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Matrix Models, Integrable Structures, and T-duality of Type 0 String Theory

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Abstract

Instanton matrix models (IMM) for two dimensional string theories are obtained from the matrix quantum mechanics (MQM) of the T-dual theory. In this paper we study the connection between the IMM and MQM, which amounts to understand T-duality from the viewpoint of matrix models. We show that type 0A and type 0B matrix models perturbed by purely closed string momentum modes (or purely winding modes) have the integrable structure of Toda hierarchies, extending the well known results for $c = 1$ string. In particular, we show that type 0A(0B) MQM perturbed by momentum modes has the same integrable structure as type 0B(0A) MQM perturbed by winding modes, which is a nontrivial check of the T-duality between the matrix models. The MQM deformed by NS-NS winding modes are used to study type 0 string in 2D black holes. We also find an intriguing connection between the IMM and the MQM via tachyon condensation. The array of alternating D-instantons and anti-D-instantons separated at the critical distance plays a key role in this picture. We discuss its implications on sD-branes in two dimensional string theories.

Contents

1. Introduction

Recently the $c = 1$ matrix quantum mechanics(for reviews see [1,2,3]) has received a lot of attention because of its new interpretation as the decoupled world volume theory of unstable D0-branes[4,5,6]. The matrix models dual to type 0 string theories were also proposed in [7,8]. For other recent developments, see [9,10,11,12,13,14,15,16,17]. The type 0B matrix quantum mechanics (MQM) describes open string tachyons living on the unstable D0-branes, whereas the type 0A MQM describes tachyonic open strings stretched between stable D0- and anti-D0-branes. Upon compactification on Euclidean time, these two matrix models are conjectured to be T-dual to each other. The exact agreement in free energy was found in [8]. However, unlike $c = 1$ matrix model which can be derived from discretizing the Polakov action on the string world sheet, such a derivation is not known for type 0 matrix models. From the matrix model point of view, the T-duality between type 0A and 0B strings seems rather mysterious. To understand this T-duality is one of the motivations of this paper.

To start, let us consider type 0A MQM with Euclidean time compactified on a circle of radius α'/R. This is supposed to be T-dual to type 0B string theory on a circle of radius

R. By decomposing the fields in type 0A MQM into their Fourier modes along the thermal circle, we get a zero-dimensional matrix model, of the form

$$\int dU d\tilde{U} \prod_n dt_n dt_n^\dagger e^{-\tilde{\beta}\mathrm{Tr}\left[(Xt_n - t_n\tilde{X} + 2\pi nRt_n)(Xt_n - t_n\tilde{X} + 2\pi nRt_n)^\dagger - a^2 t_n t_n^\dagger\right]} \tag{1.1}$$

where $U = e^{iX/R}$, $\tilde{U} = e^{i\tilde{X}/R}$ are the holonomies of the Wilson lines in type 0A MQM, $t_n, t_n^\dagger$ are the Fourier modes of the complex tachyons. This is an instanton matrix model (IMM), since the Wilson lines on the D0-branes in type 0A theory are mapped to collective coordinates $X, \tilde{X}$ for D-instantons in type 0B theory, and the tachyons on the D0-branes are mapped to tachyonic open strings stretched between D-instantons and anti-D-instantons. Similar D-instanton matrix models have been studied in the context of ten dimensional type IIB string theory[18,19]. Now having two different matrix model duals of type 0B theory, we want to understand how they are related to each other. Since one of them involves unstable D0-branes, and the other involves D- and anti-D-instantons, it is natural to suspect that tachyon condensation plays the key role of connecting the two theories. If we can show the equivalence between the IMM and MQM, we would prove the T-duality between type 0A and 0B matrix models.

The first thing we want to understand is the identification between operators in IMM and in MQM, in particular the ones that correspond to closed string momentum and winding modes. This was well understood in the context of $c = 1$ string theory. The momentum modes in MQM are represented as asymptotic perturbations of the fermi sea, whereas the winding modes are identified with the holonomy of the gauge fields[20,21]. $c = 1$ string perturbed purely by momentum modes[22,23], or by winding modes[21], has the integrable structure of Toda lattice hierarchy, with the integrable flow generated by the corresponding closed string perturbations. In particular, the perturbed grand canonical partition function is shown to be the τ-function of the corresponding integrable hierarchy. The integrable structure appearing in $c = 1$ string theory is subject to constraints, known as the string equation[24]. These constraints can be equivalently thought of as imposing an initial condition on the flow of τ-functions, which is the unperturbed partition function. Since the unperturbed $c = 1$ MQM on radius R and α'/R have the same free energy, it follows from the integrable structures that the grand partition function of two theories perturbed respectively by momentum modes and winding modes also agree.

We shall generalize this approach to type 0 string theories. In type 0B MQM, the symmetric and antisymmetric perturbations of the fermi sea decouple.[1] They generate two independent Toda flows, subject to different string equations. Consequently the perturbed partition function is the product of two τ-functions, associated with the symmetric and antisymmetric perturbations respectively. On the type 0B IMM side, one can integrate out the tachyons and get a unitary matrix model only in terms of the "holonomies", or the collective coordinates of the instantons. This is T-dual to the "twisted partition" discussed in [20,21] in the case of $c = 1$ string. As we will see, the perturbed grand canonical partition function of the IMM indeed has a similar structure, provided a nontrivial identification of the perturbation parameters with those of MQM. We will use it to identify the operators in IMM that are dual to NS-NS and R-R scalars in spacetime.

Type 0A MQM and IMM perturbed by momentum modes also have the structure of Toda hierarchy. This is very similar to the case of $c = 1$ string, except that the string equations for type 0A MQM and IMM are nonperturbatively well defined. The combination of these shows that both type 0A and type 0B theories perturbed purely by momentum modes, or purely by winding modes, are integrable. This also directly verifies that the IMM and MQM are equivalent at least when only momentum modes or winding modes are present. The perturbations involving both momentum and winding modes are more complicated, since in that case we would lose integrability.

An alternative attempt to connect the IMM with MQM, which is more direct and intuitive, is via tachyon condensation. Consider turning on an open string tachyon profile on the D0-brane world volume $T(X) \sim \lambda \cos(X/\sqrt{2\alpha'})$, where X is the Euclidean time. As well known [26,27] this corresponds to a marginal deformation in the worldsheet CFT. With sufficiently large λ, it takes a D0-brane into an array of alternating D- and anti-D-instantons separated at the critical distance. It is natural to expect that, the MQM expanded near this tachyon profile should be the same as the IMM expanded near the configuration of an array of D-instantons.

On the MQM side, this periodic tachyon profile effectively discretizes the Euclidean time circle to a periodic lattice of spacing $\pi\sqrt{2\alpha'}$. It is well known[1] that the MQM on a time lattice of spacing ϵ with ϵ less than a critical distance a is exactly equivalent to the continuum theory, provided proper redefinition of the parameters. In our case $a =$

[1] As remarked in [25], this doesn't mean that the NS-NS and R-R scalars decouple, because of the nonlinearity of bosonization.

$\pi\sqrt{2\alpha'}$ is the same critical distance at which the tachyonic open string stretched between D- and anti-D-instantons becomes massless. The array of D-instantons is classically a stationary configuration in the IMM. If we integrate out the (complex) tachyons connecting the instantons, we expect an instability that drives the D- and anti-D-instantons toward each other. On the other hand, the collective coordinates of the instantons are eigenvalues of Hermitian matrices in the IMM. They effectively repel each other and fill up a "sea" of D-instantons. We have a large number of D-instantons distributed in a periodic effective potential $V(X)$ along the thermal circle, and the D-instanton array corresponds to a critical point where the instantons are sitting at the top of the potential. There is a phase transition when the "instanton sea" merges the top of the potential, and *this is the critical point that defines the double scaling limit of the IMM.* This is analogous to the case of $c = 1$ matrix model, where the double scaling limit is defined as the limit that the fermi level approaches the top of the tachyon potential. By relating the collective coordinates of the array of D-instantons to the open string tachyons in the discretized MQM at the sites on the time lattice, we will show that the two matrix models become the same in the limit of critical distance.

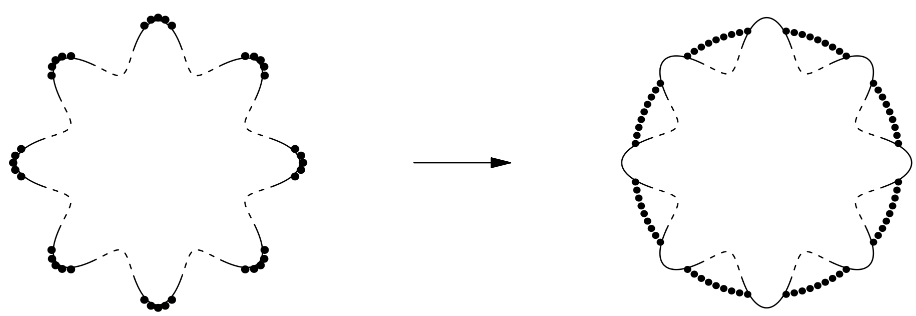

Fig. 1: Schematic picture of the array of D-instantons condensing along the Euclidean time circle into the "instanton sea".

It was proposed in [28] (see also [29,30]) that sD-branes ($\lambda = 1/2$ s-brane) are described by an array of D- and anti-D-instantons along the Euclidean time, separated at the critical distance (see also [30]). In the sense described above, the IMM can be thought of as the world volume theory of sD-branes, and sD-branes play the same role in the IMM as D0-branes in the MQM. We compute the closed string fields sourced by sD-branes from IMM, and find exact agreement with calculation using ZZ boundary states. In fact, we can reproduce the $(1,1)$ ZZ boundary state in Liouville [31] and super-Liouville theory [32,33] from $c = 1$ IMM and type 0B IMM in a very simple manner.

The $c = 1$ matrix model deformed by the lowest winding modes is conjectured to describe bosonic string in a 2D Euclidean black hole [21]. The conjecture is extended to type 0 theory in [34]. The exact free energy of the deformed type 0 matrix models can in principal be obtained by solving the Hirota differential equations. Using the technique of [21], we can solve for the genus 0 and genus 1 terms in the perturbative expansion of the free energy. They are of the form[21,35]

$$\mathcal{F} = -2\pi(R - R_H)M + (s_1 + 1)\ln M + \cdots \tag{1.2}$$

where M, depending on the coefficient λ of the winding mode deformation, is interpreted as the mass of the black hole, and R_H is the asymptotic radius in Euclidean time. In the T-dual $\mathcal{N} = 2$ Liouville theory [36,37], λ is essentially the coefficient of the Liouville term. We find from the genus 0 piece of the free energy that $M \propto \lambda^4$, which agrees with the general behavior expected from $\mathcal{N} = 2$ Liouville theory. The density of states for the black hole has Hagedorn growth

$$\rho(M) \sim M^{s_1} e^{\beta_H M} \tag{1.3}$$

We find that $s_1 = -17/12$ for type 0B string and $s_1 = -13/12$ for type 0A string (uncharged black hole).

This paper is organized as follows. In section 2 we derive the IMM for $c = 1$ and type 0 strings. In section 3, we review the integrable structure in the S-matrix of $c = 1$ MQM, and extend them to the case of type 0A and type 0B MQM. We show that the same integrable structures appear in the corresponding IMMs, and that they are subject to the same string equations by computing the unperturbed free energies. Section 4 studies the connection between IMM and MQM from the viewpoint of tachyon condensation. In section 5, we compute the closed string fields sourced by sD-branes from IMM and compare them to the results obtained from ZZ boundary states. The integrable structures of type 0 theories are applied to computing the free energy of the 2D Euclidean black hole in section 6.

2. The Instanton Matrix Model

In this section we derive the instanton matrix model of $c = 1$ string and type 0 string theories from the (gauged) matrix quantum mechanics of the T-dual theory. We will set $\alpha' = 1$ in the case of $c = 1$ string, and keep α' explicitly in most of the discussions on type 0 theories, unless otherwise indicated.

