1(s) = \exp(-p \Gamma(1 - \delta)s^\delta).$$

This does not depend on the density of the network. Since the "fading" is not Rayleigh, the success probability cannot be derived from the Laplace transform.
Channel inversion with fading

Let each node have its destination at distance 1. If the fading is fully compensated, the received signal power $S = 1$. The interference is

$$I = \sum_{x \in \Phi} \frac{h_{xo}}{h_{xz}} \|x\|^{-\alpha},$$

where $h_{xo}$ is the fading from node $x$ to the receiver at the origin, and $h_{xz}$ is the fading from node $x$ to its own receiver. The success probability

$$p_s = \mathbb{P}(S > I \theta) = \mathbb{P}(I < \theta^{-1})$$

cannot be calculated in closed-form.

If all fading is Rayleigh, an immediate problem is that $\mathbb{E}(h^{-1}) = \infty$, i.e., finite power is not sufficient for full channel inversion.
Channel inversion with fading

For Rayleigh fading, $H = \frac{h_{xo}}{h_{xz}}$ is distributed as

$$F_H(x) = \mathbb{P}(H \leq x) = \frac{x}{x + 1}.$$ 

The relevant metric for the success probability is the $\delta$-th moment $H^\delta$. It turns out that full channel inversion decreases the success probability.

However, fractional channel inversion helps. Let the transmit power at each transmitter be $P_x = h_x^{-s}$ for $0 \leq s \leq 1$. $s = 0$ means no power control, while $s = 1$ means full channel inversion.

It is shown in [JWA08b] that $s = 1/2$ is optimal. This also solves the problem of infinite mean transmit power, since in this case $\mathbb{E}(P) = \sqrt{\pi}$.

This value of $s$ minimizes $\mathbb{E}(X^{-s})\mathbb{E}(X^{s-1})$ for all non-negative RVs.
Bandwidth Partitioning

Optimum number of subbands [JWA08a]

Given a total bandwidth $B$, what number of subbands $N$ should be chosen to maximize the number of concurrent links in the network?

Given a rate of transmission $R_T$, the corresponding SIR threshold is:

$$R_T = \frac{B}{N} \log(1 + \theta(N)) \implies \theta(N) = \exp\left(\frac{NR_T}{B}\right) - 1.$$  

Let $b = \frac{NR_T}{B}$ be the spectral efficiency. Using the transmission capacity framework, we can find the $b_{\text{opt}}$ that maximizes the total density of concurrent transmissions given an outage constraint:

$$\lambda(b, \epsilon) \propto \frac{b}{(e^b - 1)^\delta} \implies b_{\text{opt}} = \mathcal{W} \left( -\frac{1}{\delta} e^{-1/\delta} \right) + \frac{1}{\delta} \quad (\text{nats/s/Hz}),$$

which is exactly the spectral efficiency that maximized the probabilistic throughput on slide 42! Hence $N_{\text{opt}} \approx (\alpha - 2)B/R$. 

Section Outline

1. Interference and Outage in PPPs
2. Throughput
3. Other Applications
4. Summary
   - Interference and outage
   - Campbell’s theorem and the pgfl
Summary (Part I)

Interference and outage

- The boundedness of the path loss model has a drastic impact on the distribution of the interference.
- On the other hand, the success probability is not affected significantly by the path loss model. Since for large interference, there is an outage anyway, the heavy tail does not matter much.
- In fact, the SIR has a very benign distribution. For Rayleigh fading, it is just a Weibull distribution:

\[
P(SIR < x) = 1 - \exp(-cx^\delta).
\]

The mean is

\[
E(SIR) = \frac{1}{R^\alpha} \frac{\Gamma(1 + 1/\delta)}{(\lambda p C(\alpha))^{1/\delta}}, \quad C(\alpha) = \pi \Gamma(1 + \delta) \Gamma(1 - \delta).
\]
Outage

Since

\[ p_s = \exp(-\lambda \pi R^2 \theta^\delta C(\alpha)), \]

the success probability equals the void probability that there is no node in \( b(o, R') \) with

\[ R' = R \theta^{\delta/2} \sqrt{C(\alpha)}. \]

Generalization to \( d \) dimensions

Most results generalize to \( d \) dimensions in a very straightforward manner. There are only two changes needed in all results:

- Replace \( \pi \lambda \) by \( c_d \lambda \), where \( c_d \) is the volume of the \( d \)-dimensional unit sphere.
- Replace \( \delta = 2/\alpha \) by \( \delta = d/\alpha \).
**Campbell’s theorem**

Let $f$ be non-negative function. Then

$$\mathbb{E} \sum_{x \in \Phi} f(x) = \int_{\mathbb{R}^d} f(x) \Lambda(dx).$$

**Example (Number of points in $A$)**

Take a stationary PPP and let $f(x) = 1(x \in A)$. Then we know that

$$\mathbb{E} \sum_{x \in \Phi} f(x) = \mathbb{E} \Phi(A).$$

Campbell’s theorem gives

$$\mathbb{E} \sum_{x \in \Phi} f(x) = \lambda \int_A dx = \lambda |A|$$

as expected.
Probability generating functional (pgfl)

The pgfl of a PPP $\Phi$ with intensity measure $\Lambda$ is

$$G[\nu] = \mathbb{E} \prod_{x \in \Phi} \nu(x) = \exp \left( - \int_{\mathbb{R}^2} [1 - \nu(x)] \Lambda(dx) \right).$$

Example (Void probability of $A$)

Let $\nu(x) = 1 - 1(x \in A)$. Then $G[\nu] = 1$ only if all points in $\Phi$ are outside of $A$, i.e., $G[\nu]$ is the void probability. We have

$$G[\nu] = \mathbb{E} \prod_{x \in \Phi} \nu(x) = \exp(-\lambda \int_A dx) = \exp(-\lambda |A|),$$

as expected from the void probability in the Poisson process.
Part II

Random Geometric Graphs and Percolation
Part II Overview

5 Gilbert’s Disk Graph

6 Bond Percolation on Lattice

7 Percolation on the Disk Graph

8 Secrecy Graphs

9 Summary
Gilbert's Disk Graph

Section Outline

5 Gilbert’s Disk Graph
   • Definition
   • Connectivity

6 Bond Percolation on Lattice

7 Percolation on the Disk Graph

8 Secrecy Graphs

9 Summary
Gilbert’s Disk Graph [Gil61]

Definition (Gilbert’s disk graph)

Take a stationary PPP of intensity $\lambda$ as the vertices of a random geometric graph and connect two vertices by an edge if they are within distance $r$ of each other. The resulting graph $G_{\lambda,r}$ is called a disk graph.

