ON DECOMPOSITION OF FUZZY $A$–CONTINUITY

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ABSTRACT In this paper, we introduce and study the notion of fuzzy $C$– sets and fuzzy $C$–continuity. We also prove a mapping $f : X \to Y$ is fuzzy $A$– continuous if and only if it is both fuzzy semi-continuous and fuzzy $C$– continuous.

1. INTRODUCTION

In the classical paper [10] of 1965, Zadeh generalized the usual notion of a set by introducing the important and useful notion of fuzzy sets. Since then, this notion has had tremendous effect on both pure and applied mathematics in different respects. Recently El-Naschie has shown in [4] and [5] that the notion of fuzzy topology may be relevant to quantum particle physics in connection with string theory and $\varepsilon^\infty$ theory. In 1986, Tong [7] introduced the notion of $A$– sets and $A$–continuous mappings in topological spaces and proved that a mapping is continuous if and only if it is both $\alpha$–continuous and $A$–continuous. In 1990, Ganster [5] established a decomposition of $A$–continuity: A mapping $f : X \to Y$ is $A$–continuous if and only if it is both semi-continuous and LC-continuous. Erguang and Pengfei [4] introduced the notion of $C$– sets and $C$–continuity and obtained another decomposition of $A$–continuity: A mapping $f : X \to Y$ is $A$–continuous if and only if it is both semi-continuous and $C$–continuous. Recently, Rajamani and Ambika [6] introduced the notion of fuzzy $A$– sets and fuzzy $A$–continuity and obtained a decomposition of fuzzy continuity.

In this paper, we transform the notions of $C$–set and $C$–continuity to fuzzy topological settings and obtain a decomposition of fuzzy $A$–continuity.

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2. PRELIMINARIES

Throughout this paper, \( X \) and \( Y \) denote fuzzy topological spaces \((X, \tau)\) and \((Y, \sigma)\) respectively on which no separation axioms are assumed. Let \( \lambda \) be a fuzzy set in a fuzzy topological spaces \( X \). The fuzzy interior of \( \lambda \), fuzzy closure of \( \lambda \) and fuzzy preclosure of \( \lambda \) are denoted by \( \text{Int}(\lambda) \), \( \text{Cl}(\lambda) \) and \( \text{pcl}(\lambda) \) respectively.

Now, we recall some definitions and results which are used in this paper.

**DEFINITION 2.1:** A fuzzy set \( \lambda \) in a fuzzy topological space \( X \) is called

(a) fuzzy semi-open [1] if \( \lambda \leq \text{cl}(\text{int}(\lambda)) \);
(b) fuzzy pre-open [2] if \( \lambda \leq \text{int}(\text{cl}(\lambda)) \);
(c) fuzzy regular-open [1] if \( \lambda = \text{int}(\text{cl}(\lambda)) \).

The complements of the above mentioned fuzzy open sets are called their respective fuzzy closed sets.

**DEFINITION 2.2:** A fuzzy set \( \lambda \) in a fuzzy topological space \( X \) is called a fuzzy \( A \)-set [6] if \( \lambda = \alpha \land \beta \), where \( \alpha \) is a fuzzy open set and \( \beta \) is a fuzzy regular closed set.

**DEFINITION 2.3:** A map \( f : X \rightarrow Y \) is said to be

(a) fuzzy continuous [3] if \( f^{-1}(\mu) \) is fuzzy open in \( X \), for every fuzzy open set \( \mu \) in \( Y \);
(b) fuzzy semi-continuous [1] if \( f^{-1}(\mu) \) is fuzzy semi-open in \( X \), for every fuzzy open set \( \mu \) in \( Y \);
(c) fuzzy pre-continuous [2] if \( f^{-1}(\mu) \) is fuzzy pre-open in \( X \), for every fuzzy open set \( \mu \) in \( Y \).

The collection of all fuzzy \( C \)-sets and fuzzy semi-open sets in \( X \) will be denoted by \( FC(X, \tau) \) and \( FSO(X, \tau) \) respectively.

3. FUZZY \( C \)-SETS

**DEFINITION 3.1:** A fuzzy set \( \lambda \) in a fuzzy topological space \( X \) is called a fuzzy \( C \)-set if \( \lambda = \alpha \land \beta \), where \( \alpha \) is fuzzy open and \( \beta \) is fuzzy pre-closed in \( X \).