2.1. The IMM for $c=1$ string

Consider the gauged $c=1$ MQM with Euclidean time x compactified on a circle of radius $R'=1/R$. The matrix model action is

$$S=\int_0^{2\pi/R} dx \mathrm{Tr}\left[\frac{1}{2}(\partial_x\Phi+i[A,\Phi])^2-\frac{1}{2}\Phi^2\right] \tag{2.1}$$

where the tachyon $\Phi(x)$ and the gauged field $A(x)$ are Hermitian $N\times N$ matrices. Let us fix the gauge $\partial_x A=0$, which sets A to its zero mode $A^{(0)}$ and introduces the Fadeev-Popov determinant

$$\int db\, dc \exp(\mathrm{Tr} b\partial_x D_x c)=\prod_{i<j}\left(\frac{\sin[(a_i-a_j)/2R]}{(a_i-a_j)/2R}\right)^2 \tag{2.2}$$

where a_i's are the eigenvalues of $A^{(0)}$. Let us decompose $\Phi(x)$ into its momentum modes $\Phi^{(n)}$ along the thermal circle. The action is now written as

$$S=\frac{2\pi}{R}\sum_{n\in\mathbf{Z}}\mathrm{Tr}\left[\frac{1}{2}\left(nR\Phi^{(n)}+[A^{(0)},\Phi^{(n)}]\right)\left(nR\Phi^{(-n)}-[A^{(0)},\Phi^{(-n)}]\right)-\frac{1}{2}\Phi^{(n)}\Phi^{(-n)}\right] \tag{2.3}$$

In the T-dual theory $X\equiv 2\pi A^{(0)}$ is the collective coordinate for D-instantons, $\Phi^{(n)}$ are the open string tachyons stretched between D-instantons with (relative) winding number n. T-duality acts on the inverse "Planck constant" β as $\beta\to\beta R$. We end up with an instanton matrix model action

$$\begin{aligned}S=\frac{\beta}{2\pi}\sum_{n\in\mathbf{Z}}\mathrm{Tr}\Bigg[\frac{1}{2}&\left(2\pi nR\Phi^{(n)}+[X,\Phi^{(n)}]\right)\left(2\pi nR\Phi^{(-n)}-[X,\Phi^{(-n)}]\right)\\&-\frac{1}{2}(2\pi)^2\Phi^{(n)}\Phi^{(-n)}\Bigg]\end{aligned} \tag{2.4}$$

The path integral measure is modified by the FP determinant (2.2). If we diagonalize X in terms of its eigenvalues x_i, the measure factor involving X becomes the measure for unitary matrices

$$\prod_{i<j}\sin^2\left(\frac{x_i-x_j}{2R}\right) \tag{2.5}$$

In other words, the natural variable to be integrated over is the "holonomy" $U=e^{iX/R}$. As well known[38,39], this can be thought of as the contribution from D-instantons at x_i together with all of their images at $x_i+2\pi nR$.

In the infinite radius limit $R\to\infty$, we can drop all the modes $\Phi^{(n)}$ with nonzero winding since they become infinitely "massive". The action becomes simply

$$S=\frac{\beta}{4\pi}\mathrm{Tr}([X,\Phi^{(0)}]^2-4\pi^2(\Phi^{(0)})^2). \tag{2.6}$$

An alternative instanton matrix model for $c=1$ string, the Liouville matrix model, was studied in [40]. It would be nice to understand its relation to the IMM we are proposing.

2.2. The IMM for Type 0 String Theories

Let us start with the MQM of type 0A theory in two dimensions, which is the decoupled world volume theory of (stable) D0- and anti-D0-branes. In the background with no net D0-brane charges, the matrix model has $U(N) \times U(N)$ gauge symmetry. This is the case we will be concerned with. We have the $U(N) \times U(N)$ gauge field A_0 and bifundamental tachyon ϕ,

$$A_0 = \begin{pmatrix} A & 0 \\ 0 & \tilde{A} \end{pmatrix}, \qquad \phi = \begin{pmatrix} 0 & t \\ t^\dagger & 0 \end{pmatrix} \tag{2.7}$$

The Lagrangian is

$$L = \mathrm{Tr}\left[(D_0 t)^\dagger (D_0 t) + \frac{1}{2\alpha'} t^\dagger t\right] \tag{2.8}$$

where $D_0 t = \partial_0 t + At - t\tilde{A}$. Again, we want to compactify the Euclidean time on a circle of radius $R' = \alpha'/R$, and rewrite the matrix quantum mechanics in terms of a matrix integral over the Fourier modes of A and t. The D0- and anti-D0-branes becomes D- and anti-D-instantons; the zero modes of A and $\tilde{A}$ (Wilson lines) become the collective coordinates of the D- and anti-D-instantons; the Fourier modes of t and $t^\dagger$ are open strings stretched between D- and anti-D-instantons, which can also wind around the circle. We expect this model to be equivalent to type 0B MQM at radius R in the double scaling limit.

Let us fix the gauge $\partial_0 A_0 = 0$, which sets A and $\tilde{A}$ to their zero modes $A^{(0)} \equiv X/2\pi\alpha'$ and $\tilde{A}^{(0)} \equiv \tilde{X}/2\pi\alpha'$. As before the gauge fixing introduces the FP determinant

$$\prod_{i<j} \left(\frac{\sin[(x_i - x_j)/2R]}{(x_i - x_j)/2R}\right)^2 \left(\frac{\sin[(\tilde{x}_i - \tilde{x}_j)/2R]}{(\tilde{x}_i - \tilde{x}_j)/2R}\right)^2 \tag{2.9}$$

where x_i and $\tilde{x}_i$ are the eigenvalues of X and $\tilde{X}$ respectively. In terms of the Fourier modes $t^{(n)}$ of $t(x)$, the action is

$$S = \frac{\beta}{\pi(2\alpha')^{3/2}} \sum_{n\in\mathbf{Z}} \mathrm{Tr}\left[\left(2\pi n R t^{(n)} + X t^{(n)} - t^{(n)}\tilde{X}\right)\left(2\pi n R t^{(n)\dagger} + t^{(n)\dagger} X - \tilde{X} t^{(n)\dagger}\right) - a^2 t^{(n)} t^{(n)\dagger}\right] \tag{2.10}$$

where we exhibited the (type 0B) inverse "Planck constant" β, $a = \pi\sqrt{2\alpha'}$ is the critical distance. There was a factor $2\pi\alpha'/R$ coming from the integral over Euclidean time, but under T-duality $\beta \to \beta R/\sqrt{2\alpha'}$ [8], so the factors of R cancel out. If we further gauge fix

X to the diagonal form, the usual measure factor $\Delta(x)^2\Delta(\tilde{x})^2$ is converted to the measure for unitary matrices

$$\prod_{i<j} \sin^2(\frac{x_i - x_j}{2R}) \sin^2(\frac{\tilde{x}_i - \tilde{x}_j}{2R}) \tag{2.11}$$

Therefore the natural variables to be integrated over are the "holonomies" $U = e^{2\pi i X/R}$, $\tilde{U} = e^{2\pi i \tilde{X}/R}$. The difference disappears in the infinite radius limit $R \to \infty$. In this limit all the winding modes are very massive and we can drop them, so the action simplifies to

$$S = \frac{\beta}{\pi(2\alpha')^{3/2}} \mathrm{Tr}\left[(Xt - t\tilde{X})(t^\dagger X - \tilde{X}t^\dagger) - a^2 tt^\dagger\right] \tag{2.12}$$

Even though in the decompactification limit the integral over $X, \tilde{X}$ is simply a Gaussian, we shouldn't integrate out them directly. One reason is that, the determinant coming from integration over $X, \tilde{X}$ would completely cancel the F-P determinant coming from diagonalizing t. We would then naively conclude that the eigenvalues of t do not repel each other, and it is unclear how to define the double scaling limit. What we have done wrong is reminiscent to the case of gauged MQM, where integrating out the gauge fields naively cancels the Vandermonde determinants of the eigenvalues at every time. This is incorrect, since if we carefully discretize the Euclidean time, the gauge fields appear as "link variables", while the tachyon fields are associated with the "sites". At the end, the F-P determinants of the eigenvalues at the initial and end points of the time evolution are not completely cancelled out. Their effect is nothing but to antisymmetrize the wave functions. Similarly in the IMM before integrating out X and $\tilde{X}$, we should either restrict to a finite interval of time, or compactify Euclidean time on a finite circle.

There is another way to understand this subtlety. An unstable D0-brane with $\lambda = 1/2$ rolling tachyon profile is an array of alternating D-instantons and anti-D-instantons along the thermal circle, separated at the critical distance $a = \pi\sqrt{2\alpha'}$ [28]. Suppose $2\pi R = 2ma$ for some integer m. Naturally we expect that the $U(N)$ MQM to be equivalent to the IMM with gauge group $U(mN) \times U(mN)$. Therefore in the limit $R \to \infty$, we must correspondingly take $N \to \infty$. We will come back to this point in section 4.

Clearly we can interchange the role of type 0A and 0B MQM in the above discussion. We will then obtain an IMM of type 0A theory. Formally this model has the same action as the IMM for $c = 1$ string. Presumably the double scaling limits are defined differently in these two models.

3. Integrable Structures in Two Dimensional String Theories

It is well known that $c = 1$ string deformed by closed string momentum modes (or winding modes) have the integrable structure of Toda chain hierarchy[22,24,23,41]. The operators corresponding to the momentum modes generate Toda flows, and the perturbed grand canonical partition function is identified with a τ-function. In this section we will show that similar integrable structures appear in type 0A and 0B string theories. Subsection 3.1 is a review of some results on the exact S-matrix and integrable structures of "theory I" ($c = 1$ matrix model), essentially following [23,42]. There is nothing new in this first subsection, but it sets up the conventions that will be used in the rest of the section. In subsection 3.2, we study the S-matrix and integrable structures of "theory II" (type 0B MQM). Most of these results are already known, but we will derive the string equations for theory II explicitly. In subsection 3.3, we show that type 0A MQM also has the Toda integrable structure. This integrable structure is, in some sense, more natural than that of "theory I" since type 0A theory is non-perturbatively well-defined. Subsection 3.4 studies the perturbed partition function of type 0B IMM. We will indeed recover the structure similar to that of type 0B MQM, and find a dictionary translating closed string modes in spacetime to operators in the IMM. In subsection 3.5, we compute the free energy of IMM by explicitly performing the matrix integral. The case for $c = 1$ IMM was essentially done in [20,21]. In the case of type 0B IMM, we find that the free energy factorizes into two pieces, which precisely agree with the contribution to type 0B MQM free energy from symmetric and antisymmetric fluctuations of the fermi sea. By matching the unperturbed free energies, we conclude that the string equations for the integrable structure of IMM are the same as those for MQM.

3.1. "Theory I"

It is convenient to define light cone variables

$$\hat{x}_\pm = \frac{\hat{x} \pm \hat{p}}{\sqrt{2}}, \qquad [\hat{x}_+, \hat{x}_-] = -i \tag{3.1}$$

In "theory I" ($c = 1$ string) these variables are restricted to the region $x_\pm > 0$, and for now we will not worry about nonperturbative effects. The Hamiltonian of the free fermions is

$$\hat{H}_0 = -\frac{1}{2}(\hat{x}_+\hat{x}_- + \hat{x}_-\hat{x}_+) \tag{3.2}$$

Since $\hat{x}_+, \hat{x}_-$ are conjugate variables, we can write the wave function of a state in either x_+ or x_- representation. The two wave functions $\psi_+(x_+)$ and $\psi_-(x_-)$ are related by a Fourier transform

$$\begin{aligned}\psi_-(x_-) &= (\hat{S}\psi_+)(x_+) \\ &= \int_0^\infty dx_+ K(x_-, x_+)\psi_+(x_+)\end{aligned} \tag{3.3}$$

where the integration kernel K is

$$K(x_-, x_+) = \sqrt{\frac{2}{\pi}}\cos(x_- x_+) \tag{3.4}$$

We could have also chosen $K(z_-, z_+) = \sqrt{2/\pi}\sin(z_- z_+)$ instead. This ambiguity reflects the fact that perturbatively the two sides of the fermi sea decouple. The energy eigenstates have wave functions

$$\psi^E_\pm(x_\pm) = \frac{1}{\sqrt{2\pi}} x_\pm^{\pm iE - \frac{1}{2}} \tag{3.5}$$

Note that we have chosen a convenient basis for ψ^E_+ and ψ^E_-, but they are *not* related by (3.3). In fact, it is straightforward to show that

$$\hat{S}^{\pm 1}\psi^E_\pm(x_\pm) = \mathcal{R}^{\pm 1}(E)\psi^E_\mp(x_\mp), \qquad \mathcal{R}(E) = \sqrt{\frac{2}{\pi}}\cosh\left(\frac{\pi}{2}(i/2 - E)\right)\Gamma(iE + \tfrac{1}{2}) \tag{3.6}$$

Essentially $\hat{S}$ is the S-matrix, and $\mathcal{R}(E)$ is the reflection coefficient (phase shift).