Example: $\lambda = 1$, $r = 1$

Interpretation

In the absence of interference, the condition $\text{SNR} > \theta$ defines a maximum communication radius $r$

$$r = \left(\frac{P}{\theta W}\right)^{1/\alpha}.$$
Connectivity

Let \( \Phi \) be a PPP of intensity 1 on \([0, \sqrt{n}]^2\) so that \( \mathbb{E} N = n \).

What communication radius \( r_c \) guarantees that \( G_r(n) \) is connected whp as \( n \to \infty \)?

Formally, we want

\[
\lim_{n \to \infty} \mathbb{P}[G_{r_c}(n) \text{ connected}] = 1.
\]
Minimum transmission radius for connectivity

A necessary condition for connectivity is that no node is isolated. The expected number of isolated nodes

\[ \mathbb{E}N_{isol} = n \mathbb{P}(\text{typical node is isolated}) = n \exp(-\pi r^2) \]

needs to go to 0 as \( n \) grows. So we need \( \pi r^2 \gtrsim \log n \) for connectivity.

[Pen97] showed that indeed the isolated nodes determine the connectivity such that

\[ \pi r^2 = \log n + \omega(n), \]

for \( \omega(n) \to \infty \) (arbitrarily slowly) is enough.

Setting \( \pi r^2 = \log n + c \) would result in a \( N_{isol} \sim \text{Po}(\exp(-c)) \).

So \( r \) needs to grow with \( \sqrt{\log n} \) to keep the network connected!

But a constant power is enough to keep an infinite number of nodes connected. Percolation theory gives bounds on this critical threshold.
Section Outline

5 Gilbert’s Disk Graph

6 Bond Percolation on Lattice
   • Model
   • Critical probability

7 Percolation on the Disk Graph

8 Secrecy Graphs

9 Summary
Bond Percolation on Lattice

Lattice with open and closed bonds

Take the set of vertices to be the points in $\mathbb{Z}^2$, and put edges among all nearest neighbors. Make edges *open* (passable) with probability $p$ or *closed* (blocked) with probability $1 - p$.

Example (Bond percolation)

![Graph examples](image-url)

$p = 0.3$

$p = 0.6$
Critical probability

Let $u \leftrightarrow v$ stand for the existence of an (open) path between $u, v \in \mathbb{Z}^2$. The open cluster $C(v)$ is the set of all vertices that are connected to $v$ by an open path:

$$C(v) = \{ u \in \mathbb{Z}^2 : u \leftrightarrow v \}$$

The central quantity is the percolation probability

$$\psi(p) = \mathbb{P}(o \leftrightarrow \infty) = \mathbb{P}(|C(o)| = \infty).$$

The lattice model exhibits a phase transition, i.e., there exists a critical value $p_c$ such that $\psi = 0$ for $p < p_c$ and $\psi > 0$ for $p > p_c$.

The critical probability is defined as

$$p_c \triangleq \sup\{ p : \psi(p) = 0 \}.$$

By Kolmogorov’s 0-1 law, there exists an infinite component w.p. 1 as soon as $p > p_c$. 

\[ \text{psi(p)} \]

\[ p \]

\[ p_c \]

\[ 1 \]
Two larger examples

\begin{align*}
p &= 0.45 \\
p &= 0.55
\end{align*}

This indicates that the critical probability is near $1/2$. 
Lower bounding the critical probability

If there is an infinite cluster, then for any $n$, there exists a (self-avoiding) path of length $n$:

$$\psi(p) \leq \mathbb{P}(\exists \text{ a path of length } n \text{ starting at } o) \quad \forall n \in \mathbb{N}.$$ 

If all edges were open, the number of $\kappa(n)$ of paths of length $n$ is smaller than $4 \cdot 3^{n-1}$.

Each path exists with probability $p^n$, so by the union bound

$$\mathbb{P}(\exists \text{ a path of length } n \text{ starting at } o) \leq 4 \cdot 3^{n-1} p^n.$$ 

If $p < 1/3$, this goes to 0 as $n \to \infty$. So $p_c \geq 1/3$. 
Upper bounding the critical probability

Take the dual lattice whose vertices are at \((\mathbb{Z} + 1/2)^2\). Place an edge if it does not intersect an open edge in the original lattice. If a component is finite in the original lattice, it must be surrounded by a circuit in the dual lattice. This way it can be established that \(p_c \leq 1 - 1/(2\sqrt{3}) \approx 0.71\).

Critical probability

Harry Kesten showed in 1980 that the critical probability for bond percolation on the square lattice is \(p_c = 1/2\) — 25 years after this was conjectured.
Section Outline

5 Gilbert’s Disk Graph

6 Bond Percolation on Lattice

7 Percolation on the Disk Graph
   - Definition
   - Bounding the critical radius

8 Secrecy Graphs

9 Summary
Percolation on the Disk Graph

The critical radius

Take a PPP of intensity $\lambda$ and add a node at the origin $o$. Let

$$\psi(r) = P(o \leftrightarrow \infty) = P(|C(o)| = \infty).$$

and define

$$r_c \triangleq \sup\{r : \psi(r) = 0\}.$$

This indicates that the critical radius is near 1.2.
Lower bounding the critical radius

Populate the set $C(o)$ of nodes connected to the origin step by step. Start with $C = \{o\}$. Then at each step add all nodes that share an edge with an element of $C$.

Each node has on average $\lambda \pi r^2$ edges, so the process can be compared with a Galton-Watson branching process. If $\lambda \pi r^2 < 1$, the (independent) branching process dies out, so our process does too.

So $r_c > 1/\sqrt{\lambda \pi}$. For $\lambda = 1$, $r_c > 0.5642$.

Upper bounding the critical radius

For the upper bound, we use the result for bond percolation.

Divide the plane into squares of size $c = r/(2\sqrt{2})$.