**PROPOSITION 3.2:** Every fuzzy \( A \)-set is a fuzzy \( C \)-set.

**REMARK 3.3:** The converse of the Proposition 3.2. need not be true as seen from the following example.

**EXAMPLE 3.4:** Let \( X = \{a, b, c\} \), Define \( \alpha_1, \alpha_2, \alpha_3 : X \rightarrow [0, 1] \) by
\[
\begin{align*}
\alpha_1(a) &= 0.3 & \alpha_2(a) &= 0.4 & \alpha_3(a) &= 0.7 \\
\alpha_1(b) &= 0.4 & \alpha_2(b) &= 0.5 & \alpha_3(b) &= 0.6 \\
\alpha_1(c) &= 0.4 & \alpha_2(c) &= 0.5 & \alpha_3(c) &= 0.6
\end{align*}
\]
Let \( \tau = \{0, 1, \alpha_1, \alpha_2\} \). Then \((X, \tau)\) is a fuzzy topological space. Now, \( \alpha_3 \) is a fuzzy \( C \)-set but not a fuzzy \( A \)-set.

**REMARK 3.5:** The concepts of fuzzy \( C \)-sets and fuzzy semi-open sets are independent as shown by the following examples.
EXAMPLE 3.6: Let $X = \{a, b, c\}$, Define $\alpha_1, \alpha_2, \alpha_3 : X \rightarrow [0, 1]$ by

- $\alpha_1(a) = 0.2 \quad \alpha_2(a) = 0.3 \quad \alpha_3(a) = 0.3$
- $\alpha_1(b) = 0.3 \quad \alpha_2(b) = 0.3 \quad \alpha_3(b) = 0.3$
- $\alpha_1(c) = 0.3 \quad \alpha_2(c) = 0.4 \quad \alpha_3(c) = 0.3$

Let $\tau = \{0, 1, \alpha_1, \alpha_3\}$. Then $(X, \tau)$ is a fuzzy topological space. Now, $\alpha_3$ is a fuzzy semi-open set but not a fuzzy $C-$ set.

EXAMPLE 3.7: Let $X = \{a, b, c\}$, Define $\alpha_1, \alpha_2, \alpha_3 : X \rightarrow [0, 1]$ by

- $\alpha_1(a) = 0.4 \quad \alpha_2(a) = 0.6 \quad \alpha_3(a) = 0.5$
- $\alpha_1(b) = 0.5 \quad \alpha_2(b) = 0.7 \quad \alpha_3(b) = 0.6$
- $\alpha_1(c) = 0.6 \quad \alpha_2(c) = 0.8 \quad \alpha_3(c) = 0.7$

Let $\tau = \{0, 1, \alpha_1, \alpha_2\}$. Then $(X, \tau)$ is a fuzzy topological space. Now, $\alpha_3$ is a fuzzy $C-$ set but not a fuzzy semi-open set.

LEMMA 3.8: Let $\alpha$ be a fuzzy set in a fuzzy topological space $X$. Then $\alpha \in FC(X, \tau)$ if and only if $\alpha = \lambda \wedge pcl(\alpha)$ for some fuzzy open set $\lambda$.

Proof: Let $\alpha \in FC(X, \tau)$. Then $\alpha = \lambda \wedge \mu$ where $\lambda$ is fuzzy open and $\mu$ is fuzzy pre-closed. Now, $\alpha \leq \lambda$ and $\alpha \leq \mu$, we have $pcl(\lambda) \leq pcl(\mu) = \mu$, since $\mu$ is fuzzy pre-closed in $X$. Thus $pcl(\alpha) \leq \mu$.

Therefore $\lambda \wedge pcl(\alpha) \leq (\lambda \wedge \mu) = \alpha \leq \lambda \wedge pcl(\alpha)$. (i.e.,) $\lambda \wedge pcl(\alpha) = \alpha$.

Converse part is obvious.

THEOREM 3.9: Let $\alpha$ be a fuzzy set in a fuzzy topological space $X$. Then $\alpha = \lambda \wedge cl(int(\alpha))$ for some fuzzy open set $\lambda$ if and only if $\alpha \in FC(X, \tau) \wedge FSO(X, \tau)$.

Proof: Let $\alpha = \lambda \wedge cl(int(\alpha))$ for some fuzzy open set $\lambda$ in $X$. Then $\alpha \leq cl(int(\alpha))$. So $\alpha$ is fuzzy semi-open in $X$. Let $\beta = cl(int(\alpha))$, then $\beta$ is fuzzy regular closed. Since every fuzzy regular closed set is fuzzy pre-closed, $\beta$ is fuzzy pre-closed which implies $\alpha$ is fuzzy $C-$ set. Thus $\alpha \in FC(X, \tau) \wedge FSO(X, \tau)$.