The closed string momentum modes correspond to operators in MQM of the form[43]

$$V_{\pm k/R}(x) = e^{\pm ikx/R}\mathrm{Tr}X_\pm(x)^{k/R} \tag{3.7}$$

where x is the Euclidean time variable, R is the radius of the thermal circle, and the allowed momenta are $\pm k/R$ for integer k. A general perturbation of momentum modes is described by a potential

$$V_\pm(x_\pm) = R\sum_{k\geq 1} t_{\pm k} x_\pm^{k/R} \tag{3.8}$$

In the $c = 1$ MQM deformed by (3.8), the energy eigenstates are given by "dressed" wave functions

$$\Psi^E_\pm(x_\pm) = e^{\mp i\varphi_\pm(x_\pm, E)}\psi^E_\pm(x_\pm) \equiv \mathcal{W}_\pm\psi^E_\pm(x_\pm) \tag{3.9}$$

where the phases $\varphi_\pm$ are of the form

$$\varphi_\pm(x_\pm; E) = V_\pm(x_\pm) + \frac{1}{2}\phi(E) - R\sum_{k\geq 1}\frac{1}{k}v_{\pm k}(E)x_\pm^{-k/R} \tag{3.10}$$

In above $\phi(E)$ is a constant phase shift. The terms involving $v_{\pm k}$ vanish as $x_\pm \to \infty$. Semiclassically the perturbation (3.8) corresponds to deformed fermi sea profile

$$\begin{aligned} x_+x_- &= \mu + x_\pm \partial\varphi_\pm(x_\pm;\mu) \\ &= \mu + \sum_{k\geq 1} kt_{\pm k} x_\pm^{k/R} + \sum_{k\geq 1} v_{\pm k}(E) x_\pm^{-k/R} \end{aligned} \tag{3.11}$$

The compatibility of the two equations with + and − signs in the subscripts puts constraints on $v_{\pm k}$ in terms of $t_{\pm k}$'s. This comes from the requirement that $\Psi^E_\pm$ are the wave functions of the same state in $x_\pm$-representations. They are related by $\hat{S}\Psi^E_+ = \Psi^E_-$, or equivalently

$$\mathcal{W}_- = \mathcal{W}_+ \hat{\mathcal{R}} \tag{3.12}$$

Note that this constraint is essentially determined from the reflection coefficient $\mathcal{R}(E)$.

To see the integrable structure of Toda lattice hierarchy, we shall recast the above in terms of operators on the E-space. For example, it follows from (3.5) that $\hat{x}_\pm$ are represented as shift operators $\hat{\omega}^{\pm 1} = e^{\mp i\partial_E}$. One can define a Lax pair

$$\begin{aligned} L_\pm &= \mathcal{W}_\pm \hat{\omega}^{\pm 1} \mathcal{W}_\pm^{-1} = e^{\mp i\phi/2} \hat{\omega}^{\pm 1} \left(1 + \sum_{k\geq 1} a_{\pm k} \hat{\omega}^{\mp k/R} \right) e^{\pm i\phi/2} \\ M_\pm &= -\mathcal{W}_\pm \hat{E} \mathcal{W}_\pm^{-1} = \sum_{k\geq 1} kt_{\pm k} L_\pm^{k/R} - \hat{E} + \sum_{k\geq 1} v_{\pm k} L_\pm^{-k/R} \end{aligned} \tag{3.13}$$

which satisfy the commutation relation

$$[L_\pm, M_\pm] = \pm i L_\pm \tag{3.14}$$

Recall that the dressing operators $\mathcal{W}_\pm$ in terms of $\hat{E}$ and $\hat{\omega}$ are of the form

$$\mathcal{W}_\pm = e^{\mp i\phi/2} \left(1 + \sum_{k\geq 1} w_{\pm k} \hat{\omega}^{\mp k/R} \right) e^{\mp iR \sum_{k\geq 1} t_{\pm k} \hat{\omega}^{\pm k/R}} \tag{3.15}$$

The integrable flow equation of the Lax operators is

$$\partial_{t_n} L_\pm = [H_n, L_\pm] \tag{3.16}$$

where

$$H_n = (\partial_{t_n} \mathcal{W}_+) \mathcal{W}_+^{-1} = (\partial_{t_n} \mathcal{W}_-) \mathcal{W}_-^{-1} \tag{3.17}$$

By the virtue of (3.12), the generator of the Toda flow H_n is the same for both L_+ and L_-. By a standard argument (for example, see [23]), it follows from the structure of (3.15) and (3.13) that H_n are of the upper or lower triangular form

$$\begin{aligned} H_n &= (L_+^{n/R})_> + \frac{1}{2}(L_+^{n/R})_0, \\ H_{-n} &= (L_-^{n/R})_< + \frac{1}{2}(L_-^{n/R})_0. \end{aligned} \qquad (n>0) \tag{3.18}$$

It also follows from (3.16) that the zero-curvature conditions on H_n are automatically satisfied.

The "almost lower triangular" structure of lax operators $L_\pm$ (3.13), together with (3.16) and the important relation (3.18), define a Toda lattice hierarchy. The two Lax operators $L_+[t_n]$ and $L_-[t_n]$ are in addition constrained by (3.12). Using the functional relation $R(E-i) = (-E+i/2)R(E)$, the constraints on Lax operators can be expressed in terms of the so called string equations, in this case given by

$$\begin{aligned} M_+ = M_- &= \frac{1}{2}(L_+L_- + L_-L_+), \\ [L_+, L_-] &= -i. \end{aligned} \tag{3.19}$$

These conditions define a constrained Toda hierarchy.

The parameters $\phi, v_{\pm k}$ appearing in $\mathcal{W}_\pm$ are determined in terms of $t_{\pm k}$ via the Toda flow (3.16) and the initial condition $L_\pm|_{t_n=0} = e^{\mp i\phi_0/2}\hat{\omega}^{\pm 1}e^{\pm i\phi_0/2}$ where $e^{i\phi_0(E)} = \mathcal{R}(E)$. They are related to the Toda τ-function $\tau(E;t_n)^2$ by

$$v_n = \frac{\partial \ln \tau}{\partial t_n}, \qquad \phi(E) = i \ln \frac{\tau(E+i/2R)}{\tau(E-i/2R)} \tag{3.20}$$

where we suppressed the dependence on t_n's. The τ-function satisfies Hirota's bilinear equations [46,44], an infinite set of differential equations in t_n's that completely determine $\tau(E;t_n)$ provided the initial condition $\tau|_{t_n=0}(E)$ for all E. The initial condition for $\tau(E;t_n)$ is essentially equivalent to the constraints imposed by the string equations.

To see how the τ-function is related to the free energy, let us compute the density of states in terms of $\phi(E)$. We shall introduce a cutoff at $x = \sqrt{2\Lambda}$, which is a wall

2 The τ-function appearing here is denoted $\tau'[t]$ in [44]. In section 3.4 we will introduce another τ-function $\tau[t]$ defined using vertex operators, following [45,21]. They are related by $\tau'[t] = \exp(\sum_{n>0} nt_nt_{-n}) \cdot \tau[t]$. The grand canonical partition function is given by $\tau'[t]$. Also the standard form of the τ-function, $\tau_l[t]$, is related to $\tau(E;t_n)$ by $\tau_l[t] = \tau(\mu + il/R; t_n)$.

that reflects all momenta. In light cone variables this imposes a boundary condition at $x_+ = x_- = \sqrt{\Lambda}$,

$$\Psi_+^E(\sqrt{\Lambda}) = \Psi_-^E(\sqrt{\Lambda}) \tag{3.21}$$

$\Psi_\pm^E$ as defined in (3.9) have asymptotic behavior

$$\Psi_\pm^E(\sqrt{\Lambda}) \sim e^{\mp i\phi(E)/2}(\sqrt{\Lambda})^{\pm iE} \times (E \text{ independent piece}) \tag{3.22}$$

It follows form (3.21) that the allowed energies E_n satisfy $i\phi(E_n) - iE_n \ln\Lambda = 2\pi in$. The density of energy eigenstates $\rho(E)$ is then

$$\rho(E) = \frac{\ln\Lambda}{2\pi} - \frac{1}{2\pi}\frac{d\phi(E)}{dE} \tag{3.23}$$

The free energy is given by

$$\begin{aligned} \mathcal{F} &= \int_{-\infty}^{\infty} dE \rho(E) \ln(1 + e^{-2\pi R(E+\mu)}) \\ &= -R\int_{-\infty}^{\infty} dE \frac{\phi(E)}{1 + e^{2\pi R(E+\mu)}} \\ &= i\sum_{n\geq 0} \phi\left(-\mu + (n + \tfrac{1}{2})i/R\right) \\ &= \ln\tau(-\mu; t_n) \end{aligned} \tag{3.24}$$

where in the third line we closed the contour in the upper half plane which picks up the poles at $E = -\mu + (n + \frac{1}{2})i/R$, and in the last line we used the relation (3.20). In our convention μ is the negative chemical potential, which is positive when the fermi level is below the top of the potential ($E = 0$). We see that the perturbed grand partition function is $\mathcal{Z}_\mu[t_n] = \tau(-\mu; t_n)$.

3.2. "Theory II"

Now let us turn to type 0B MQM, also known as "theory of type II"[47]. For the unperturbed Hamiltonian there are two sets of eigenfunctions that classically correspond to fermions in the left and right sector of the fermi sea

$$\begin{aligned} \psi^E_{\pm,>}(x_\pm) &= \frac{1}{\sqrt{2\pi}}\frac{x_\pm^{\pm iE-\frac{1}{2}}}{\sqrt{1+e^{2\pi E}}}, \qquad (x_\pm > 0) \\ \psi^E_{\pm,<}(x_\pm) &= \frac{1}{\sqrt{2\pi}}\frac{(-x_\pm)^{\pm iE-\frac{1}{2}}}{\sqrt{1+e^{2\pi E}}}, \qquad (x_\pm < 0) \end{aligned} \tag{3.25}$$

In the other half of the real $x_\pm$ axis the wave functions are defined by analytic continuation, explicitly

$$\begin{aligned} \psi^E_{\pm,>}(-x_\pm) &= \pm i e^{\pi E}\psi^E_{\pm,>}(x_\pm), \\ \psi^E_{\pm,<}(x_\pm) &= \pm i e^{\pi E}\psi^E_{\pm,<}(-x_\pm), \qquad x_\pm > 0. \end{aligned} \tag{3.26}$$

The wave functions in x_+ and x_- representations, ψ_+ and ψ_-, are not merely related by a reflection because of the quantum tunnelling. The operator $\hat{S}$ relating ψ_+ to ψ_- is defined as (3.3) but with a different kernel

$$K(x_-, x_+) = \frac{1}{\sqrt{2\pi}} e^{ix_+x_-} \tag{3.27}$$

$\hat{S}$ acts on the energy eigenstates as [23]

$$\begin{pmatrix} \hat{S}\psi^E_{+,>}(x_-) \\ \hat{S}\psi^E_{+,<}(x_-) \end{pmatrix} = \mathcal{R}(E) \begin{pmatrix} 1 & -ie^{\pi E} \\ -ie^{\pi E} & 1 \end{pmatrix} \begin{pmatrix} \psi^E_{-,>}(x_-) \\ \psi^E_{-,<}(x_-) \end{pmatrix} \tag{3.28}$$

where

$$\mathcal{R}(E) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\pi}{2}(E-i/2)} \Gamma(iE + \tfrac{1}{2}) \tag{3.29}$$

Perturbatively the left and right side of the fermi sea decouple. This is reflected in the exponential suppression factor $e^{\pi E}$ $(E<0)$ when we analytically continue $x_\pm \to -x_\pm$. $\hat{S}$ is diagonalized by symmetric and antisymmetric eigenfunctions

$$\psi^E_{\pm,s} = \frac{\psi^E_{\pm,>} + \psi^E_{\pm,<}}{\sqrt{2}}, \quad \psi^E_{\pm,a} = \frac{\psi^E_{\pm,>} - \psi^E_{\pm,<}}{\sqrt{2}} \tag{3.30}$$

Then we have

$$\begin{aligned} \hat{S}\psi^E_{+,s}(x_-) &= R_s(E)\psi^E_{-,s}(x_-), \qquad R_s(E) = \sqrt{\frac{2}{\pi}} \cosh[\frac{\pi}{2}(i/2 - E)]\Gamma(iE + \tfrac{1}{2}), \\ \hat{S}\psi^E_{+,a}(x_-) &= R_a(E)\psi^E_{-,a}(x_-), \qquad R_a(E) = \sqrt{\frac{2}{\pi}} \sinh[\frac{\pi}{2}(i/2 - E)]\Gamma(iE + \tfrac{1}{2}). \end{aligned} \tag{3.31}$$

Note that $x_\pm$ act on $\psi^E_{\pm,>}$ as $\hat{\omega}^{\pm 1} = e^{\mp i\partial_E}$ and on $\psi^E_{\pm,<}$ as $-\hat{\omega}^{\pm 1} = -e^{\mp i\partial_E}$, or equivalently

$$\hat{x}_\pm \psi^E_{\pm,s} = \psi^{E\mp i}_{\pm,a}, \qquad \hat{x}_\pm \psi^E_{\pm,a} = \psi^{E\mp i}_{\pm,s} \tag{3.32}$$