Each square corresponds to a potential edge in the bond percolation model. The bond is open if there is at least one point of the PPP in the square, which happens with probability $p = 1 - \exp(\lambda c^2)$. 
Upper bounding the critical radius

If \( p = 1 - \exp(\lambda c^2) > 1/2 \), or \( \lambda > \log 2/c^2 \), the bond model percolates. If two edges are adjacent in the bond model, then two points of the PPP are located in squares that touch in at least a corner. Since the distance between them is at most \( 2\sqrt{2}c = r \), the two points are connected in \( G_{\lambda,r} \). So we have \( \lambda r_c^2 < 8 \log 2 \), or

\[
r_c < \sqrt{8 \log 2/\lambda} ; \quad \text{for } \lambda = 1 : \quad r_c < 2.355.
\]
The critical radius

We have shown (for $\lambda = 1$):

$$0.564 < r_c < 2.355.$$ 

The best known analytical bounds are $0.833 < r_c < 1.83$, so we’re not too far off.

In [BBW05], the bounds

$$1.1979 < r_c < 1.1988$$

were established with 99.99% confidence. This corresponds to a mean number of neighbors of $\lambda \pi r_c^2 \approx 4.51$ per node.

So, a positive fraction of nodes can be connected at constant power level.

At this level of connectivity, the fraction of isolated nodes is

$$\exp(-4.51) \approx 0.011.$$
Section Outline

5 Gilbert’s Disk Graph

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8 Secrecy Graphs
   - Definition
   - Power- and secrecy-limited regimes
   - Percolation in the Poisson model

9 Summary
Secrecy Graphs [Hae08]

The Poisson-Poisson secrecy graph

Let $\Phi$ be a PPP of users or "good guys" of intensity 1, and let $\Psi$ be a PPP of eavesdroppers of intensity $\lambda$. The secrecy graph $\vec{G}_{\lambda,r} = (\Phi, \vec{E})$ includes all directed edges for which $\vec{xy}$ if $\|x - y\| < r$ and $y$ is closer to $x$ than any eavesdropper.

This graph contains only edges along which secure communication is possible.

**good guys.** $\bullet$ are receivers only.
**eavesdroppers.** $\lambda = 0.3$.
$r = \infty$ (no power constraint, only secrecy constraints)
Secrecy Graphs

Power- and secrecy-limited regimes

\[ \mathbb{E} N(r) = \frac{1}{\lambda} (1 - \exp(-\lambda \pi r^2)) \, . \]

Mean out-degree of \( \bar{G}_{1/10, r} \).

The inflection point in \( \mathbb{E} N(r) \) marks the boundary between the **power-limited** and the **secrecy-limited** regime.

In the **power-limited** regime, the degree distribution is close to Poisson.

At the inflection point, \( r = r_T = (2\pi \lambda)^{-1/2} = \sqrt{5/\pi} \approx 1.26 \).
Percolation in the Poisson model

Critical radius and density

- With $\psi(\lambda, r)$ being the probability that the component containing the origin (or any arbitrary fixed node) is infinite, the percolation threshold radius for $\hat{G}_r$ is [BBW05]

$$r_G \triangleq \sup \{ r : \psi(0, r) = 0 \} \approx 1.19,$$

- For radii larger than $r_G$, we define

$$\lambda_c(r) \triangleq \inf \{ \lambda : \theta(\lambda, r) = 0 \}, \quad r > r_G.$$

This is the smallest density of eavesdroppers that ensures that the network is partitioned into many small components.
### Oriented Percolation

#### Fact (Out-percolation of $\vec{G}_{\lambda,r}$)

$\lambda_c(r)$ is monotonically increasing for $r > r_G$, and we have

$$0 < \lim_{r \to \infty} \lambda_c(r) < \infty.$$  

In other words, there exists a $\lambda_\infty$ such that for $\lambda > \lambda_\infty$, $\vec{G}_{\lambda,r}$ does not out-percolate for any $r$.

This follows from the fact that for fixed $r$ the mean degree $\mathbb{E}N(\lambda)$ is continuously decreasing to 0. For intensities smaller than $\lambda_\infty$, we define

$$r_c(\lambda) \triangleq \sup \{ r : \theta(\lambda, r) = 0 \}, \quad \lambda \leq \lambda_\infty.$$
Numerical investigation

\[ \lambda_c(r) \approx \lambda_\infty - \exp(a - br), \quad r > r_G \]

where: \( \lambda_\infty \approx 0.1499 \quad a = 2\sqrt{2}, \quad b = 4 \)
Two larger examples ($r = \infty$)

- $\lambda = 0.1$ (percolates)
- $\lambda = 0.2$ (does not percolate)
Section Outline

5 Gilbert’s Disk Graph
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8 Secrecy Graphs
9 Summary
Connectivity and percolation

- Full connectivity in Poisson networks requires increasing transmit power with growing network size. The main obstacle to connectivity are isolated nodes.

- Percolation can be achieved at finite power. This means that there exists an infinite component of connected nodes somewhere in the network. In the Poisson case, this component is unique.

- Critical radii or densities for percolation are unknown in most cases, but good bounds can usually be given.

- Percolation models have been extended to include interference [DFM⁺06] and, more recently, to secrecy.
Part III

Multi-hop Analysis of Poisson Networks
Part III Overview

10 Routing in Poisson Networks

11 Correlation and Local Delay in Poisson Networks

12 Information Propagation in Poisson Networks

13 Summary
Section Outline

10 Routing in Poisson Networks
   - Network model
   - Random access transport capacity
   - Delay analysis with queueing

11 Correlation and Local Delay in Poisson Networks

12 Information Propagation in Poisson Networks

13 Summary
Multihop Network Model

PPP network model of $M$-hop routes

- $\times$: PPP of intensity $\lambda$ of sources.
- $\circ$: Destinations, at fixed distance $R$ in random direction.
- $\oplus$: $M - 1 = 2$ relays per route placed equidistantly on the SD line.

