ICFD 25
15-16 September 2008

Viscous Fingering: Theory, Simulation & Modelling

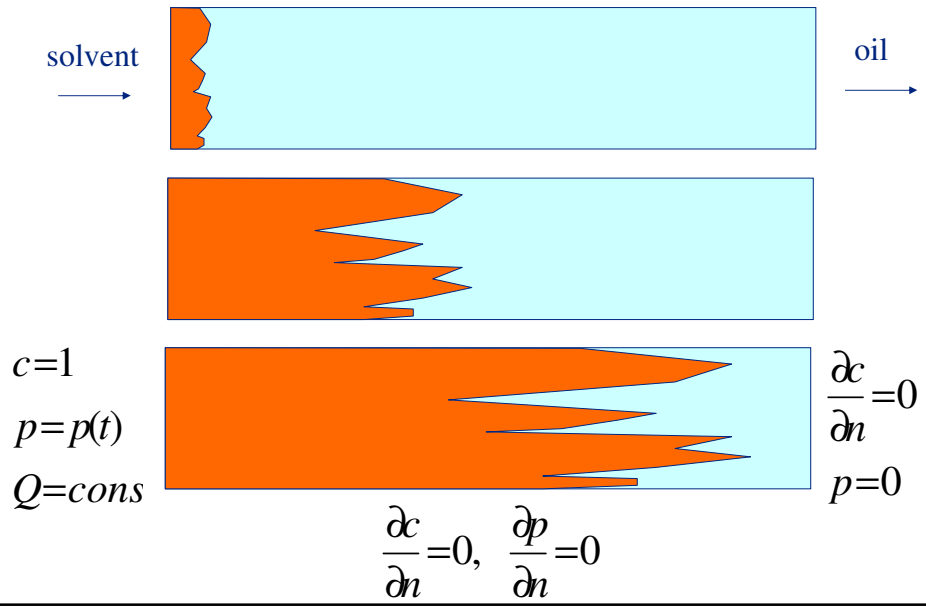
Chris L. Farmer
Schlumberger, Abingdon Technology Center
and
OCIAM, Mathematical Institute
University of Oxford

Schlumberger

© 2008 Schlumberger. All rights reserved.

Schlumberger

Viscous fingering



Contents

- **The Peaceman Theory**
- **The moving point method**
- **A *very* simple theory**
- **Homogenisation**

Schlumberger

Peaceman's miscible displacement equations

$$\frac{\varphi \partial c}{\partial t} + \nabla.(\mathbf{u}c) = \nabla.(D\nabla c), \quad 0 \le c \le 1$$

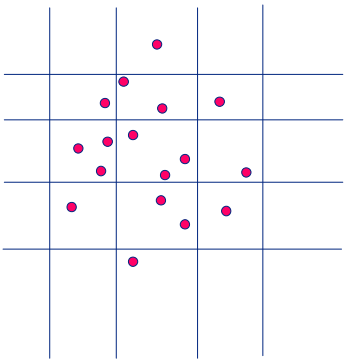
$$\nabla.\mathbf{u} = 0$$

$$\mathbf{u} = -\frac{\mathbf{k}(\mathbf{x}) \bullet \nabla p}{\mu(c)}$$

$$\mu(c) = \mu_o(1-c) + \mu_s c, \qquad \mu_s << \mu_o$$

The Moving Point Method

- Garder, Peaceman and Pozzi, SPE, 1964 (GPP)

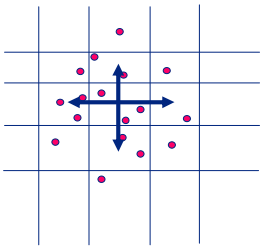


$$\nabla.\mathbf{u} = 0$$

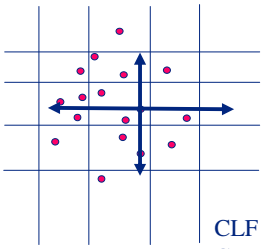
$$\mathbf{u} = -\frac{\mathbf{k}(\mathbf{x}) \bullet \nabla p}{\mu(c)}$$

$$\frac{d\mathbf{x}_p}{dt} = \frac{\mathbf{u}(\mathbf{x}_p, t)}{\varphi(\mathbf{x}_p)}, \qquad \frac{dC_p}{dt} = \nabla.(D\nabla c)|_{x=x_p}, \qquad c_i^n = \sum_{\substack{\text{points} \\ \text{in the} \\ \text{cell}}} C_p$$