Now let us turn on perturbations by closed string momentum modes. The dressed wave functions are of the form

$$\begin{aligned} \Psi^E_{\pm,s}(x_\pm) &= e^{\mp\varphi^s_\pm(\hat{\omega}^{\pm 1};E)}\psi^E_{\pm,s}(x_\pm) \equiv \mathcal{W}^s_\pm \psi^E_{\pm,s}(x_\pm), \\ \Psi^E_{\pm,a}(x_\pm) &= e^{\mp\varphi^a_\pm(\hat{\omega}^{\pm 1};E)}\psi^E_{\pm,a}(x_\pm) \equiv \mathcal{W}^a_\pm \psi^E_{\pm,a}(x_\pm), \end{aligned} \tag{3.33}$$

where

$$\varphi^s_\pm(\hat\omega^{\pm 1};E) = R\sum_{k\geq 1} t^s_{\pm k}\hat\omega^{\pm k/R} + \frac{1}{2}\phi^s(E) - R\sum_{k\geq 1}\frac{1}{k}v^s_{\pm k}(E)\hat\omega^{\mp k/R}$$
$$\varphi^a_\pm(\hat\omega^{\pm 1};E) = R\sum_{k\geq 1} t^a_{\pm k}\hat\omega^{\pm k/R} + \frac{1}{2}\phi^a(E) - R\sum_{k\geq 1}\frac{1}{k}v^a_{\pm k}(E)\hat\omega^{\mp k/R} \tag{3.34}$$

$t^s_{\pm k}, t^a_{\pm k}$ parameterize the symmetric and antisymmetric part of the asymptotic perturbations of the fermi sea, respectively. $\mathcal{W}_\pm$ again have the structure of (3.15). The Lax pairs $(L_\pm, M_\pm)$ are defined the same way as (3.13), and satisfy the relations (3.14), (3.16), (3.17). Now we have two sets of generators of the flow, H^s_n and H^a_n (associated to t^a_n and t^a_n). Since $L_\pm$ and $\hat\omega$ are block diagonal in the (ψ_s, ψ_a) basis,

$$H_n = \begin{pmatrix} H^s_n & 0 \\ 0 & H^a_n \end{pmatrix} \tag{3.35}$$

still satisfy the upper or lower triangular relations (3.18). They define two independent Toda flows, associated to the symmetric and antisymmetric perturbations of the fermi sea.

However, $\hat\omega^{\pm 1}$ is no longer the same as $\hat x_\pm$ since the latter interchanges ψ_s with ψ_a. Consequently the functional constraint on the reflection coefficients is modified to

$$R_s(E-i) = (-E+i/2)R_a(E),$$
$$R_a(E-i) = (-E+i/2)R_s(E). \tag{3.36}$$

So the strings equations (3.19) no longer hold in the case of "theory II". They are modified to

$$M^{s,a}_+ = M^{s,a}_-,$$
$$L^s_+L^s_- = \tanh[\frac{\pi}{2}(i/2 - M^s_\pm)]\,(-M^s_\pm + i/2),$$
$$L^s_-L^s_+ = \coth[\frac{\pi}{2}(i/2 - M^s_\pm)]\,(-M^s_\pm - i/2),$$
$$L^a_+L^a_- = \coth[\frac{\pi}{2}(i/2 - M^a_\pm)]\,(-M^a_\pm + i/2),$$
$$L^a_-L^a_+ = \tanh[\frac{\pi}{2}(i/2 - M^a_\pm)]\,(-M^a_\pm - i/2). \tag{3.37}$$

Perturbatively, i.e. in the limit $e^{\pi E} \ll 1$, these equations reduce to (3.19). They define two independent constrained Toda hierarchies, with the Lax operators acting on ψ_s and ψ_a respectively. The perturbed grand canonical partition function is the product of the two τ-functions,

$$\mathcal{Z}_\mu[t^s, t^a] = \tau_s(\mu; t^s)\tau_a(\mu; t^a) \tag{3.38}$$

As remarked in [25], this doesn't mean that the NS-NS and R-R closed string modes decouple from each other. The symmetric and antisymmetric perturbations of the fermi sea are mixtures of NS-NS and R-R fields in spacetime, due to the nonlinearity of bosonization.

3.3. Type 0A theory

Type 0A MQM can be represented by non-relativistic free fermions moving in a two dimensional upside-down harmonic oscillator potential. The Hamiltonian is

$$\hat{H} = \frac{1}{2}(\hat{p}_x^2 + \hat{p}_y^2) - \frac{1}{4\alpha'}(\hat{x}^2 + \hat{y}^2) \tag{3.39}$$

The theory has different independent sectors labelled by net D0-brane charge q, which is the same as the angular momentum $\hat{J} = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$ [8]. We shall mostly focus on the case where there is no net D0-brane charge, namely the $J = 0$ sector.

It is again convenient to define light cone variables

$$\hat{x}_\pm = \frac{\frac{1}{\sqrt{2\alpha'}}\hat{x} \pm \hat{p}_x}{\sqrt{2}}, \quad \hat{y}_\pm = \frac{\frac{1}{\sqrt{2\alpha'}}\hat{y} \pm \hat{p}_y}{\sqrt{2}} \tag{3.40}$$

and

$$\hat{z}_\pm = \hat{x}_\pm + i\hat{y}_\pm, \quad \hat{\bar{z}}_\pm = \hat{x}_\pm - i\hat{y}_\pm \tag{3.41}$$

We have commutators

$$\begin{aligned}
&[\hat{x}_+, \hat{x}_-] = [\hat{y}_+, \hat{y}_-] = i/\sqrt{2\alpha'}, \\
&[\hat{z}_+, \hat{z}_-] = [\hat{\bar{z}}_+, \hat{\bar{z}}_-] = 0, \\
&[\hat{z}_+, \hat{\bar{z}}_-] = [\hat{\bar{z}}_+, \hat{z}_-] = 2i/\sqrt{2\alpha'}.
\end{aligned} \tag{3.42}$$

In light cone variables the Hamiltonian is written as

$$\begin{aligned}
\hat{H} &= -(\hat{x}_+\hat{x}_- + \hat{y}_+\hat{y}_- - \frac{i}{\sqrt{2\alpha'}}) \\
&= -\frac{1}{2}(\hat{z}_+\hat{\bar{z}}_- + \hat{\bar{z}}_+\hat{z}_- - \frac{2i}{\sqrt{2\alpha'}}) \\
&= \frac{i}{\sqrt{2\alpha'}}(z_+\frac{\partial}{\partial z_+} + \bar{z}_+\frac{\partial}{\partial \bar{z}_+} + 1)
\end{aligned} \tag{3.43}$$

where the last line is written in $(z_+, \bar{z}_+)$-representation. In addition, we have commutation relations

$$[\hat{J}, \hat{z}_\pm] = z_\pm, \quad [\hat{J}, \hat{\bar{z}}_\pm] = -\hat{\bar{z}}_\pm, \tag{3.44}$$

or in $(z_+, \bar{z}_+)$ representation

$$\hat{J} = z_+\frac{\partial}{\partial z_+} - \bar{z}_+\frac{\partial}{\partial \bar{z}_+} \tag{3.45}$$

The wave function of a state can be expressed either in $(z_+, \bar{z}_+)$ representation or in $(z_-, \bar{z}_-)$ representation, denoted by $\psi_+(z_+, \bar{z}_+)$ and $\psi_-(z_-, \bar{z}_-)$ respectively. Since we restrict our wave functions to have zero angular momentum, it must be of the form $\psi_\pm(z_\pm, \bar{z}_\pm) = \psi_\pm(z_\pm \bar{z}_\pm)$. The energy eigenstates are given by (setting $\alpha' = 2$)

$$\psi_\pm^E \sim z_\pm^{\pm iE - \frac{1}{2}} \bar{z}_\pm^{\pm iE - \frac{1}{2}} \tag{3.46}$$

The wave functions in $(z_+, \bar{z}_+)$ and $(z_-, \bar{z}_-)$ representations are related by

$$\begin{aligned} \psi_-(z_-, \bar{z}_-) &= (\hat{S}\psi_+)(z_-, \bar{z}_-) \\ &= \int dz_+ d\bar{z}_+ K(\bar{z}_-, z_+) K(z_-, \bar{z}_+) \psi_+(z_+, \bar{z}_+) \end{aligned} \tag{3.47}$$

where $K(z_-, z_+) = \frac{1}{\sqrt{2\pi}} e^{iz_- z_+}$. Acting on energy eigenstates, we have

$$\hat{S}\psi_+^E = \mathcal{R}(E)\psi_-^E, \quad \mathcal{R}(E) = \frac{\Gamma(iE + \frac{1}{2})}{\Gamma(-iE + \frac{1}{2})} \tag{3.48}$$

This is the same as the phase shift found in [8].

Now we can study perturbations of the fermi sea. Let us consider dressed wave functions

$$\Psi_\pm^E = e^{\mp\varphi(z_\pm \bar{z}_\pm; E)} \psi_\pm^E \equiv \mathcal{W}_\pm \psi_\pm^E \tag{3.49}$$

where the phases $\varphi_\pm$ have Laurent expansion

$$\varphi_\pm(z_\pm \bar{z}_\pm; E) = \frac{1}{2}\phi(E) + R \sum_{k \geq 1} t_{\pm k} (z_\pm \bar{z}_\pm)^{k/R} - R \sum_{k \geq 1} \frac{1}{k} v_{\pm k} (z_\pm \bar{z}_\pm)^{-k/R} \tag{3.50}$$

$t_{\pm k}$ parameterize the asymptotic perturbation by momentum modes of NS-NS scalars, corresponding to the operator

$$V_{p=k/R} = \mathrm{Tr}(Z_\pm \bar{Z}_\pm)^{|k|/R} \tag{3.51}$$

in type 0A MQM, where the sign of the subscripts depends on the sign of k. $(\hat{z}_\pm \hat{\bar{z}}_\pm)$ can be represented as shift operators $\hat{\omega}^{\pm 1}$, where $\hat{\omega}$ acts on energy eigenstates as $\hat{\omega}^{\pm 1} \psi_\pm^E = \psi_\pm^{E \mp i}$. We have $\hat{\omega} = e^{-i\partial_E}$ and the commutation relations

$$[\hat{\omega}^\pm, -\hat{E}] = \pm i \hat{\omega}^\pm \tag{3.52}$$

As before we can define a Lax pair

$$L_\pm = \mathcal{W}_\pm \hat{\omega}^\pm \mathcal{W}_\pm^{-1}, \quad M_\pm = -\mathcal{W}_\pm \hat{E} \mathcal{W}_\pm^{-1} \tag{3.53}$$

The dressing operators $\mathcal{W}_\pm$ satisfy the constraint

$$\mathcal{W}_- = \mathcal{W}_+ \cdot \mathcal{R}(\hat{E}) \tag{3.54}$$

This again defines the structure of constrained Toda lattice hierarchy. But the string equation (3.19) is modified to the following

$$M_+ = M_-, \quad [L_+, L_-] = 2iM_\pm, \quad \{L_+, L_-\} = 2M_\pm^2 - \frac{1}{2}. \tag{3.55}$$

The density of states, and hence the free energy, is related to the phase $\phi(E)$ in the standard way. To compute the density of states, we shall introduce a cutoff at $x^2 + y^2 = \Lambda$. The cutoff wall reflects all the momenta, so we have $xp_x + yp_y = 0$ as well. Further we demand the vanishing of angular momentum $xp_y - yp_x = 0$. The combination of these is equivalent to $z_+\bar{z}_+ = z_-\bar{z}_- = \Lambda$. We can impose a boundary condition at the wall

$$\Psi_+^E(\Lambda) = \Psi_-^E(\Lambda) \tag{3.56}$$

It follows that the density of states depends on E the same way as in (3.23).

It is not hard to generalize the above construction to sectors of nonzero net D0-brane charge q. These backgrounds are identified as extremal black holes in type 0A string theory[12]. In this case, the energy eigenstates (carrying angular momentum q) are

$$\psi_\pm^E \sim z_\pm^{\pm iE + q/2 - \frac{1}{2}} \bar{z}_\pm^{\pm iE - q/2 - \frac{1}{2}} \tag{3.57}$$

The reflection coefficients are computed from (3.47) to be

$$\mathcal{R}(E) = \frac{\Gamma(iE + \frac{q+1}{2})}{\Gamma(-iE + \frac{q+1}{2})} \tag{3.58}$$

The Lax operators are defined as before, but the string equations become

$$M_+ = M_-, \quad [L_+, L_-] = 2iM_\pm, \quad \{L_+, L_-\} = 2M_\pm^2 + \frac{q^2 - 1}{2}. \tag{3.59}$$

Interestingly, $\{L_+, L_-, M \equiv M_\pm\}$ form a representation of $sl(2, \mathbf{R})$, with isospin $l = (|q| - 1)/2$.