Conversely, let $\alpha \in FC(X, \tau) \wedge FSO(X, \tau)$. Then $\alpha \in FC(X, \tau)$ and $\alpha \in FSO(X, \tau)$. Since $\alpha \in FC(X, \tau), \alpha = \lambda \wedge pcl(\alpha)$, using Lemma 3.8. Thus $\alpha = \lambda \wedge cl(int(\alpha))$ for some fuzzy open set $\lambda$.

4. DECOMPOSITION OF FUZZY $A-$CONTINUITY

DEFINITION 4.1: A mapping $f : X \rightarrow Y$ is called fuzzy $A-$continuous [6] if $f^{-1}(\mu)$ is a fuzzy $A-$ set in $X$, for every fuzzy open set $\mu$ in $Y$.

DEFINITION 4.2: A mapping $f : X \rightarrow Y$ is called fuzzy $C-$continuous if $f^{-1}(\mu)$ is a fuzzy $C-$ set in $X$, for every fuzzy open set $\mu$ in $Y$.

PROPOSITION 4.3: Every fuzzy $A-$ continuous function is fuzzy $C-$ continuous.

REMARK 4.4: The converse of Proposition 4.3 need not be true as shown by the following example.

EXAMPLE 4.5: Let $X = \{a, b, c\}, Y = \{x, y, z\}$ and $\alpha_1, \alpha_2$ and $\alpha_3$ are fuzzy sets defined as follows:

- $\alpha_1(a) = 0.3 \quad \alpha_2(a) = 0.4 \quad \alpha_3(a) = 0.7$
- $\alpha_2(b) = 0.4 \quad \alpha_2(b) = 0.5 \quad \alpha_3(b) = 0.6$
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\( \alpha_1(c) = 0.4 \quad \alpha_2(c) = 0.5 \quad \alpha_3(c) = 0.6 \)

Let \( \tau_1 = \{0, 1, \alpha_1, \alpha_2\}, \tau_2 = \{0, 1, \alpha_3\} \). Then the mapping \( f : (X, \tau_1) \rightarrow (Y, \tau_2) \) defined by \( f(a) = x, \quad f(b) = y \) and \( f(c) = z \) is fuzzy \( C^- \) continuous but not fuzzy \( A^- \) continuous.

**REMARK 4.6:** The concepts of fuzzy \( C^- \) continuity and fuzzy semi-continuity are independent as shown by the following examples.

**THEOREM 4.7:** Let \( X = \{a, b, c\}, Y = \{x, y, z\} \) and \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) are fuzzy sets defined as follows:

\[
\begin{align*}
\alpha_1(a) &= 0.2 \quad \alpha_2(a) = 0.3 \quad \alpha_3(a) = 0.3 \\
\alpha_1(b) &= 0.3 \quad \alpha_2(b) = 0.3 \quad \alpha_3(b) = 0.3 \\
\alpha_1(c) &= 0.3 \quad \alpha_2(c) = 0.4 \quad \alpha_3(c) = 0.3
\end{align*}
\]

Let \( \tau_1 = \{0, 1, \alpha_1, \alpha_2\}, \tau_2 = \{0, 1, \alpha_3\} \). Then the mapping \( f : (X, \tau_1) \rightarrow (Y, \tau_2) \) defined by \( f(a) = x, \quad f(b) = y \) and \( f(c) = z \) is fuzzy semi-continuous but not fuzzy \( C^- \) continuous.

**EXAMPLE 4.8:** Let \( X = \{a, b, c\}, Y = \{x, y, z\} \) and \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) are fuzzy sets defined as follows:

\[
\begin{align*}
\alpha_1(a) &= 0.4 \quad \alpha_2(a) = 0.6 \quad \alpha_3(a) = 0.5 \\
\alpha_1(b) &= 0.5 \quad \alpha_2(b) = 0.7 \quad \alpha_3(b) = 0.6 \\
\alpha_1(c) &= 0.6 \quad \alpha_2(c) = 0.8 \quad \alpha_3(c) = 0.7
\end{align*}
\]

Let \( \tau_1 = \{0, 1, \alpha_1, \alpha_2\}, \tau_2 = \{0, 1, \alpha_3\} \). Then the mapping \( f : (X, \tau_1) \rightarrow (Y, \tau_2) \) defined by \( f(a) = x, \quad f(b) = y \) and \( f(c) = z \) is fuzzy \( C^- \) continuous but not fuzzy semi-continuous.

**THEOREM 4.9:** A mapping \( f : X \rightarrow Y \) is fuzzy \( A^- \) continuous if and only if it is both fuzzy semi-continuous and fuzzy \( C^- \) continuous.

**Proof:** Follows from Theorem 3.9.

**REFERENCES**