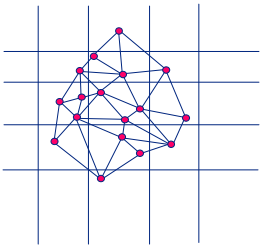
Diffusion & Convergence



GPP 64
Central Differences
on cells



CLF 85
Central Differences
on moving points



Rees & Morton 91
Diffusion on triangles

Exact solutions and ill-posedness

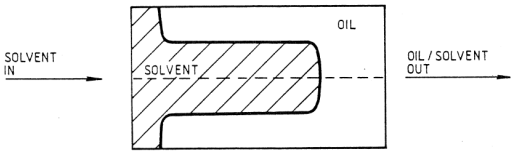


FIG. 4 SCHEMATIC OF THE JACQUARD AND SEGUIER EXACT RESULT FOR THE GROWTH OF A SINGLE VISCOUS FINGER.

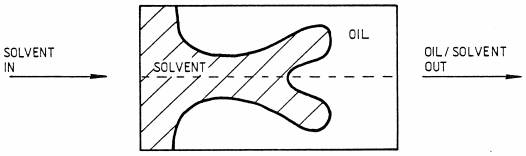
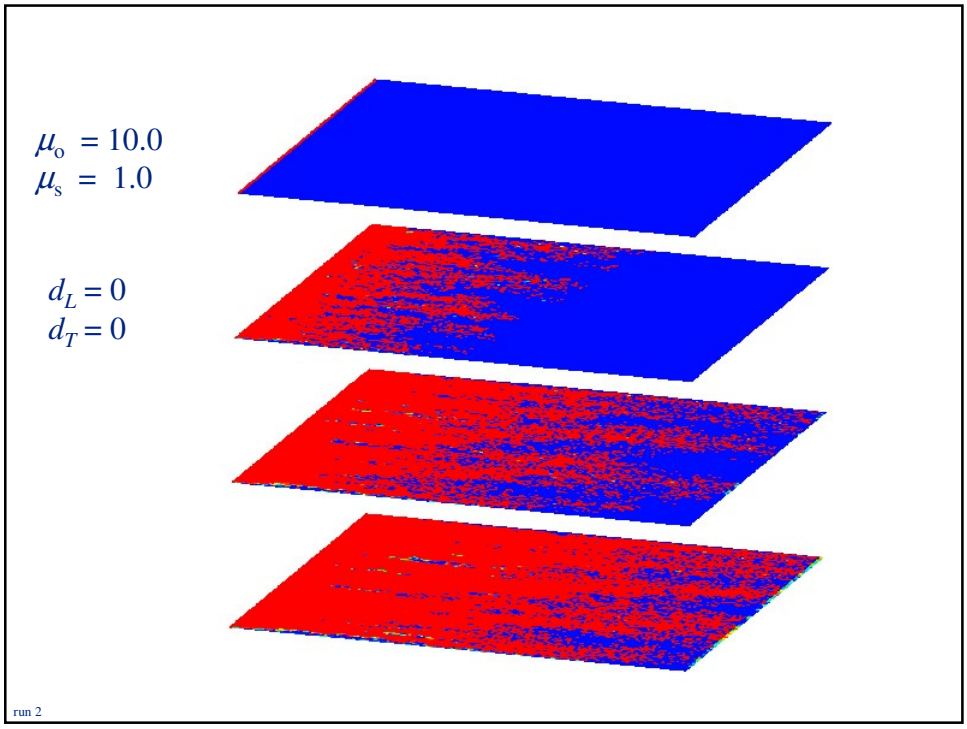
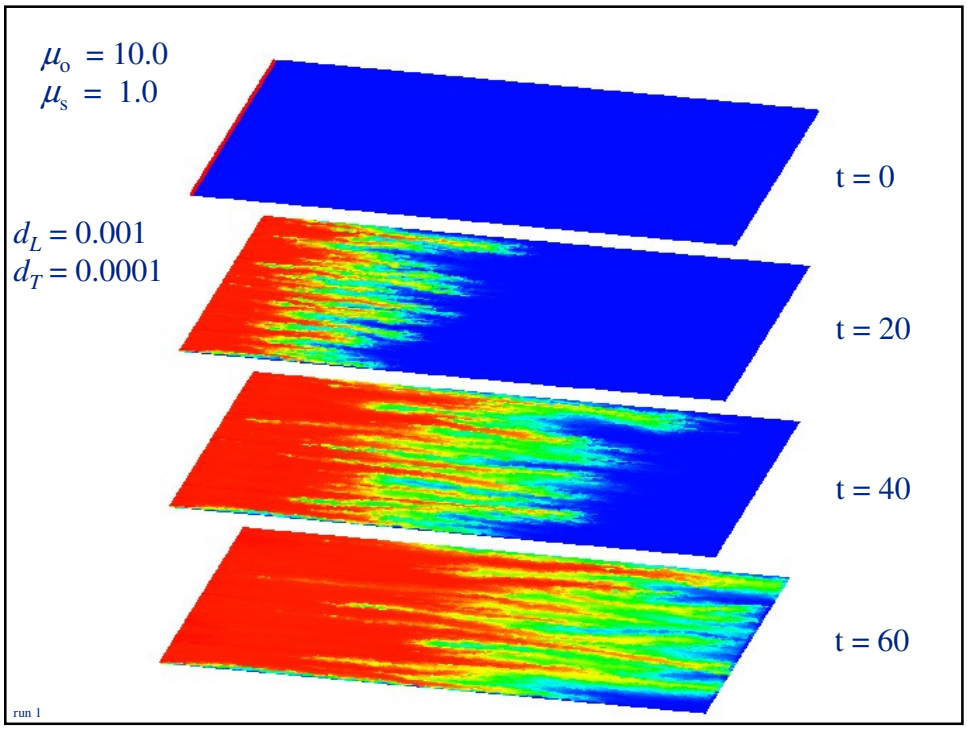
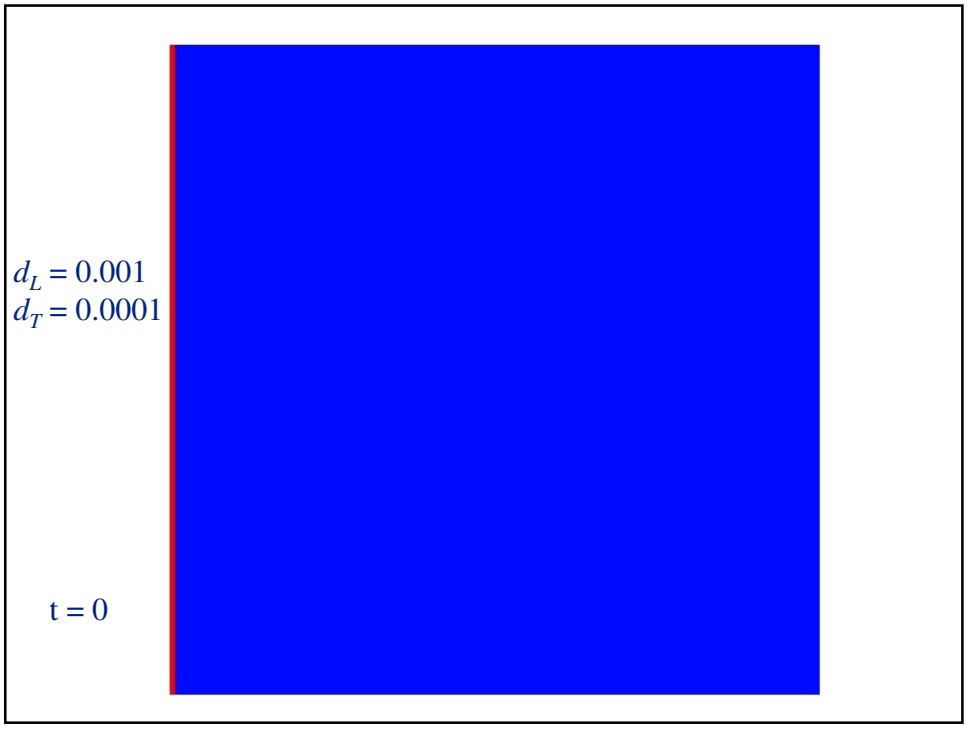
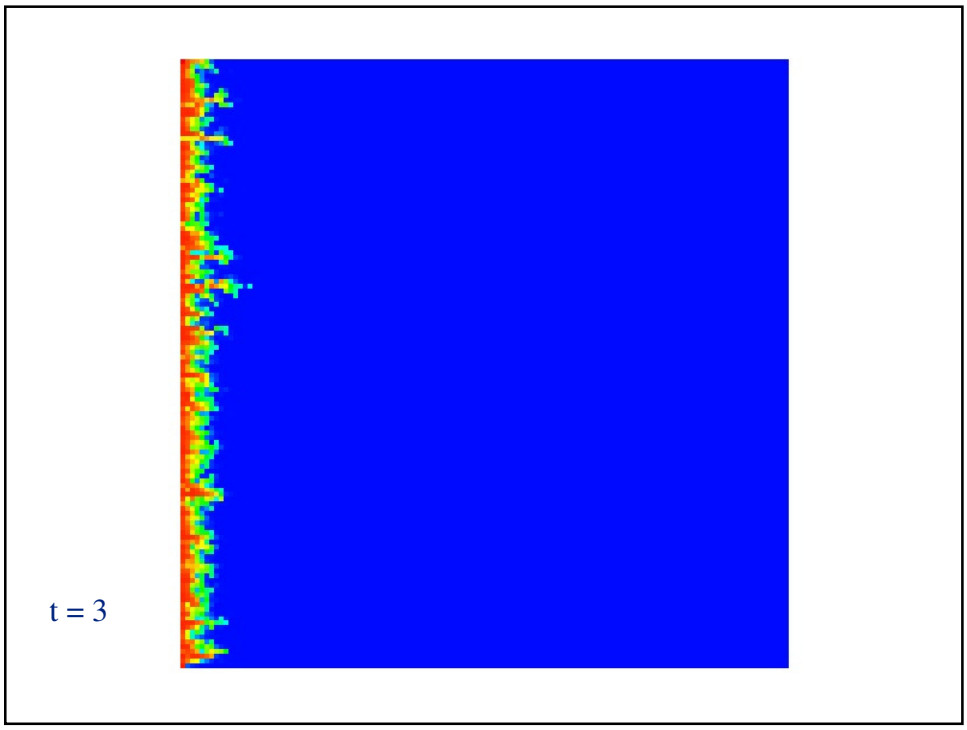