3.4. Type 0B IMM perturbed by closed string momentum modes

Following [20,21], we shall integrate out the tachyons $t^{(n)}$, $t^{(n)\dagger}$ in the type 0B IMM (2.10) by analytic continuation, and obtain a matrix integral in terms of the eigenvalues of X and $\tilde{X}$,

$$\begin{aligned}
&\int \prod dx_i d\tilde{x}_i \prod_{i<j} \sin^2(\frac{x_i - x_j}{2R}) \sin^2(\frac{\tilde{x}_i - \tilde{x}_j}{2R}) \prod_{i,j} \prod_{n=-\infty}^{\infty} \frac{1}{(2\pi n R + x_i - \tilde{x}_j)^2 - a^2} \\
&= \int \prod dx_i d\tilde{x}_i \prod_{i<j} \sin^2(\frac{x_i - x_j}{2R}) \sin^2(\frac{\tilde{x}_i - \tilde{x}_j}{2R}) \prod_{i,j} \frac{1}{\sin^2[(x_i - \tilde{x}_j)/2R] - \sin^2(a/2R)}
\end{aligned} \tag{3.60}$$

Roughly speaking we have a system of N eigenvalues x_i and $\tilde{x}_j$, they repel eigenvalues of the same type through the F-P determinant and interact through some effective potential. The analytic continuation made above is quite naive. As a trade off, (3.60) is not unambiguously defined, and we have to give a correct contour prescription. This will be considered in the next section, for now we still formally work with (3.60).

Let us write the partition function in terms of the integral over eigenvalues of $U, \tilde{U}$, $z_j = e^{ix_j/R}, \tilde{z}_j = e^{i\tilde{x}_j/R}$,

$$\begin{aligned}
Z_N &= \frac{1}{(N!)^2} \prod_{k=1}^{N} \oint \frac{dz_k}{2\pi i z_k} \oint \frac{d\tilde{z}_k}{2\pi i \tilde{z}_k} \prod_{i<j} |z_i - z_j|^2 |\tilde{z}_i - \tilde{z}_j|^2 \\
&\quad \times \prod_{i,j} \frac{1}{|z_i q^{1/2} - \tilde{z}_j q^{-1/2}| \cdot |z_i q^{-1/2} - \tilde{z}_j q^{1/2}|} \\
&= \frac{1}{(N!)^2} \prod_{k=1}^{N} \oint \frac{dz_k}{2\pi i} \oint \frac{d\tilde{z}_k}{2\pi i} \prod_{i \neq j} (z_i - z_j)(\tilde{z}_i - \tilde{z}_j) \\
&\quad \times \prod_{i,j} \frac{1}{(z_i q^{1/2} - \tilde{z}_j q^{-1/2})(z_i q^{-1/2} - \tilde{z}_j q^{1/2})}.
\end{aligned} \tag{3.61}$$

where $q = e^{ia/R} = e^{\pi i \sqrt{2\alpha'}/R}$. Consider the perturbation by momentum modes (winding modes in the T-dual type 0A theory)

$$\sum_{n \in \mathbf{Z}} \lambda_n \mathrm{Tr} U^n + \tilde{\lambda}_n \mathrm{Tr} \tilde{U}^n \tag{3.62}$$

The generating functional is then

$$
\begin{aligned}
Z_N[\lambda,\tilde{\lambda}] &= \frac{1}{(N!)^2}\prod_{k=1}^{N}\oint\frac{dz_k}{2\pi i}\oint\frac{d\tilde{z}_k}{2\pi i}e^{u(z_k)+\tilde{u}(\tilde{z}_k)}\prod_{i\neq j}(z_i-z_j)(\tilde{z}_i-\tilde{z}_j)\\
&\quad\times\prod_{i,j}\frac{1}{(z_iq^{1/2}-\tilde{z}_jq^{-1/2})(z_iq^{-1/2}-\tilde{z}_jq^{1/2})}\\
&= \frac{1}{(N!)^2}\prod_{k=1}^{N}\oint\frac{dz_k}{2\pi i}\oint\frac{d\tilde{z}_k}{2\pi i}e^{u(z_k)+\tilde{u}(\tilde{z}_k)}\\
&\quad\times\det{}_{ij}\left(\frac{1}{z_iq^{1/2}-\tilde{z}_jq^{-1/2}}\right)\det{}_{ij}\left(\frac{1}{z_iq^{-1/2}-\tilde{z}_jq^{1/2}}\right)
\end{aligned}
\tag{3.63}
$$

where

$$
u(z)=\sum_n\lambda_nz^n,\quad \tilde{u}(\tilde{z})=\sum_n\tilde{\lambda}_n\tilde{z}^n,
\tag{3.64}
$$

and we have used the Cauchy identity

$$
\frac{\Delta(a)\Delta(b)}{\prod_{i,j}(a_i-b_j)}=\det{}_{ij}\left(\frac{1}{a_i-b_j}\right)
\tag{3.65}
$$

It is most convenient to consider the grand canonical partition function

$$
Z_\mu[\lambda,\tilde{\lambda}]=\sum_{N=0}^{\infty}e^{\pi\sqrt{2\alpha'}\mu N}Z_N[\lambda,\tilde{\lambda}]
\tag{3.66}
$$

where μ is the chemical potential of type 0B theory, related to the one of the T-dual type 0A theory by $\mu'=\mu R/\sqrt{2\alpha'}$. For reasons that will become clear shortly, let us define

$$
\lambda_n=2t_n+(q^n+q^{-n})\tilde{t}_n,\quad \tilde{\lambda}_n=-2\tilde{t}_n-(q^n+q^{-n})t_n
\tag{3.67}
$$

and a "τ-function"

$$
\begin{aligned}
\tau_l[t,\tilde{t}] &= e^{-\sum_n n[2t_nt_{-n}+2\tilde{t}_n\tilde{t}_{-n}+(q^n+q^{-n})(t_n\tilde{t}_{-n}+\tilde{t}_nt_{-n})]}\sum_{N=0}^{\infty}(q^{2l}e^{\pi\sqrt{2\alpha'}\mu})^NZ_N[t,\tilde{t}]\\
&= e^{-\sum_n n[2t_nt_{-n}+2\tilde{t}_n\tilde{t}_{-n}+(q^n+q^{-n})(t_n\tilde{t}_{-n}+\tilde{t}_nt_{-n})]}Z_{\mu+2il/R}[t,\tilde{t}]
\end{aligned}
\tag{3.68}
$$

One might hope that the grand canonical partition function $Z_\mu[t,\tilde{t}]$ is the τ-function of some integrable hierarchy. This is not quite true. The perturbations that generate the integrable flows are related to $t_n,\tilde{t}_n$'s through some bosonization maps, as we will show below.

It is useful to rewrite the partition function in vertex operator formalism, analogous to the case of $c = 1$ string [21]. One can introduce two independent 2D chiral bosons $\varphi_{1,2}(z)$, with mode expansion

$$\varphi_{1,2}(z) = \hat{q}_{1,2} + \hat{p}_{1,2} \ln z + \sum_{n \neq 0} \frac{H_n^{(1,2)}}{n} z^{-n} \tag{3.69}$$

The vacuum $|l\rangle$ is defined by

$$H_n^{(1),(2)}|l\rangle = 0 \quad (n > 0), \qquad \hat{p}_1|l\rangle = \hat{p}_2|l\rangle = l|l\rangle. \tag{3.70}$$

We could have considered more general vacuum state $|l_1, l_2\rangle$, but that would be unnecessary for our purpose. We further define

$$\begin{aligned} \phi(z) &= \varphi_1(q^{1/2}z) - \varphi_2(q^{-1/2}z), \\ \tilde{\phi}(z) &= \varphi_1(q^{-1/2}z) - \varphi_2(q^{1/2}z), \end{aligned} \tag{3.71}$$

and $H_n, \tilde{H}_n$ the corresponding creation and annihilation operators in the mode expansion of $\phi, \tilde{\phi}$,

$$\begin{aligned} H_n &= q^{-n/2} H_n^{(1)} - q^{n/2} H_n^{(2)} \\ \tilde{H}_n &= q^{n/2} H_n^{(1)} - q^{-n/2} H_n^{(2)} \end{aligned} \tag{3.72}$$

Note that ϕ and $\tilde{\phi}$ are not independent fields. We have commutation relations

$$[H_n, H_m] = 2n\delta_{n+m} = [\tilde{H}_n, \tilde{H}_m], \qquad [H_n, \tilde{H}_m] = (q^n + q^{-n})n\delta_{n+m}. \tag{3.73}$$

Using the operator

$$\hat{\mathbf{g}}' = \sum_{N=0}^{\infty} \frac{1}{(N!)^2} \left(q^{-i\mu R} \oint \frac{dz}{2\pi} : e^{\phi(z)} : \oint \frac{d\tilde{z}}{2\pi} : e^{-\tilde{\phi}(\tilde{z})} : \right)^N \tag{3.74}$$

the "τ-function" (3.68) can be written as

$$\tau_l[t, \tilde{t}] = \langle l | e^{-\sum_{n>0} t_n H_n + \tilde{t}_n \tilde{H}_n} \hat{\mathbf{g}}' \, e^{\sum_{n>0} t_{-n} H_{-n} + \tilde{t}_{-n} \tilde{H}_{-n}} | l \rangle \tag{3.75}$$

To see this, observe that the contractions among the operators e^{ϕ} and $e^{-\tilde{\phi}}$ in (3.74) give the integrand in (3.61), and commuting $\mathbf{g}'$ through $e^{-\sum_{n>0} t_n H_n + \tilde{t}_n \tilde{H}_n}$ and $e^{\sum_{n<0} t_n H_n + \tilde{t}_n \tilde{H}_n}$ give the momentum mode perturbation $e^{u+\tilde{u}}$ appearing in (3.63). The symmetry $\phi \to \phi + a$, $\tilde{\phi} \to \tilde{\phi} + a$ is respected by the vacuum state $|l\rangle$. Therefore, we can replace $\hat{\mathbf{g}}'$ by

$$\hat{\mathbf{g}} = \exp\left(q^{-i\mu R/2} \oint \frac{dz}{2\pi} : e^{\phi(z)} + e^{-\tilde{\phi}(z)} : \right) \tag{3.76}$$

in the partition function (3.75). The partition function in sectors of nonzero net D-instanton number can be obtained from the general vacuum state $|l_1, l_2\rangle$. The latter are closely related to the solitonic sectors of the type 0B bosonic Hilbert space discussed in [25]. Of course, the τ-function (3.75) does not simply factorize into two components involving only t_n and $\tilde{t}_n$ respectively, because ϕ and $\tilde{\phi}$ have nontrivial OPE.

The exponent in $\hat{\mathbf{g}}$ can be interpreted as a Hamiltonian. It is clearer to fermionize φ_1, φ_2:

$$\begin{aligned} e^{\varphi_1(z)} &\simeq \psi_1(z), \quad e^{-\varphi_1(z)} \simeq \bar{\psi}_1(z), \\ e^{\varphi_2(z)} &\simeq \psi_2(z), \quad e^{-\varphi_2(z)} \simeq \bar{\psi}_2(z). \end{aligned} \tag{3.77}$$

The vacuum $|l\rangle$ satisfies

$$\psi_{1,r}|l\rangle = \psi^\dagger_{1,-r}|l\rangle = \psi_{2,r}|l\rangle = \psi^\dagger_{2,-r}|l\rangle = 0, \quad r \in \mathbf{Z} + \tfrac{1}{2},\ r > l. \tag{3.78}$$

In terms of $\psi_{1,2}$, we can write

$$\hat{\mathbf{g}} = \exp\left\{ q^{-i\mu R/2} \oint \frac{dz}{2\pi} \left[-\bar{\psi}_2(q^{-1/2}z)\psi_1(q^{1/2}z) + \bar{\psi}_1(q^{-1/2}z)\psi_2(q^{1/2}z) \right] \right\} \tag{3.79}$$

We can interpret (3.75) as a partition function of the fermions ψ_1, ψ_2. The integral in the exponent of $\hat{\mathbf{g}}$ can be written in first-quantized form as

$$\begin{aligned} \hat{H} &= e^{\frac{i}{2}\ln q(\hat{z}\hat{p}_z + \hat{p}_z\hat{z})} P_{12} \\ &= e^{-\frac{a}{2R}(\hat{z}\hat{p}_z + \hat{p}_z\hat{z})} P_{12} \end{aligned} \tag{3.80}$$

where $P_{12} : \psi_1 \to \psi_2, \psi_2 \to -\psi_1$, and we have used $q = e^{ia/R}$. This "Hamiltonian" can be compared to the $c = 1$ string case[21], where one can write the partition function in terms of vertex operators involving a single chiral boson $\varphi(z)$ or its fermionization $\psi(z)$, and the corresponding "Hamiltonian" is $\hat{H} = e^{-\frac{a}{2R}(\hat{z}\hat{p}_z + \hat{p}_z\hat{z})}$.

Since $\psi_\pm = \frac{1}{\sqrt{2}}(\psi_1 \pm i\psi_2)$ diagonalize P_{12}, it is clear that $\tau_l[t_n = \tilde{t}_n = 0]$ factorizes as the product of partition functions involving ψ_+ and ψ_- separately. In the type 0B fermi sea picture, ψ_+, ψ_- should correspond to symmetric and antisymmetric perturbations of the fermi sea. Indeed, as we will show in the next section, the partition function involving ψ_+ (ψ_-) agrees with the partition functions in type 0B MQM involving only symmetric (antisymmetric) perturbations.