Channel access: Only one node per route transmits. This means the set of transmitters form a PPP at all times.
Random access transport capacity [AWKH10]

Retransmission scheme and delay bound

- At each hop, the packet is transmitted until received successfully.
- The number of transmissions at hop $m$ is geometric and denoted by $T_m(M)$. The total number of transmissions is
  \[
  T(M) = \sum_{m=1}^{M} T_m(M).
  \]
  The maximum number of transmissions is restricted to $A$. Hence if $T(M) \leq A$, the packet is successfully delivered. If $T(M) > A$, there is an outage, and the actual number of transmissions is \(\min\{T(M), A\}\). Thus the effective rate per route is
  \[
  \log(1 + \theta) \mathbb{P}(T(M) \leq A) / \min\{T(M), A\}.
  \]
- $M$ can be chosen from $\{1, 2, \ldots, A\} = [A]$. $M = 1$ means $A$ single-hop transmissions, while $M = A$ means no retransmissions are possible.
Definition (Random access transport capacity)

\[ C(A) = \max_{M \in [A]} \lambda R \frac{\log(1 + \theta)}{\mathbb{E} \min\{T(M), A\}} \]

\[ = \lambda R \log(1 + \theta) \max_{M \in [A]} \frac{\mathbb{P}(T(M) \leq A)}{\mathbb{E} \min\{T(M), A\}} \]

Upper bound

Since

\[ \frac{\mathbb{P}(T(M) \leq A)}{\mathbb{E} \min\{T(M), A\}} \leq \frac{1}{\mathbb{E} T(M)} \]

and \( \mathbb{E} T(M) = M/p_s(M) \),

\[ C(A) < \lambda R \log(1 + \theta) \max_{M \in [A]} \frac{p_s(M)}{M} \cdot \]

Since \( p_s(M) = \exp(-c(R/M)^2) \) there is an optimum \( M \).
Optimum number of hops

\[ M_{\text{opt}} = \arg \max_{M \in [A]} \frac{p_s(M)}{M} \]

For \( \alpha = 4 \) (and without noise),

\[ M_{\text{opt}} = \min \{ A, R \sqrt{\lambda \theta^{1/4}} \pi \} . \]

The resulting upper bound on the random access transport capacity is

\[ C < \frac{\sqrt{\lambda} \log(1 + \theta)}{\pi \sqrt{e} \theta^{1/4}} . \]

Noise can be included in the analysis [AWKH10].
Delay Analysis with Queueing [SHR\textsuperscript{+}10]

The TDMA-ALOHA channel access scheme

The previous model did not include queueing delays, since there is no queueing. If nodes are not permitted to re-transmit in each slot until successful, queueing delays become important.

Use the same network model, but relax the assumption of equal hop length and change the channel access scheme to TDMA-ALOHA: In each route, a token is passed from node to node, and the node with the token is allowed to transmit with probability $p$.

But this node only transmits when it has a packet. So even if scheduled to transmit, the node may not contribute to the interference. So we do not make the "heavy-traffic assumption", where all nodes always have packets. Only the source is backlogged.
Success probabilities

Let $p_s(n)$ be the success probability at the $n$-th hop. Packet arrivals are geometric with parameter $pp_s(1)$ (source traffic intensity), provided that $p_s(1) < p_s(n), \forall n > 1$. Then the queue of the $(n - 1)$-th relay is empty with probability $p_s(1)/p_s(n)$.

It follows that the point process of interferers is a PPP with intensity

$$\lambda_I = \lambda p_I = \frac{\lambda p}{M} \sum_{n=1}^{M} \frac{p_s(1)}{p_s(n)},$$

where $p_I$ is the probability that a node is allowed to transmit and has a packet.

So $\lambda_I$ depends on $p_s(n), n \in [M]$. But

$$p_s(n) = \mathbb{P}(\text{SIR}_n > \theta) = \exp(-\lambda_I c r_n^2),$$

which introduces an intricate inter-dependence between $\lambda_I$ and $p_s(n)$!
End-to-end delay

We obtain

\[ \lambda_I = \frac{\lambda p}{M} \sum_{n=1}^{M} \exp(-\lambda_I c(r_1^2 - r_n^2)) , \]

which is a fixed point equation for \( \lambda_I \) given \( r_n, n \in [M] \).

The end-to-end delay is

\[ D = H_1 + \sum_{n=2}^{M} H_n + Q_n = \frac{M}{pp_s(1)} + M \sum_{n=2}^{M} \frac{1 - pp_s(n)}{pp_s(n) - pp_s(1)} , \]

where \( H_n \) is the service time and \( Q_n \) is the waiting time at node \( n \).

To minimize the delay, necessarily \( r_2 = r_3 = \ldots = r_M = (R - r_1)/(M - 1) \).

So the problem is to find \( M_{\text{opt}} \) and \( r_{1,\text{opt}} \), such that \( D(M, r_1) \) is minimized.

Direct analytical optimization is not possible, but numerical evaluation is straightforward.
**Numerical results** \( \alpha = 4, \lambda = 10^{-4}, p = 0.05 \)

![Graphs](image)

**End-to-end delay**

Each jump in the right plot corresponds to a crossover point in the left curve. These are the points when \( M + 1 \) hops are better than \( M \) hops.

These results provide optimum relay locations and thus give guidelines for routing. In practice, relays will have to be chosen from a point process, so there is some deviation between the optimum and actual relay location.
A critical assumption

In both models discussed, a critical assumption was made:

The transmission success events in subsequent time slots and across hops are independent.

Strictly speaking, this is never completely true. If there is significant mobility, this assumption is more likely to hold. But in static networks, there is correlation between outage events.

Such correlations are the topic of the next section.
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10 Routing in Poisson Networks

11 Correlation and Local Delay in Poisson Networks
   - Spatiotemporal correlation
   - Outage correlation
   - Local delay in Poisson networks

12 Information Propagation in Poisson Networks

13 Summary
Correlation and Local Delay

Intuition (PPP with ALOHA probability $p$)

Since the PPP is static (common randomness), there is temporal correlation of the interference at $o$ in different time slots.

There is also spatial correlation between the interference measured at nearby points $o$ and □.
Interference correlation: Setup

- A PPP $\Phi \subset \mathbb{R}^2$ with ALOHA with transmit prob. $p$ and iid fading.
- Let $I_k(u)$ be the interference measured at $u$ in time slot $k$.

The distribution of $I_k(u)$ is the same for all $k \in \mathbb{Z}$ and $u \in \mathbb{R}^d$, but the common randomness $\Phi$ introduces dependence.