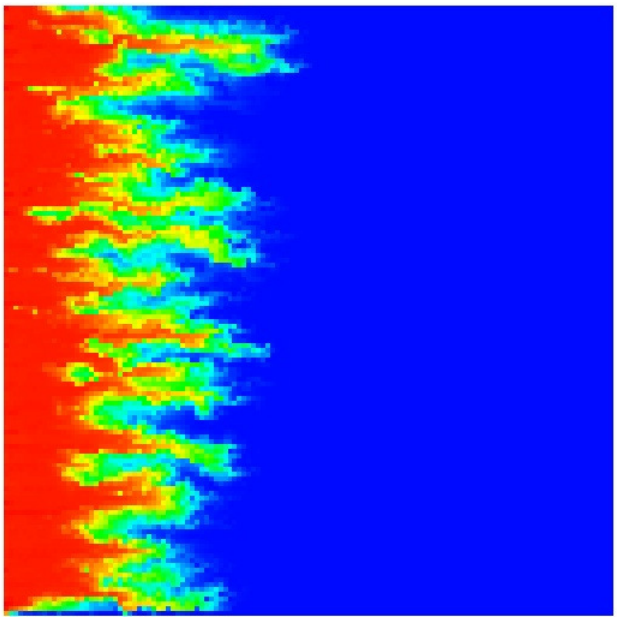
FIG. 5 SCHEMATIC OF NUMERICAL CONCENTRATION PROFILE AT ZERO AND LOW DISPERSIVITY OBTAINED WITH INITIAL CONDITIONS LIKE THAT OF FIG. 3


μ_o = 10.0
μ_s = 1.0
$d_L = 0$
$d_T = 0$
run 2

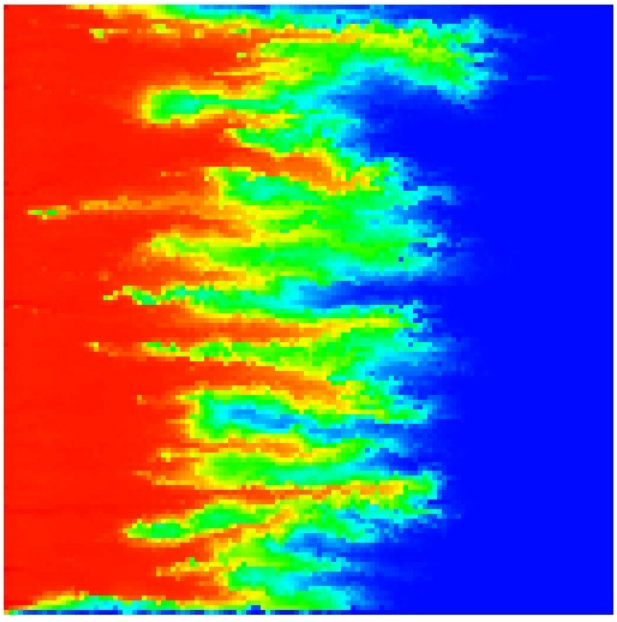

μ_o = 10.0
μ_s = 1.0
$d_L = 0.001$
$d_T = 0.0001$
t = 0
t = 20
t = 40
t = 60
run 1