In the Hamiltonian interpretation, the "initial" state in the expression (3.75) for the τ-function is

$$e^{\sum_{n>0} t_{-n}H_{-n} + \tilde{t}_{-n}\tilde{H}_{-n}}|l\rangle = e^{\sum_{n>0} t^{(1)}_{-n}H^{(1)}_{-n} + t^{(2)}_{-n}H^{(2)}_{-n}}|l\rangle \tag{3.81}$$

where

$$t_n^{(1)} = q^{-n/2} t_n + q^{n/2} \tilde{t}_n, \quad t_n^{(2)} = -q^{-n/2} \tilde{t}_n - q^{n/2} t_n \tag{3.82}$$

As we have seen, $\psi_1 \pm i\psi_2$ correspond to symmetric and antisymmetric perturbations of the fermi sea. It is natural to expect that ψ_1 and ψ_2 correspond to perturbations of the left and right sector of the fermi sea (decoupled perturbatively). Consequently the bosonized modes $H_n^{(1)}, H_n^{(2)}$ are linear combinations of momentum modes of NS-NS and R-R scalars in spacetime. From (3.67) and (3.82), the perturbation parameters $t_n^{(1)}, t_n^{(2)}$ are related to $\lambda_n, \tilde{\lambda}_n$ in IMM by

$$\begin{aligned} \lambda_n &= q^{n/2} t_n^{(1)} - q^{-n/2} t_n^{(2)} \\ \tilde{\lambda}_n &= -q^{-n/2} t_n^{(1)} + q^{n/2} t_n^{(2)} \end{aligned} \tag{3.83}$$

This also leads us to the identification

$$\begin{aligned} a_{L,n} &\sim q^{n/2} \mathrm{Tr} e^{inX/R} - q^{-n/2} \mathrm{Tr} e^{in\tilde{X}/R}, \\ a_{R,n} &\sim -q^{-n/2} \mathrm{Tr} e^{inX/R} + q^{n/2} \mathrm{Tr} e^{in\tilde{X}/R}, \end{aligned} \tag{3.84}$$

or

$$\begin{aligned} \mathrm{Tr} e^{inX/R} &\sim \frac{q^{n/2} a_{L,n} + q^{-n/2} a_{R,n}}{q^n - q^{-n}}, \\ \mathrm{Tr} e^{in\tilde{X}/R} &\sim \frac{q^{-n/2} a_{L,n} + q^{n/2} a_{R,n}}{q^n - q^{-n}}, \end{aligned} \tag{3.85}$$

where $a_{L,n} \pm a_{R,n}$ are the momentum modes of the NS-NS and R-R scalar with momenta $p = n/R$. Note that the Lorentzian energy is $E = in/R$. To analytically continue to Lorentzian signature, we have $q^n = e^{2\pi E}$. Perturbatively positive powers of q can be neglected ($E < 0$ below the top of the potential). In this limit we have approximately $\mathrm{Tr} e^{inX/R} \sim -e^{\pi E} a_{R,n}$, $\mathrm{Tr} e^{in\tilde{X}/R} \sim -e^{\pi E} a_{L,n}$. As will be shown in section 5, the relations (3.84) precisely reproduce the ZZ boundary state in super-Liouville theory.

We should remind the reader that $H_n^{(1)} \pm i H_n^{(2)}$ are *not* the bosonization of $\psi_1 \pm i\psi_2$, and that $t_n^{(1)} \pm i t_n^{(2)}$ are not the same as t_n^s, t_n^a defined in section 3.2. The operators $\mathrm{Tr} e^{inX/R}, \mathrm{Tr} e^{in\tilde{X}/R}$ in IMM correspond to linear combinations of NS-NS and R-R modes, whereas in MQM the fluctuations of the fermi sea are related to closed string fields in spacetime by bosonization. Due to this bosonization relation between $t_n, \tilde{t}_n$ and t_n^s, t_n^a, the partition function of IMM does not factorize in a manifest way. However if we replace the "initial state" (and similarly the "final state") in (3.75) by

$$e^{\sum_{n>0} t^s_{-n} H^+_{-n} + t^a_{-n} H^-_{-n}} |l\rangle \tag{3.86}$$

where $H_n^\pm$ are the modes of the bosonization of $\psi_\pm$, then the partition function will factorize into two τ-functions of constrained Toda hierarchies that depend only on t^s or t^a, just as in type 0B MQM. To show that the perturbed grand partition functions in type 0B IMM and MQM are actually the same, it remains to show that the constraints (string equations) of the IMM agree with those of MQM.

3.5. String equations for IMM

In this subsection we derive the string equations for type 0A and 0B IMM, by computing the commutator $[L_+, L_-]$ at $t_n = 0$. Using the general form of Lax operators (3.13), we know that in the absence of perturbation,

$$
\begin{aligned}
L_\pm &= e^{\mp i\phi_0/2}\hat{\omega}^{\pm 1} e^{\pm i\phi_0/2}, \\
[L_+, L_-] &= e^{i\phi_0(E+i) - i\phi_0(E)} - e^{i\phi_0(E) - i\phi_0(E-i)}.
\end{aligned}
\tag{3.87}
$$

Using the relation (3.24), we have

$$
\phi(\mu) = -i\left[\mathcal{F}(\mu + \frac{i}{2R}) - \mathcal{F}(\mu - \frac{i}{2R})\right] \tag{3.88}
$$

where $\mathcal{F}(\mu) = \ln \mathcal{Z}(\mu)$ is the unperturbed free energy. To show that the IMM string equations are the same as the ones for MQM found in previous sections, it suffices to show that $\phi(\mu)$ as given by (3.88) (with $\mathcal{F}$ evaluated from IMM) does lead to the correct reflection coefficient $\mathcal{R}(E) = e^{i\phi(E)}$.

Let us first consider $c = 1$ or type 0A IMM, which is already done in [20]. The grand partition function takes the form of a Freholm determinant

$$
\mathcal{Z}(\mu) = \det(1 + q^{i\mu R}\hat{K}) \tag{3.89}
$$

where $q = e^{2\pi i/R}$, μ is the chemical potential of the theory on radius R (the chemical potential of the T-dual theory is $\mu' = \mu R$). $\hat{K}$ is defined by

$$
(\hat{K}f)(z) = -\oint \frac{dz'}{2\pi i}\frac{f(z')}{q^{1/2}z - q^{-1/2}z'} \tag{3.90}
$$

The contour prescription of [20] (see also [41]) is to add a small imaginery part to R so that $|q| < 1$. A basis that diagonalizes $\hat{K}$ consists of the monomials z^n, with

$$
\hat{K}z^n = \begin{cases} q^{n+\frac{1}{2}}z^n, & n \geq 0 \\ 0, & n < 0 \end{cases} \tag{3.91}
$$

The free energy is given by

$$\begin{aligned}\mathcal{F}(\mu) &= \sum_{n\geq 0} \ln(1+q^{i\mu R+n+\frac{1}{2}}) \\ &= \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \frac{q^{im\mu R}}{q^{m/2}-q^{-m/2}} \\ &= \int_C \frac{dt}{t} \frac{e^{i\mu t}}{4\sinh(t/2)\sinh(t/2R)}\end{aligned} \tag{3.92}$$

where C is the contour that picks up the poles of $\sinh(t/2)$ at $t = 2\pi n i$ $(n > 0)$. For example, we can choose C to run from $-\infty$ to 0 along the real axis, and from 0 to $i\infty$ on the right of the imaginery axis, as shown in Fig. 2.

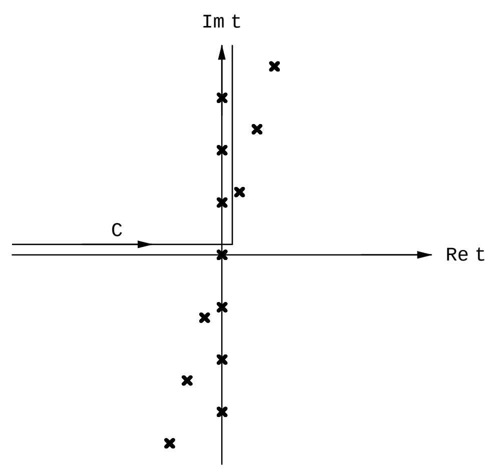

Fig. 2: The integration contour C in the complex t-plane that appears in (3.92), where R is given a small negative imaginery part.

(3.92) is however not quite right. It is in fact the free energy of a system with ordinary ("right sign") harmonic oscillator potential. Nevertheless, the integrand in the last line of (3.92) has the same structure as in the free energy computed from the true spectrum of the upside-down harmonic oscillator potential. This suggests that we modify the contour prescription, so that

$$\mathcal{F} = \mathrm{Re} \int_0^{\infty} \frac{dt}{t} \frac{e^{i\mu t}}{2\sinh(t/2)\sinh(t/2R)} \tag{3.93}$$

It then follows from (3.88) that

$$\begin{aligned}\phi(\mu) &= -\int_0^{\infty} \frac{dt}{t} \frac{\sin(\mu t)}{\sinh(t/2)} \\ &= -i \ln \frac{\Gamma(i\mu+\frac{1}{2})}{\Gamma(-i\mu+\frac{1}{2})}\end{aligned} \tag{3.94}$$

which agrees with the type 0A MQM reflection coefficient (3.48).

The change of contour in above calculation, which effectively takes us from the right sign harmonic oscillator potential to the wrong sign potential, seems rather ad hoc. It will be nice to have a clearer prescription. We will however use the same prescription to compute the free energy for type 0B IMM, and find a nontrivial agreement with the results obtained from the MQM.

In the case of type 0B IMM, we have seen from the previous subsection that the unperturbed partition function is the product of two Fredholm determinants,

$$\mathcal{Z}(\mu) = \det(1 + iq^{i\mu R/2}K)\det(1 - iq^{i\mu R/2}K) \tag{3.95}$$

where again, $\mu' = \mu R$ is the chemical potential of the T-dual type 0A MQM. The "symmetric part" and "antisymmetric part" of the free energy are

$$\begin{aligned} \mathcal{F}_s(\mu) &= \sum_{n\geq 0} \ln(1 + iq^{i\mu R/2+n+\frac{1}{2}}) \\ &= \int_C \frac{dt}{t} \frac{e^{i\mu t/2} e^{t/4}}{4\sinh(t/2)\sinh(t/2R)}, \\ \mathcal{F}_a(\mu) &= \sum_{n\geq 0} \ln(1 - iq^{i\mu R/2+n+\frac{1}{2}}) \\ &= \int_C \frac{dt}{t} \frac{e^{i\mu t/2} e^{3t/4}}{4\sinh(t/2)\sinh(t/2R)}. \end{aligned} \tag{3.96}$$

where C is the contour that picks up the poles of $\sinh(t/2)$ at $2\pi ni$ $(n > 0)$. Again, with proper modification of the contour prescription, and using (3.88), we obtain the phase shifts

$$\begin{aligned} \phi_s(\mu) &= -\int_0^\infty \frac{dt}{t} \frac{\sin(\mu t/2) e^{t/4}}{\sinh(t/2)} \\ &= -i \ln \frac{\Gamma(\frac{i\mu}{2} + \frac{1}{4})}{\Gamma(-\frac{i\mu}{2} + \frac{1}{4})}, \\ \phi_a(\mu) &= -\int_0^\infty \frac{dt}{t} \frac{\sin(\mu t/2) e^{3t/4}}{\sinh(t/2)} \\ &= -i \ln \frac{\Gamma(\frac{i\mu}{2} + \frac{3}{4})}{\Gamma(-\frac{i\mu}{2} + \frac{3}{4})}. \end{aligned} \tag{3.97}$$

Up to terms that contribute a constant to the density of states, these precisely reproduce the reflection coefficients (3.31).

Note that one can assign different chemical potentials $\mu_{\pm}$ to the symmetric and anti-symmetric perturbations. This corresponds to turning on a background RR flux [16]. The T-dual type 0A flux background should be analogously defined by giving different chemical potentials to two independent winding sectors. In this case we need to work with the grand canonical ensemble which sums up sectors of arbitrary D0-brane and anti-D0-brane numbers. We don't expect the corresponding type 0A background to have a fixed net D0-brane charge q. The thermal fluctuation in q should be of order $\mathcal{O}(e^{-\pi\beta\mu})$ instead of $\mathcal{O}(e^{-2\pi\beta\mu})$, since the "effective" chemical potential for the two winding sectors in type 0A theory on radius $1/R$ is $\mu R/2$ (μ being the chemical potential in the T-dual type 0B theory), as in (3.95). This is responsible for the mismatch at nonperturbative level between type 0A partition function with $\mu_{\pm} = \mu \pm Q$ and type 0B partition function with RR flux $q = iQ$ as found in [16].