Definition (The spatio-temporal correlation coefficient)

For path loss laws $g(x): \mathbb{R}^2 \rightarrow \mathbb{R}^+$ for which the interference has a finite second moment and $k \neq \ell$,

$$
\zeta(u, v) \triangleq \frac{\mathbb{E}[I_k(u)I_\ell(v)] - \mathbb{E}[I_k(u)]^2}{\mathbb{E}[I_k(u)^2] - \mathbb{E}[I_k(u)]^2}.
$$
Spatio-temporal correlation [GH09b]

Spatio-temporal correlation coefficient of $I_k(u)$ and $I_\ell(v)$, $k \neq \ell$:
\[
\zeta(u, v) = \frac{p \int_{\mathbb{R}^2} g(x)g(x - \|u - v\|)dx}{\mathbb{E}(h^2) \int_{\mathbb{R}^2} g^2(x)dx}.
\]

Temporal correlation

Setting $u = v$ yields the temporal correlation coefficient. For Nakagami-$m$ fading, it is simply
\[
\zeta_t = p \frac{m}{m + 1}.
\]

- The correlation is proportional to the transmit probability $p$.
- Fading helps decorrelate the interference. In Rayleigh fading, the correlation coefficient is $p/2$.
- Different MAC schemes and channels with memory exhibit stronger correlation, so this is a lower bound.
Impact of interference correlation on outage

\( g(x) = \|x\|^{-4}, \theta = 1. \) Follows from the joint success probability (pgfl)

\[
\mathbb{P}(A_u, A_v) = \exp \left( -\lambda \int_{\mathbb{R}^2} 1 - \left( \frac{p}{1 + \theta g(x)/g(z)} + 1 - p \right)^2 \, dx \right).
\]

This has an impact on retransmission schemes and end-to-end delays.
Local Delay

Basic question

How long does it take for a node in a (Rayleigh fading) Poisson network with ALOHA to successfully communicate with its nearest neighbor?

Mean delay for nearest-receiver transmission [Hae10]

High-mobility networks:

\[ D = \frac{1}{p} + \frac{\gamma}{\pi(1 - p)} \]

Static networks:

\[ D = \frac{1}{p} \frac{\pi}{\pi - \gamma p(1 - p)^{2/\alpha^2}} \]

\[ \gamma = \theta^2/\alpha^2 \pi^2 / (\alpha \sin(2\pi/\alpha)) \]

is the spatial contention parameter.

"High mobility" means that a new realization of the PPP is drawn in each time slot.
Calculation for high mobility

Transmission success events are independent. So

\[ D = (\mathbb{E}_R(p_s(R)))^{-1}, \]

where \( R \) is the distribution of the distance to the nearest receiver and \( p_s(R) \) is the success probability given \( R \).

Calculation for static network

Transmission success events are *conditionally independent* given \( \Phi \). Conditioned on the \( \Phi \), the number of transmissions until success is again geometric with parameter \( p_\Phi(R) = \mathcal{L}_I(\theta R^\alpha | \Phi) = \mathbb{E}(\exp(-\theta R^\alpha I | \Phi)) \).

\[ D(R) = \mathbb{E}_\Phi \left( \frac{1}{\mathcal{L}_I(\theta R^\alpha | \Phi)} \right) = \exp \left( \frac{p\lambda \gamma R^2}{(1 - p)^{1-2/\alpha}} \right), \]

The local delay is \( D = \mathbb{E}_R(D(R)) \).
Comparison ($\alpha = 4$)

NRT: Nearest-receiver transmission. NNT: Nearest-neighbor transmission.

- In the high-mobility case, the delay is insensitive to $p$.
- Static networks suffer from a significantly increased delay (due to correlation or lack of diversity). The min. delay is 4 times larger asymptotically.
Section Outline

10 Routing in Poisson Networks

11 Correlation and Local Delay in Poisson Networks

12 Information Propagation in Poisson Networks
   - The SIR multigraph
   - Propagation delay

13 Summary
The SIR multigraph

Let $\Phi$ be a PPP on $\mathbb{R}^2$, partitioned at each time $k \in \mathbb{N}$ into a transmitter process $\Phi_t(k)$ and a receiver process $\Phi_r(k)$ by a ALOHA. Let $1_k(x \rightarrow y) = 1$ if $x \in \Phi_t(k)$ and $y \in \Phi_r(k)$ and the following conditions hold:

- **Interference:** The disk $b(y, \beta \| x - y \|), \beta > 0$ is free from other transmitters.
- **Noise:** $\| x - y \| < \eta$.

Otherwise $1_k(x \rightarrow y) = 0$.

These two conditions approximate the condition $\text{SINR} > \theta$. They are known as the *protocol model* for communication [GK00].
Definition (The SIR multigraph)

The connectivity at time $k$ is captured by the weighted and directed random geometric graph $G(k) = (\Phi, \vec{E}_k)$ with

$$\vec{E}_k = \{(x, y): 1_k(x \to y) = 1\}.$$ 

A weight $k$ is attached to all these directed edges.

The **SINR multigraph** $\mathcal{G}$ is the edge-union of these snapshot graphs:

$$\mathcal{G}(0, n) = \left(\Phi, \bigcup_{k=0}^{n} \vec{E}_k\right)$$

$\mathcal{G}(0, n)$ captures all information about the network from time 0 to time $n$.

Definition (Causal path)

A causal path is a directed path $\{x_0, \vec{e}_0, x_1, \vec{e}_1, \ldots, \vec{e}_{q-1}, x_q\}$ with strictly increasing edge weights.
Propagation Delay [GH10]

Single-hop delay

Add a node at the origin, and let

\[ T_O = \min\{k : \sum_{x \in \Phi} 1_k(o \rightarrow x) > 0\} \]

be the number of time slots to connect to any node.

If \( \eta < \infty \), \( \mathbb{E}(T_O) = \infty \), since the origin is too far from any node with probability \( \exp(-\lambda \pi \eta^2) \).