$d_L = 0.001$
$d_T = 0.0001$
t = 0


t = 3

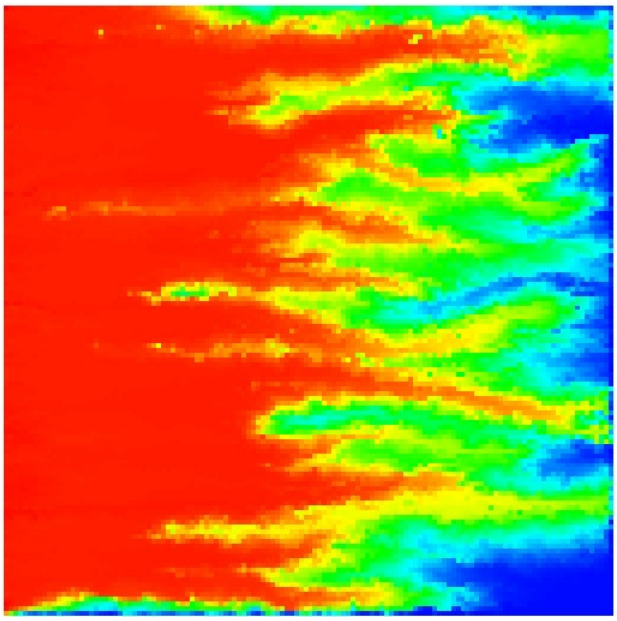


t = 20

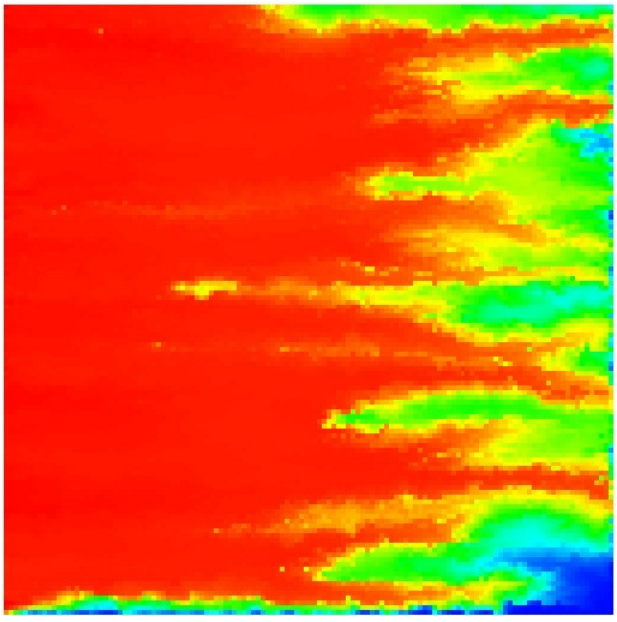


t = 40

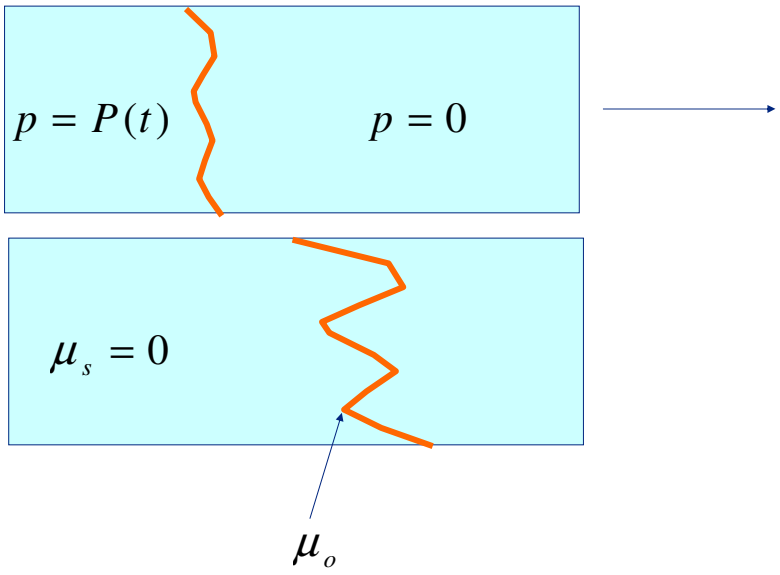
t = 60



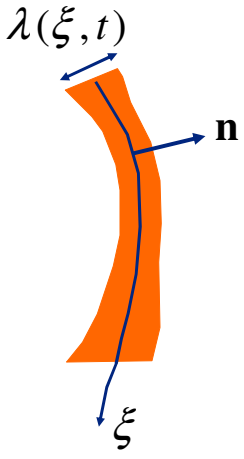
t = 80



Motion of a viscous filament: A *very* simple case



Motion of a viscous filament



$\mathbf{x} = \mathbf{x}(\xi,t)$

$\mathbf{n} = \mathbf{n}(\xi,t)$

Conservation of mass

$$\frac{\partial}{\partial t}\int_{\xi_0}^{\xi_1} \lambda(\xi,t)(x_\xi^2 + y_\xi^2)^{1/2}\, d\xi = 0$$

Darcy's law

$$\frac{\partial \mathbf{x}}{\partial t} = -\frac{k\Delta P}{\mu_o \lambda}\mathbf{n}$$

Motion of a viscous filament

$$\frac{\partial}{\partial t}\int_{\xi_0}^{\xi_1} \lambda(\xi,t)(x_\xi^2 + y_\xi^2)^{1/2}\, d\xi = 0$$

$$\frac{\partial}{\partial t}[\lambda(\xi,t)(x_\xi^2 + y_\xi^2)^{1/2}] = 0$$

$$\lambda(\xi,t)(x_\xi^2 + y_\xi^2)^{1/2} = \alpha(\xi)$$

Motion of a viscous filament

$$\mathbf{t} = (x_\xi, y_\xi), \quad \mathbf{n} = \frac{(y_\xi, -x_\xi)}{(x_\xi^2 + y_\xi^2)^{1/2}}, \quad \mathbf{n}.\mathbf{n} = 1, \quad \mathbf{n}.\mathbf{t} = 0$$