With proper contour prescription for the type 0A and 0B IMM, we have produced the correct reflection coefficients, hence the string equations of the constrained Toda lattice hierarchies. This proves the equivalence between type 0A (0B) MQM and IMM perturbed by purely momentum modes or winding modes. The contour prescriptions introduced above should be regarded as part of the definition of the double scaling limit for the IMM.

4. Connecting IMM to MQM via Tachyon Condensation

4.1. The array of D-instantons

In this section we shall attempt to connect the IMM to MQM in a more direct and intuitive way via open string tachyon condensation. To be definite let us work with type 0B string theory compactified on a thermal circle of radius $R = ma$, where m is an integer, $a = \pi\sqrt{2\alpha'}$ is the critical distance. As well known [26], turning on an open string tachyon profile

$$T(X) = \lambda \cos(\pi X/a) \tag{4.1}$$

on the Euclidean world volume of an unstable D0-brane is described by an exactly marginal deformation in the boundary CFT. For sufficiently large λ one ends up with an array of alternating D- and anti-D-instantons separated at distance a. From the point of view of matrix models, this suggests that the MQM perturbed strongly by the tachyon profile (4.1) should be identical to the IMM expanded near the configuration of the array of D-instantons. Although it is not clear to us how to describe the tachyon profile (4.1) in the

MQM in a precise way, it is very plausible that the effect of (4.1) is to put the MQM on a discrete Euclidean time lattice of spacing a.[3]

Given this picture, we want to understand how the operators on both sides are identified. When the open string tachyon is condensed, the D0-brane effectively vanishes, so the open string excitations are localized near the "sites" of the time lattice where $T(X) \sim 0$. Let us consider in the MQM a small tachyon lump $\Phi(x) \simeq \Phi_k$ near $x = (k + \frac{1}{2})a$. This will shift the zero locus of $T(X) = \lambda \cos(\pi X/a) + \Phi(X)$ to $x_k \simeq (k + \frac{1}{2})a + c(-1)^k \Phi_k$, where $c = a/\pi\lambda$. So effectively the position of the kth D-instanton (or anti-instanton) is shifted by $c(-1)^k \Phi_k$. This suggests that the positions of the D-instantons in the array in IMM should be mapped to the open string tachyons on the corresponding sites in the discretized MQM. Let us denote by X_k the fluctuation of the position of the kth cluster of instantons from the array configuration. Then we are tempted to identify

$$X_k \sim (-1)^k \Phi_k \tag{4.2}$$

On the IMM side we shall integrate out the (complex) tachyons which are open strings stretched between D- and anti-D-instantons, and get an effective theory that describes only the collective coordinates X_k of the instantons. The D-instanton and anti-D-instanton want to move toward each other so that the tachyon can condense. Roughly one can think of this instability as an unstable "effective potential"[4] $V(X)$ (of periodicity a) felt by the eigenvalues of X_k's. Similar to the picture of $c = 1$ MQM, here the eigenvalues repel each other and fill up the "valleys" of the potential. In the large N limit, the eigenvalues are distributed in $2m$ cuts along the thermal circle. When there are a sufficiently large number of eigenvalues, the cuts will connect to adjacent ones and we expect a phase transition. The transition point is where the "energy" of the eigenvalues reaches the top of the potential. From (4.2) we expect that this is the critical point that defines the double scaling limit! In other words, the array of D-instantons at critical distance in IMM plays the same role as the unstable D0-brane corresponding to the top of the tachyon potential in MQM.

[3] It is well known[1] that the discretized MQM on a time lattice of spacing $\epsilon < a$ is exactly equivalent to the continuum theory up to a redefinition of parameters. In the critical limit $\epsilon = a$ the discretized MQM becomes singular. We will regularize the theory by taking ϵ slightly less than a.

[4] The picture we are describing here is rather heuristic. As we will see in the next subsection, the interaction between the eigenvalues are not simply represented by an ordinary potential plus the repulsion through the Vandermonde determinant. The phase transition occurring here as the eigenvalues merge the top of the "potential", is similar but different from the Gross-Witten transition[48], the latter being well known to describe 2D gravity coupled to $c = 0$ matter[2,49].

4.2. Effective matrix integral of the array

We will expand type 0B IMM near the configuration of an array of alternating D- and anti-D-instanton clusters at separation ϵ. It is natural to work with $\epsilon > a$ so that all the tachyons stretched between D- and anti-D-instantons are massive. This can be thought of as a kind of regularization. Eventually we will be interested in the limit $\epsilon \to a$. We will assume the radius $R = 2ma/2\pi$, so there are $2m$ clusters distributed along the thermal circle, with N instantons in each cluster. We shall write the diagonalized X and $\tilde{X}$ as

$$X = \begin{pmatrix} X_1 & & & \\ & X_2 + 2\epsilon & & \\ & & \ddots & \\ & & & X_m + (2m-2)\epsilon \end{pmatrix}, \quad \tilde{X} = \begin{pmatrix} \tilde{X}_1 + \epsilon & & & \\ & \tilde{X}_2 + 3\epsilon & & \\ & & \ddots & \\ & & & \tilde{X}_m + (2m-1)\epsilon \end{pmatrix} \tag{4.3}$$

where $X_k = \mathrm{diag}\{x_{2k-1,\alpha}\}_{\alpha=1}^{N}$ and $\tilde{X}_k = \mathrm{diag}\{x_{2k,\alpha}\}_{\alpha=1}^{N}$ are the collective coordinates of each cluster of N instantons.

We will take ϵ to be very close to a, and ignore all the massive modes with masses of order $\sim a$, so that only the "tachyons" as open strings stretched between adjacent clusters are retained. Furthermore, we shall take the double scaling limit *before* taking the limit $\epsilon \to a$, so we can assume the fluctuations $x_{j,\alpha}$ are small, since only the small fluctuations near the top of the potential are responsible for the universal behavior of the theory near the critical point. With these approximations, the (unperturbed) partition function can be written as

$$Z_N = \oint \prod_{j,\alpha} \frac{dz_{j,\alpha}}{2\pi i} \prod_{j=1}^{2m} \det{}_{\alpha\beta} \left(\frac{1}{q^{-1/2} z_{j+1,\alpha} - q^{1/2} z_{j,\beta}} \right) \tag{4.4}$$

where $z_{j,\alpha} = e^{i(x_{j,\alpha}+j\epsilon)/R}$, $q = e^{ia/R}$. We can rewrite it as

$$\begin{aligned} Z_N &= \int \prod_{j,\alpha} dx_{j,\alpha} \prod_{j=1}^{2m} \det{}_{\alpha\beta} \exp\left[-\ln \sin\left(\frac{x_{j+1,\alpha} - x_{j,\beta} + \epsilon - a}{2R}\right) \right] \\ &= \int \prod_{j,\alpha} dx_{j,\alpha} \prod_{j=1}^{2m} \det{}_{\alpha\beta} \exp\left[\frac{1}{4R^2 \sin^2(\frac{\epsilon-a}{2R})} (x_{j+1,\alpha} - x_{j,\beta})^2 + \mathcal{O}(x^3) \right] \end{aligned} \tag{4.5}$$

Let us note that there is a zero mode corresponding to the overall shift of X and $\tilde{X}$, whose origin can be traced back to the decoupled diagonal $U(1)$ in the $U(N) \times U(N)$ type 0A

MQM. We are free to fix this redundant gauge symmetry. Then we expect a tachyonic instability for each individual x_i and $\tilde{x}_j$ as they want to move towards each other.

Using the Itzykson-Zuber formula, we can rewrite (4.5) in the limit $\epsilon \to a$ as[5]

$$\begin{aligned}
&\int \prod_{j=1}^{2m} dX_j d\Omega_j \, e^{\frac{1}{(\epsilon-a)^2} \sum_j \mathrm{Tr}(X_{j+1}-\Omega_j X_j \Omega_j^\dagger)^2} \\
&\sim \int \prod_{j=1}^{2m} dX_j d\Omega_j \delta(X_{j+1} - \Omega_j X_j \Omega_j^\dagger) \qquad (4.6) \\
&= \int \prod_{j=1}^{2m} dX_j dY_j d\Omega_j \, e^{i \sum_j \mathrm{Tr}(X_{j+1}-\Omega_j X_j \Omega_j^\dagger) Y_j}
\end{aligned}$$

In fact, by integrating out all but one of the X_j's, (4.6) reduces to the three matrix model of [41]

$$\int dX_+ dX_- d\Omega \, e^{i\mathrm{Tr}(X_+X_- - qX_+\Omega X_- \Omega^\dagger)} \qquad (4.7)$$

with $q = e^{2\pi i R/\sqrt{2\alpha'}}$. Our prescription is to compute correlators with imaginery R, and then analytically continue to real values of $R (= m\sqrt{2\alpha'}$ in above).

4.3. The discretized MQM

Now we put the MQM on a discretized time lattice of spacing ϵ, where $\epsilon < a$. In order for the discretized MQM to be exactly equivalent to the continuum MQM, we need to have discretized propagator

$$\int dU_{i,i+1} \exp\left(-\beta' \epsilon \mathrm{Tr}\left[\left(\frac{U_{i,i+1}\Phi_{i+1}U_{i,i+1}^{-1} - \Phi_i}{\epsilon}\right)^2 - \tfrac{1}{2}\omega'^2(\Phi_{i+1}^2 + \Phi_i^2) \right] \right) \qquad (4.8)$$

where $dU_{i,i+1}$ is the Haar measure over $U(N)$, and the parameters β', ω' are related to those of the continuum model by

$$\omega' \epsilon = 2 \sin \frac{\pi\epsilon}{2a}, \qquad \beta' = \beta \frac{\pi\epsilon/a}{\sin(\pi\epsilon/a)} \qquad (4.9)$$

We can diagonalize Φ_i into its eigenvalues $\lambda_{i,\alpha}$ at each step, so the propagator simplifies to

$$\int dU \exp\left(-\frac{\beta'}{\epsilon} \left[\sum_\alpha (\lambda_{i+1,\alpha}^2 + \lambda_{i,\alpha}^2) - 2\sum_{\alpha,\beta} \lambda_{i,\alpha}\lambda_{i+1,\beta} U_{\alpha\beta} U_{\alpha\beta}^* - \tfrac{1}{2}\omega'^2\epsilon^2 \sum_\alpha (\lambda_{i,\alpha+1}^2 + \lambda_{i,\alpha}^2) \right] \right) \qquad (4.10)$$

[5] To compare with the three matrix model, we are really taking ϵ to differ from a by a small imaginery number.

Using the Itzykson-Zuber formula

$$\int_{U(N)} dU\, e^{\sum_{\alpha,\beta} U_{\alpha\beta} U^*_{\alpha\beta} x_\alpha y_\beta} = \prod_{k=1}^{N-1} k! \frac{\det_{\alpha\beta} e^{x_\alpha y_\beta}}{\Delta(x)\Delta(y)} \tag{4.11}$$

The propagator becomes

$$\exp\left(-\frac{\beta'}{\epsilon}(1-\tfrac{1}{2}\omega'^2\epsilon^2)\sum_\alpha(\lambda^2_{i+1,\alpha}+\lambda^2_{i,\alpha})\right)\frac{\det_{\alpha\beta}(e^{\frac{2\beta'}{\epsilon}\lambda_{i+1,\alpha}\lambda_{i,\beta}})}{\Delta(\lambda_{i+1})\Delta(\lambda_i)} \tag{4.12}$$

The partition function is given by

$$Z_N = \int \prod d\lambda_{i,\alpha} \prod_{i=1}^{2m} \det_{\alpha\beta} \exp\left\{-\frac{\beta'}{\epsilon}\left[(\lambda_{i+1,\alpha}-\lambda_{i,\beta})^2 - \tfrac{1}{2}\omega'^2\epsilon^2(\lambda^2_{i+1,\alpha}+\lambda^2_{i,\beta})\right]\right\} \tag{4.13}$$

Note that we haven't made any approximation in above manipulations. Under the identification (4.9), (4.13) is *exactly* the same as the path integral of the continuum MQM.

In the limit $\epsilon \sim a$, (4.13) is approximately

$$Z_N = \int \prod d\lambda_{i,\alpha} \prod_{i=1}^{2m} \det_{\alpha\beta} \exp\left\{-\frac{\beta}{a-\epsilon}\left[-(\lambda_{i+1,\alpha}+\lambda_{i,\beta})^2 + \frac{\pi^2}{2a^2}(a-\epsilon)^2(\lambda^2_{i+1,\alpha}+\lambda^2_{i,\beta})\right]\right\} \tag{4.14}$$

Under the identification (4.2), (4.14) approaches the path integral of IMM expanded near the array (4.5), up to rescalings of ϵ and λ_i's which can be absorbed into a shift in the chemical potential. This confirms the connection between the discretized MQM and the IMM expanded around the D-instanton array in the critical limit.