So focus on the interference-limited regime where \( \eta = \infty \). In this case, \( \mathbb{E}(T_O) \) is finite and can be lower bounded.
Path formation time

The path formation time from $x$ to $y$ is

$$T(x, y) = \min\{k : G(0, k) \text{ has a path from } x \text{ to } y\}.$$ 

Similarly, define

$$T_n(x, y) = \min_{k>n}\{k - n : G(n, k) \text{ has a path from } x \text{ to } y\}.$$ 

Since $T$ is sub-additive, for $0 < p < 1$, i.e.,

$$T(o, y) \leq T(o, x) + T_{T(o,x)}(x, y),$$

the propagation time constant

$$\mu = \lim_{x \to \infty} \frac{\mathbb{E} T(o, x)}{x}$$

is finite. It is infinite with noise, but if the disk graph $G_{\lambda, \eta}$ percolates and we only consider nodes in the infinite component, again $\mu$ is finite.
Numerical results

Mean delay from $o$ to $x$

Depending on the distance, a different $p$ is optimum. The mean hop length of fast routes increases with $\|x\|$.

This framework allows reverse engineering to find good routing protocols.
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10 Routing in Poisson Networks

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12 Information Propagation in Poisson Networks

13 Summary
Summary (Part III)

Multi-hop analysis

- Multihop extensions and end-to-end analyses are possible in the Poisson case. They are based on the single-hop success probabilities.

- A common assumption is that transmission success events are spatially and temporally independent.

- A calculation of the correlation coefficient shows that this assumption is reasonable for small transmit probabilities and with fading. For larger $p$, there is enough dependence in static networks that the local delay becomes infinite.

- Another common assumption is that nodes are always backlogged. A more careful analysis considers queues and the fact that nodes do not transmit if they do not have packets.

- The SINR multigraph captures the dynamic connectivity of the network. Using methods from first-passage percolation, the propagation speed of a prioritized packet can be bounded.
Part IV

Single-hop Analysis of General Networks
Part IV Overview

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15 Palm Theory

16 Interference in Poisson Cluster Processes

17 Outage Probability in General Networks

18 Summary
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14 General Point Processes
   - Attraction and repulsion
   - Examples
   - Motion-invariant point processes

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18 Summary
General Point Processes

Simon Denis Poisson, 1781-1840.

Is the analytical treatment of wireless networks restricted to his model?
Motivation

The transmitter process is only a PPP if the process of all nodes is PPP and ALOHA is used as the MAC protocol. In all other cases, the Poisson model is at best an approximation.

Two typical cases:

- Nodes form a PPP, but a CSMA-type MAC is used. This leads to repulsion among the transmitters, i.e., a more regular process.
- Nodes form a cluster process. This usually leads to attraction between the transmitters as well, i.e., a more clustered process.

From regular to clustered processes

- Lattice:
  - Zero interaction;
  - Complete spatial randomness
- Hardcore PPs
- PPP
- Clustered PPs

- Repulsion
- Attraction
Examples

Example (Matern Hard-Core Process of Type I)

Take a homogeneous PPP of intensity $\lambda_p$ and remove all points that are within distance $r$ of each other. The resulting process has intensity $\lambda = \lambda_p \exp(-\lambda_p \pi r^2)$.

Remarks:
- Imposes a minimum distance $r$.
- Process is stationary.
- Obtained by dependent thinning.
- Repulsion or inhibition.
- Possible model for CSMA.

$\lambda_p = 1, \ r = 1$. Red points are retained.
Example (Neyman-Scott Cluster Processes)

Poisson cluster processes resulting from homogeneous independent clustering applied to a PPP.

- Parent points \( \Phi_p = \{x_1, x_2, \ldots \} \) form a PPP of intensity \( \lambda_p \).
- Clusters \( N^x \) are of the form \( N^{x_i} = N_i + x_i \) for each \( x_i \in \Phi_p \).
- The \( N_i \) are a family of iid finite point sets with distribution \( F(x) \) independent of the parent process. The complete process is given by

\[
\Phi = \bigcup_{x \in \Phi_p} N^x.
\]

- The intensity of the cluster process is \( \lambda = \lambda_p \bar{c} \), where \( \bar{c} \) is the average number of points per cluster.
- If the number of points per cluster is Poisson, the process is called a *Poisson cluster process*. 
Example (Special Neyman-Scott processes: Matern and Thomas cluster processes)

**Matern cluster process** (parameters $\lambda_p$, $\bar{c}$, and $a$):
Daughter points are iid *uniformly* distributed in a ball of radius $a$ around the parent.

**Thomas cluster process** (parameters $\lambda_p$, $\bar{c}$, and $\sigma$):
Daughter points are iid *symmetrically normally* distributed with variance $\sigma^2$ around the parent, *i.e.*, each child cluster forms an inhomogeneous PPP with intensity

$$
\lambda(x) = \frac{\bar{c}}{2\pi\sigma^2} \exp\left(-\|x\|^2/2\sigma^2\right), \quad (2\text{-dim.})
$$

so that the mean number of children per parent is $\bar{c}$.
Useful to model tactical networks (soldiers, troops, platoons, ...), human cocktail parties, networks with closely cooperating nodes (virtual MIMO).
Comparison of Thomas cluster process and PPP on $[-5, 5]^2$:

Thomas process

PPP

$\lambda_p = 1$, $\bar{c} = 5$, and $\sigma = 0.2$

$\lambda = 5$. 
Comparison of Thomas cluster process and PPP on $[-5, 5]^2$:

Almost a Thomas process

$\lambda, \bar{c}, \sigma = ?$

PPP

$\lambda = 5.$
Example (Poisson hole process)

Take a homogeneous PPP $\Psi$ of intensity $\lambda_p$ and a second PPP $\Phi$ of intensity 1. Remove all points in $\Phi$ that are within distance $r$ of any point in $\Psi$. The resulting Poisson hole process has intensity $\lambda = \exp(-\lambda_p \pi r^2)$.

- Process is stationary.
- Obtained by dependent thinning.
- Possible model for cognitive networks. $\Psi$ are the primary and $\Phi$ the secondary users. The secondary users who are allowed to transmit form the hole process.

$\lambda_p = 0.2$, $r = 1$. Blue points $\circ$ form the hole process.
Motion-invariant Point Processes

Some definitions

- **Stationarity**: \( \{x_i\} \) and \( \{x_i + x\} \) have the same distribution \( \forall x \in \mathbb{R}^d \).
- **Isotropy**: The same holds for all rotations about the origin.
- **Motion-invariance**: Stationarity plus isotropy. The stationary (homogeneous) PPP is motion-invariant.
- **Void probabilities**: The probabilities \( \mathbb{P}(\Phi(B) = 0) \) tell us everything about a point process.