$$\frac{\partial \mathbf{x}}{\partial t} = -\frac{k\Delta P}{\mu_o \lambda}\mathbf{n}$$

$$= -\frac{k\Delta P}{\mu_o \lambda}\frac{(y_\xi, -x_\xi)}{(x_\xi^2 + y_\xi^2)^{1/2}}$$

$$= -\frac{k\Delta P}{\mu_o \alpha}(y_\xi, -x_\xi)$$

$$\frac{\partial x}{\partial t} = a(\xi)\frac{\partial y}{\partial \xi}$$

$$\frac{\partial y}{\partial t} = -a(\xi)\frac{\partial x}{\partial \xi}$$

Motion of a viscous filament

$$\frac{\partial x}{\partial t} = a(\xi)\frac{\partial y}{\partial \xi}$$
$$\frac{\partial y}{\partial t} = -a(\xi)\frac{\partial x}{\partial \xi}$$

$$a(\xi) = 1$$

$$\frac{\partial x}{\partial t} = \frac{\partial y}{\partial \xi}$$
$$\frac{\partial y}{\partial t} = -\frac{\partial x}{\partial \xi}$$

Viscous filaments = Cauchy-Riemann equations

$$\frac{\partial x}{\partial t} = \frac{\partial y}{\partial \xi}$$
$$\frac{\partial y}{\partial t} = -\frac{\partial x}{\partial \xi}$$

Solution 1.

$$x = t$$
$$y = \xi$$

Solution 2.

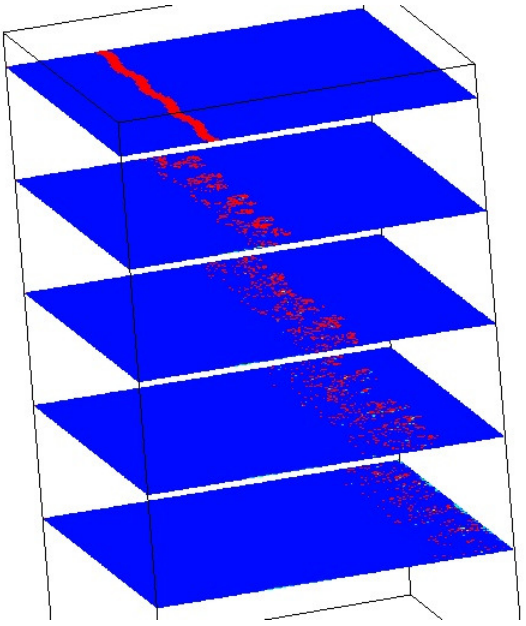
$$x = e^{nt}\cos(n\xi)$$
$$y = e^{nt}\sin(n\xi)$$

$\therefore$ Small wavelengths grow faster than long wavelengths

$$(n = \frac{2\pi}{l})$$

Numerical simulations

$\mu_o = 10.0$
$\mu_s = 1.0$



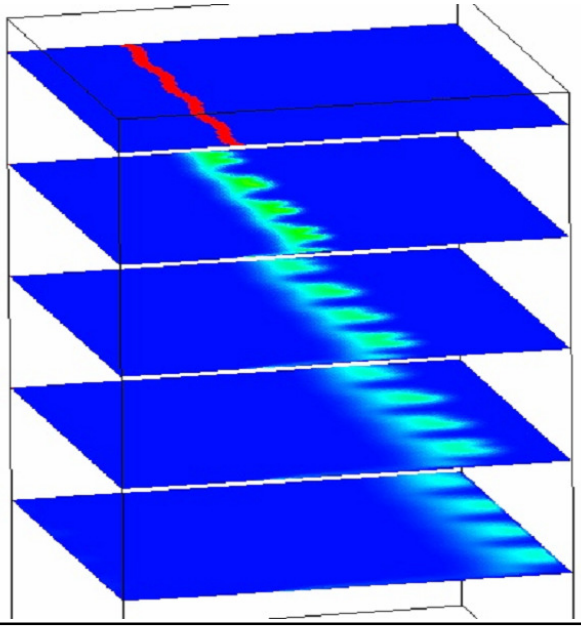
(Run 6)