All point processes we consider are motion-invariant.

Palm theory

The analysis of all non-Poisson point processes requires proper conditioning on the process having a node somewhere, typically at the origin. Such events have probability 0, so some care is required. This is the topic of Palm theory.
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15 Palm Theory
   - Motivation
   - Palm distribution
   - Second-order statistics

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18 Summary
Palm Theory

Motivation

Take a motion-invariant point process $\Phi_t$ of transmitters. Assume we are interested in interference. Where do we measure? We could take an arbitrary point $z \in \mathbb{R}^2$:

$$I(z) = \sum_{x \in \Phi_t} h_x g(\|x - z\|).$$

Since $\Phi_t$ is stationary, the distribution of $I(z)$ does not depend on $z$. And $\mathbb{E}(I(z))$ is the same for all point processes with the same intensity (Campbell’s theorem).

However, we are not interested in an arbitrary location, but either

- in a point of a process $\Phi \supset \Phi_t$, or
- a point near a transmitter, which would be the desired transmitter and thus does not contribute to the interference.
Motivation: Matern hard core process

The mean interference measured at a point $x$ is smaller than at an arbitrary point since there are no nodes nearby.

But measuring at $x \in \Phi$ means conditioning that $\Phi$ has a point at $x$.

Since $\Phi$ is stationary, we can always condition on $o \in \Phi$. 
Motivation: Cluster process

Conditioned on $o \in \Phi$, this means that there is a cluster near the origin.

Measuring $I(z)$ for small $\|z\|$ means that the interference is likely to be high, as there are more transmitters close.

The node at the origin could be the desired transmitter, so it does not contribute to the interference.

Conditioning on a node at $o$ but disregarding its impact yields the reduced Palm distribution.
Palm Distribution

Definition

Let \( Y \) be a set of point processes with a certain property, such as having no point in \( b(o, r) \). Saying that \( \Phi \in Y \) means that \( \Phi \) has this property. Formally, \( Y \) is an element of the \( \sigma \)-algebra of point processes.

The Palm distribution \( \mathbb{P}_o \) is defined as

\[
\mathbb{P}_o(\Phi \in Y) \triangleq \mathbb{P}(\Phi \in Y \mid o),
\]

and the reduced Palm distribution \( \mathbb{P}^I_o \) is

\[
\mathbb{P}^I_o(\Phi \in Y) \triangleq \mathbb{P}(\Phi \setminus \{o\} \in Y \mid o).
\]

The corresponding expectations are \( \mathbb{E}_o \) and \( \mathbb{E}^I_o \).

Slivnyak’s theorem

For a PPP: \( \mathbb{P}^I_o \equiv \mathbb{P} \Rightarrow \mathbb{E}^I_o \equiv \mathbb{E} \)
Second-order Statistics

Reduced second moment measure

The first-order statistic of a stationary point process is its intensity $\lambda$. The second moment measure plays a role similar to the variance. The reduced second moment measure $\lambda \mathcal{K}_2(B)$ is the mean number of points in $B \setminus \{o\}$ given that $o \in \Phi$: $\lambda \mathcal{K}_2(B) = \mathbb{E}_{o}^{\setminus} \Phi(B)$. There is a corresponding density, the second-order product density $\varrho^{(2)}$:

$$
\lambda \mathcal{K}_2(B) = \frac{1}{\lambda} \int_{B} \varrho^{(2)}(x)dx
$$

$\varrho^{(2)}$ measures the probability that there are two points separated by $x$; it is the density pertaining to the second-order factorial moment measure:

$$
\alpha^{(2)}(A \times B) = \mathbb{E} \left( \sum_{x \neq y, \in \Phi} 1_{A}(x)1_{B}(y) \right) = \int_{A} \int_{B} \varrho^{(2)}(x - y)dydx
$$
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Interference in Poisson Cluster Processes

Conditional mean interference

For a path loss law \( g(x) : \mathbb{R}^2 \to \mathbb{R}^+ \) and a stationary PP \( \Phi \),

\[
\mathbb{E}_{\circ}^o I(z) = \mathbb{E}_{\circ}^o \left[ \sum_{x \in \Phi} h_x g(x - z) \right] = \mathbb{E}[h] \lambda \int_{\mathbb{R}^2} g(x - z) \mathcal{K}_2(dx)
\]

where \( \mathcal{K}_2 \) is the \textit{reduced second moment measure}.

Mean interference for Thomas cluster process [GH09a]

\[
\mathbb{E}_{\circ}^o I(z) = \mathbb{E}(I_{\text{PPP}}) + \frac{\bar{c}}{4\pi \sigma^2} \int_{\mathbb{R}^2} g(x - z) \exp \left( -\frac{||x||^2}{4\sigma^2} \right) dx
\]

Note: \( g(x) \) must be bounded as \( ||x|| \to 0 \) for this to be finite.
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17 Outage Probability in General Networks
   - Problem formulation
   - Outage in the high-SIR regime

18 Summary
Outage Probability in General Networks

Setup (1)

Start with a motion-invariant point process of density \( \lambda \).
The MAC scheme selects a subset of nodes as transmitters $\Phi_\eta$, for $0 \leq \eta \leq 1$ s.t. the density of the transmitter point process is $\lambda_t = \eta \lambda$. 
Let one **transmitter** be the **receiver** under consideration.
Add a virtual transmitter at unit distance, with unit transmit power.

⇒ What is the outage probability from $T$ to $R$ as $\eta \to 0$?
Problem formulation

- Take a general motion-invariant PP of intensity $\lambda$ and a MAC scheme that can tune the intensity of transmitters $\lambda_t$ from 0 to $\lambda$.
- Let $\eta \triangleq \lambda_t / \lambda$. What is $p_s(\eta) = \mathbb{P}(\text{SIR} > \theta)$ for Rayleigh fading as $\eta \to 0$ (high-SIR asymptotics)?

Questions

Is the outage probability near $\eta = 0$ convex or concave?
Can the network accommodate some spatial reuse without affecting the outage probability?
Outage in the High-SIR Regime

Result [GGH10]

For all *reasonable* MAC schemes, ∃ unique parameters γ > 0 and 1 ≤ κ ≤ α/2 s.t.

\[ p_s(\eta) \sim 1 - \gamma \eta^\kappa, \quad \eta \to 0. \]

Moreover, \( p_s(\eta) \geq 1 - \gamma \eta^\kappa \).