Numerical simulations

$\mu_o = 10^6$
$\mu_s = 1.0$

$d_L = 0.001$
$d_T = 0.001$

$k_x = 1.0$
$k_y = 1.0$



t = 0

t = 20

t = 40

t = 60

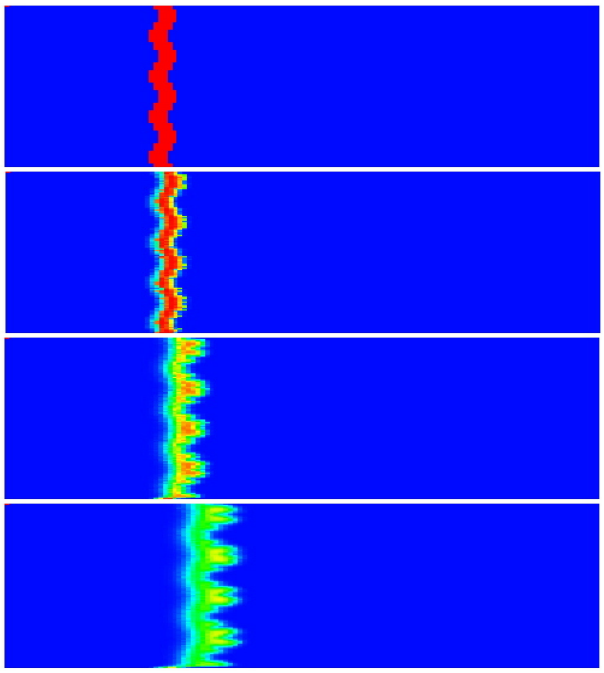
t = 80

run 7

$\mu_o = 10^6$
$\mu_s = 1.0$

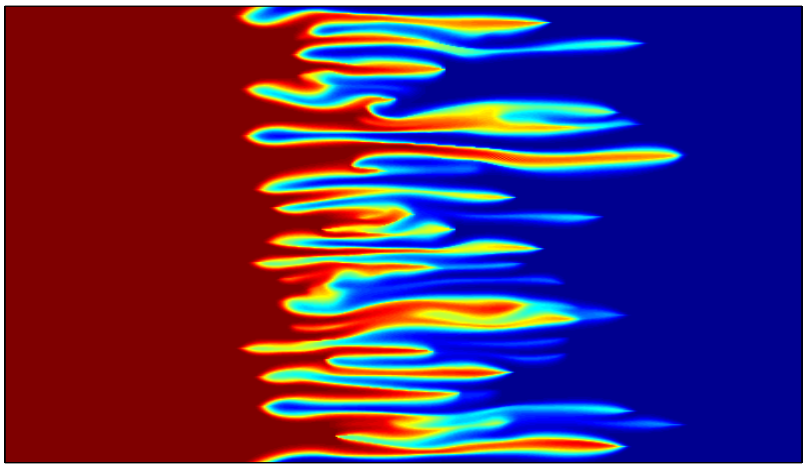
$d_L = 0.001$
$d_T = 0.001$

$k_x = 1.0$
$k_y = 1.0$



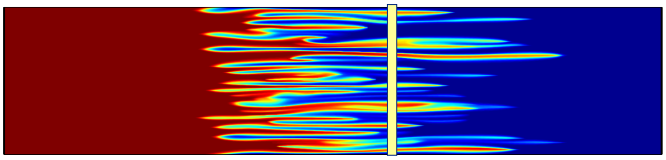
run 7

Finite Volume simulations by R. Booth, 2007



The Koval Model

$$\bar{c}(x,t) = \frac{1}{W}\int c(x,y,t)\,dy$$

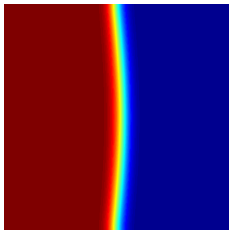


$$\varphi\frac{\partial \bar{c}}{\partial t} + \nabla.(\bar{u}f(\bar{c})) = 0$$

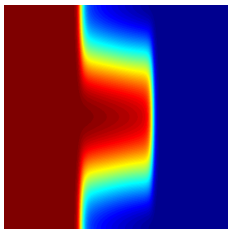
$$f(\bar{c}) \;?$$
$$\tilde{\mu}(\bar{c}) \;?$$

$$\nabla.\bar{u} = 0, \qquad \bar{u} = -\frac{\bar{K}}{\tilde{\mu}(\bar{c})}\nabla\bar{p}$$

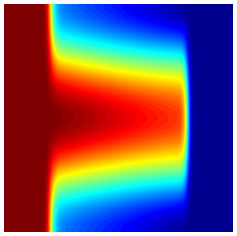
CFD analysis for the Koval model



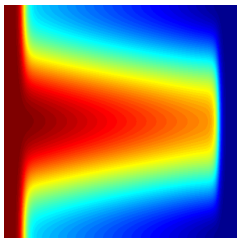
(a) $t = 0$



(b) $t = 1$



(c) $t = 2$



(d) $t = 3$

Figure 3.5: Numerical simulations of solutions of (3.23) in the moving frame = x – t. Red represents the less viscous solvent (c = 1) and blue represents the more viscous oil (c = 0). The dashed, black lines represent the locations of the shocks representing the root and tip regions. We have taken M = 2 and Pe* = 1/100 for this simulation.

Concluding Discussion

- Viscous fingering is a fascinating area, and of practical importance.
- Even *very* simple models can display fingering - the Cauchy-Riemann equations provide a simple model.
- Simulations with low numerical diffusion will mimic reality.
- Simulation helps to determine effective medium models such as the Koval Model.

Schlumberger

References

Garder, A.O., Peaceman, D.W. & Pozzi,A.L. SPEJ, March 1964, 26-36.

CLF. UKAEA Report No. AEEW R 1628, June 1983.

CLF. A Moving Point Method for Arbitrary Peclet Number Multi-Dimensional Convection-Diffusion Equations.
IMA Journal of Numerical Analysis, 5 , 465-480, 1985.

CLF. Moving Point Techniques. NATO Advanced Study Institute on 'Fundamentals of Transport Phenomena in Porous Media'. 14-23 July 1985, Newark, Delaware, USA. Proceedings published in 1987 by Martinus Nijhoff, Ed. J. Bear and M.Y. Corapcioglu, pp 953-1004.

Rees, M.D. and Morton, K.W. (1991). Moving point, particle and free Lagrange methods for convection-diffusion equations. SIAM J. Sci. Statist. Comp. **12**(3), 547-572.

Farmer, C.L. and Howison, S.D. (2005).
The motion of a viscous filament in a porous medium or Hele-Shaw cell: a physical realisation of the Cauchy-Riemann Equations.
Applied Mathematics Letters, 19 (4), April 2006, 356-361.

Booth, R. Miscible Flow Through Porous Media. DPhil, Oxford, Sept. 2008.

Schlumberger