A MAC scheme is reasonable iff \( \lim_{\eta \to 0} p_s(\eta) = 1 \).

\( \gamma(\alpha, \theta) \) is the *spatial contention* parameter that captures the spatial reuse capability of a network. The smaller the better.

\( \kappa(\alpha) \) is the *interference scaling parameter* and measures the *coordination level* of the MAC. The larger the better.
Result (from previous slide)

\[ p_s(\eta) \sim 1 - \gamma \eta^\kappa \quad (\eta \to 0) \]

Discussion

- For all networks that use ALOHA, \( \kappa = 1 \). We know the result for the PPP:
  \[ p_s = \exp(-\eta \gamma) \implies \kappa = 1. \]
- For lattices with TDMA, \( \kappa = \alpha/2 \).
- CSMA with sensing range \( \Theta(\eta^{-1/2}) \) also achieves \( \kappa = \alpha/2 \) (hard-core process).
- Conjecture: For most types of fading, for all \( 0 \leq \eta \leq 1 \),
  \[ 1 - \gamma \eta^\kappa \leq p_s(\eta) \leq \frac{1}{1 + \gamma \eta^\kappa}. \]
Reasonable and unreasonable ALOHA for clustered point process

Reasonable: 20% of nodes transmit.

Unreasonable: 20% of clusters transmit.
Starting with a PPP of intensity $\lambda = 0.3$, the hard-core distance is adjusted to get $\lambda_t = 0.3\eta$. $\gamma \approx 1.95$ can be analytically determined, and $\kappa = \alpha/2 = 2$. The asymptotic expression provides a good bound for $\eta \in [0, 0.3]$. 
Reasonable TDMA on square lattice

- For $\eta = 1/9$
  - Minimum distance increases with $\eta^{-1/2}$.

- For $\eta = 1/16$
Unreasonable TDMA on square lattice

Minimum distance does not increase with decreasing $\eta$. \(\lim_{\eta \to 0} p_s(\eta) < 1.\) Actually, \(p_s(\eta)\) decreases with decreasing $\eta$. 

$\eta = 1/9$

$\eta = 1/16$. 
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Summary (Part IV)

Analysis of general point processes

- The analysis of non-Poisson point processes is difficult due to the dependence among the node locations.
- The analysis requires the use of Palm theory and higher-order statistics such as the reduced second moments measures and second-order product densities.
- By considering the right asymptotic regimes, sharp statements are still possible. In particular, the success probability is in great generality

\[ p_s(\eta) \sim 1 - \gamma \eta^\kappa, \quad \eta \to 0, \]

for a spatial contention parameter \( \gamma \) and an interference scaling exponent \( \kappa \).
Space is the critical resource; the network geometry greatly affects the performance of ad hoc networks.

The uniform PPP is great to work with, but it is time to consider other, often more realistic node distributions.

Stochastic geometry, in particular Palm theory, offers the tools to analyze more general networks.

Interference correlation affects efficiency of ARQ and routing but is a greatly under-investigated topic.

The theory is applicable to wireless networks with infrastructure, such as multihop extensions of cellular systems.
The road to the analysis of general networks

Parts I,II:

PPP single-hop; graphs

Part III:

PPP multi-hop/e2e

Part IV:

general PP single-hop

general PP multi-hop

(no results yet...)

From green to red: Increasing dependence in time and space.
Outlook

Ongoing work

- Analysis of end-to-end throughput and delay (including queueing delays)
- From analysis to synthesis: Routing, MAC, power control?
- MIMO networks: How to use antennas?
- Cooperative communications in large networks: Interference cancellation, information-theoretic relaying, broadcasting and multiple-access
- Cognitive radio networks
- Femtocells
- Inclusion of secrecy constraints (secrecy graph)
- Mobility: Temporal coherence of the point process
Books and Tutorial and Survey Articles I

### Stochastic geometry


### Random Graphs and Percolation

Applications to wireless networks


See also http://users.ece.utexas.edu/~jandrews/stochgeom/.
Ad Hoc Networks: To Spread or not to Spread?
Available at http://www.nd.edu/~mhaenggi/pubs/commag07.pdf.

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Continuum percolation with steps in the square or the disc.

Percolation in the Signal-to-Interference Ratio Graph.

Outage Probability of General Ad Hoc Networks in the High-Reliability Regime.


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Poisson Processes.
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Throughput and Transmission Capacity of Ad Hoc Networks with Channel State Information.

An Overview of the Transmission Capacity of Wireless Networks.
*IEEE Transactions on Communications*, 2010.

Transmission Capacity of Wireless Ad Hoc Networks with Outage Constraints.
# Symbols

## List of Symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[n]$</td>
<td>The set ${1, 2, \ldots, n}$</td>
</tr>
<tr>
<td>$\Phi$</td>
<td>Point process (and counting measure)</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>Intensity of (stationary) point process</td>
</tr>
<tr>
<td>$o$</td>
<td>Origin in $\mathbb{R}^2$</td>
</tr>
<tr>
<td>$\mathcal{L}(s)$</td>
<td>Laplace transform</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Path loss exponent</td>
</tr>
<tr>
<td>$g(r)$ or $g(x)$</td>
<td>Path loss law; typically $g(r) = r^{-\alpha}$</td>
</tr>
<tr>
<td>$P$</td>
<td>Transmit power</td>
</tr>
<tr>
<td>$S$</td>
<td>Received power</td>
</tr>
<tr>
<td>$I$</td>
<td>Interference power</td>
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<tr>
<td>$W$</td>
<td>Noise power</td>
</tr>
<tr>
<td>$h$</td>
<td>Fading coefficient</td>
</tr>
<tr>
<td>$R$</td>
<td>Transmission distance</td>
</tr>
<tr>
<td>$p$</td>
<td>Transmit probability in slotted ALOHA</td>
</tr>
<tr>
<td>$\theta$</td>
<td>SINR threshold for successful reception</td>
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<tr>
<td>$p_s$</td>
<td>Success probability of a transmission</td>
</tr>
<tr>
<td>$R_T$</td>
<td>Transmission rate (spectral efficiency)</td>
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