Similar to the manipulation of (4.6), we can rewrite (4.14) as

$$\int \prod_{j=1}^{2m} dX_j dY_j d\Omega_j e^{i\sum_j \mathrm{Tr}(X_{j+1}-e^{-i\pi(1-\epsilon/a)}\Omega_j X_j \Omega_j^\dagger)Y_j} \tag{4.15}$$

which again reduces to the three matrix model. The contour prescription of giving R a small imaginery part so that $|q| = |e^{2\pi i R}| < 1$, corresponds to taking $\epsilon - a$ to be a small imaginery number.

5. On SD-branes and ZZ Boundary States

It was pointed out in [28] that sD-branes ($\lambda = 1/2$ s-brane) can be described as an array of D-instantons along Euclidean time separated at the critical distance. In the previous section we have argued that sD-branes play the same role in the IMM as D0-branes in MQM. In this section we study the closed string fields sourced by sD-branes from IMM. We find agreement with calculations from ZZ boundary states. This is very much in the same spirit as the calculation of D-brane decay into closed strings from MQM [6] as opposed to the boundary state approach. In fact, we will reproduce the $(1,1)$ ZZ boundary states from IMM in a very simple manner.

5.1. sD-brane in $c=1$ string theory

In this subsection we study the closed string fields in spacetime dual to the array configuration in the IMM for $c=1$ string. The configuration of an array of D-instantons separated at distance $a = 2\pi\sqrt{\alpha'}$ $(2\pi R = ma)$ is described by

$$X = \begin{pmatrix} \frac{1}{2}a & & & \\ & \frac{3}{2}a & & \\ & & \ddots & \\ & & & (m-\frac{1}{2})a \end{pmatrix} \tag{5.1}$$

The array configuration can be expressed in a gauge invariant way as

$$\mathrm{Tr}U^k = \begin{cases} (-1)^{k/m} m, & \text{if } m|k, \\ 0, & \text{otherwise.} \end{cases} \tag{5.2}$$

for all $k \in \mathbf{Z}$, where $U = e^{iX/R}$. The operator $\mathrm{Tr}U^k$ is mapped to vertex operator[21]

$$\mathrm{Tr}U^k \leftrightarrow \frac{1}{(q^{k/2}-q^{-k/2})}\frac{\Gamma(|p|)}{\Gamma(-|p|)}\int e^{ipX+(2-|p|)\phi}, \qquad p = k/R \tag{5.3}$$

Therefore (5.2) corresponds to the condensation of momentum modes

$$\langle V_l \rangle = \langle e^{ilX+(2-|l|)\phi} \rangle = (q^{ml/2}-q^{-ml/2})\frac{\Gamma(-|l|)}{\Gamma(|l|)}(-1)^l, \quad l \in \mathbf{Z}. \tag{5.4}$$

Formally $\Gamma(-|l|) = \frac{(-1)^l}{|l|!}\Gamma(0)$ is divergent, and since $q = e^{2\pi i/R} = e^{2\pi i/m}$, $(q^{ml/2}-q^{-ml/2})$ is zero. As discussed earlier (also as in [20]), to make the IMM well-defined, we should add a small imaginery part to R, say $R \to R(1-i\epsilon)$ so that $|q|<1$,

$$q^{ml/2}-q^{-ml/2} \to (-1)^l(-2\pi l\epsilon)$$

At the same time, we have

$$\Gamma(-|l|) \to \Gamma(-\frac{|l|}{1-i\epsilon}) = \frac{(-1)^l}{|l|!}\frac{1}{(-i\epsilon)|l|}$$

The factors of ϵ cancel, and we end up with spacetime tachyon profile

$$\begin{aligned} T(X,\phi) &= \sum_l \frac{1}{l}\langle V_l\rangle \cos(lX)e^{l\phi} \\ &\sim 2\sum_{l=0}^{\infty}\frac{(-1)^l}{(l!)^2}\mu^{l/2}\cos(lX)e^{l\phi} \\ &= J_0(2\mu^{1/4}e^{\frac{\phi+iX}{2}}) + J_0(2\mu^{1/4}e^{\frac{\phi-iX}{2}}) \end{aligned} \tag{5.5}$$

This result precisely agrees with the closed string fields sourced by sD-branes computed from ZZ boundary states [50].

In fact, we can consider a much simpler case, - a single D-instanton sitting at $X=0$. This is described by $\mathrm{Tr}U^k = 1$ for all $k \in \mathbf{Z}$ in the IMM. As above, it corresponds to a condensation of closed string momentum modes

$$\langle V_{p=k/R}\rangle = (q^{k/2} - q^{-k/2})\frac{\Gamma(-|p|)}{\Gamma(|p|)} = 2i\sin(\pi p)\frac{\Gamma(-|p|)}{\Gamma(|p|)} \tag{5.6}$$

This is nothing but the ZZ boundary state (times the Dirichlet boundary state in Euclidean time direction), up to a normalization factor. The agreement between IMM calculation and ZZ boundary state is very reminiscent to the calculation of [6].

5.2. sD-brane in type 0B string theory

An sD-brane in type 0B theory is described by an array of alternating D- and anti-D-instantons separated at critical distance $a = \pi\sqrt{2\alpha'}$. In this subsection we will set $\alpha' = 2$ for convenience. The radius is then $R = 2ma/2\pi = 2m$, and we have $q = e^{ia/R} = e^{\pi i/m}$. The D-instantons are located $X = \frac{1}{2}a, \frac{5}{2}a, \cdots, (2m-\frac{3}{2})a$, whereas the anti-D-instantons are located at $\tilde{X} = \frac{3}{2}a, \frac{7}{2}a, \cdots, (2m-\frac{1}{2})a$. They can be described in the following gauge invariant way

$$\mathrm{Tr}U^k = \begin{cases} e^{i\frac{k\pi}{2m}}m, & \text{if } m|k, \\ 0, & \text{otherwise.} \end{cases} \qquad \mathrm{Tr}\tilde{U}^k = \begin{cases} e^{-i\frac{k\pi}{2m}}m, & \text{if } m|k, \\ 0, & \text{otherwise.} \end{cases} \tag{5.7}$$

From (3.84), they are mapped to NS-NS and R-R vertex operators

$$\begin{aligned}
\frac{1}{2}(\mathrm{Tr}U^k + \mathrm{Tr}\tilde{U}^k) &\leftrightarrow \frac{1}{(q^{k/2} - q^{-k/2})}\frac{\Gamma(|p|)}{\Gamma(-|p|)}\int e^{ipX+(1-|p|)\phi} \\
\frac{1}{2}(\mathrm{Tr}U^k - \mathrm{Tr}\tilde{U}^k) &\leftrightarrow \frac{1}{(q^{k/2} + q^{-k/2})}\frac{\Gamma(\frac{1}{2}+|p|)}{\Gamma(\frac{1}{2}-|p|)}\int e^{ipX+(1-|p|)\phi} S\bar{S}
\end{aligned} \tag{5.8}$$

where $p = k/R$, and $S, \bar{S}$ are spin fields. On RHS of (5.8) we have included the corresponding leg factors. (5.7) corresponds to condensation of closed string modes

$$\begin{aligned}
\langle V_{NS,p=l}\rangle &= (q^{ml} - q^{-ml})\frac{\Gamma(-|l|)}{\Gamma(|l|)}(-1)^l \\
&\to l\frac{(-1)^l}{(|l|!)^2}, \\
\langle V_{R,p=l+\frac{1}{2}}\rangle &= i(q^{m(l+1/2)} + q^{-m(l+1/2)})\frac{\Gamma(\frac{1}{2} - |l+\frac{1}{2}|)}{\Gamma(\frac{1}{2} + |l+\frac{1}{2}|)}(-1)^l \\
&\to \frac{l+\frac{1}{2}}{|l+\frac{1}{2}|}\cdot\frac{(-1)^l}{[(|l+\frac{1}{2}| - \frac{1}{2})!]^2}
\end{aligned} \tag{5.9}$$

where $l \in \mathbf{Z}$, and we have again used the prescription $R \to R(1-i\epsilon)$ to regularize the singular terms. The NS-NS and R-R scalars in spacetime are

$$\begin{aligned}
T(X,\phi) &\sim 2\sum_{l=0}^{\infty}\frac{(-1)^l}{(l!)^2}(2\mu)^l\cos(lX)e^{l\phi} \\
&= J_0(2\sqrt{2\mu}e^{\frac{\phi+iX}{2}}) + J_0(2\sqrt{2\mu}e^{\frac{\phi-iX}{2}}), \\
V(X,\phi) &\sim 2\sum_{l=0}^{\infty}\frac{(-1)^l}{(l!)^2}\frac{1}{l+\frac{1}{2}}(2\mu)^{l+\frac{1}{2}}\sin((l+\tfrac{1}{2})X)e^{(l+\frac{1}{2})\phi}.
\end{aligned} \tag{5.10}$$

In particular, we find using (5.10) that (with proper normalization)

$$\begin{aligned}
\int_{-\infty}^{\infty} d\phi\,\partial_X V(X,\phi) = \int_{-\infty}^{\infty} d\phi\,\frac{1}{4}\Big[&\sqrt{2\mu}e^{\frac{\phi+iX}{2}}J_0(2\sqrt{2\mu}e^{\frac{\phi+iX}{2}}) \\
&+\sqrt{2\mu}e^{\frac{\phi-iX}{2}}J_0(2\sqrt{2\mu}e^{\frac{\phi-iX}{2}})\Big] = \frac{1}{2}
\end{aligned} \tag{5.11}$$

This is the conserved s-charge of the sD-brane[28].

Extending [50], we can compute the closed string fields in the weak coupling region from the ZZ boundary state in super-Liouville theory. Our convention is $b = 1, Q = 2$, so that $\hat{c}_L = 1 + 2Q^2 = 9$. The disk 1-point function for the tachyon $T(P)$ and RR scalar $V(P)$ are [32,33,8]

$$
\begin{aligned}
\psi_{NS}(P) &= -i\sqrt{2\pi}\sinh(\pi P)\frac{\Gamma(iP)}{\Gamma(-iP)}(2\mu)^{-iP}, \\
\psi_R(P) &= \sqrt{2\pi}\cosh(\pi P)\frac{\Gamma(\frac{1}{2}+iP)}{\Gamma(\frac{1}{2}-iP)}(2\mu)^{-iP}.
\end{aligned}
\tag{5.12}
$$

The closed string fields sourced by the sD-brane are (in momentum space)

$$
\begin{aligned}
T(P,t) &= \frac{1}{4\pi E}\psi_{NS}(P)\sum_{n\geq 0} e^{-(n+\frac{1}{2})aE}(e^{-iEt}+e^{iEt}) \\
&= \frac{1}{4\pi E}\psi_{NS}(P)\frac{\cos Et}{\sinh(aE/2)}, \\
V(P,t) &= \frac{1}{4\pi E}\psi_R(P)\sum_{n\geq 0}(-1)^n e^{-(n+\frac{1}{2})aE}(e^{-iEt}-e^{iEt}) \\
&= \frac{-i}{4\pi E}\psi_R(P)\frac{\sin Et}{\cosh(aE/2)},
\end{aligned}
\tag{5.13}
$$

where $E = |P|$. Translated into Liouville coordinates,

$$
\begin{aligned}
T(\phi,t) &\sim -i\int_{-\infty}^{\infty} dP e^{-iP\phi}\frac{\cos Pt}{P}\frac{\Gamma(iP)}{\Gamma(-iP)}(2\mu)^{-iP} \\
&= J_0(2\sqrt{2\mu}e^{\frac{\phi+t}{2}}) + J_0(2\sqrt{2\mu}e^{\frac{\phi-t}{2}}), \\
V(\phi,t) &\sim -i\int_{-\infty}^{\infty} dP e^{-iP\phi}\frac{\sin Pt}{P}\frac{\Gamma(\frac{1}{2}+iP)}{\Gamma(\frac{1}{2}-iP)}(2\mu)^{-iP} \\
&= 2\sum_{l=0}^{\infty}\frac{(-1)^l}{(l!)^2}\frac{1}{l+\frac{1}{2}}(2\mu)^{l+\frac{1}{2}}\sinh((l+\tfrac{1}{2})t)e^{(l+\frac{1}{2})\phi}
\end{aligned}
\tag{5.14}
$$

which precisely agree with (5.10).

Again, we can consider the simpler case of a single D-instanton at $X = 0$. This is described in IMM as $\mathrm{Tr}U^k = 1, \mathrm{Tr}\tilde{U}^k = 0$ for all $k \in \mathbf{Z}$. It then follows from (5.8) that

$$
\begin{aligned}
\langle V_{NS,p=k/R}\rangle &= (q^{k/2}-q^{-k/2})\frac{\Gamma(-|p|)}{\Gamma(|p|)} = 2i\sin(\pi p)\frac{\Gamma(-|p|)}{\Gamma(|p|)} \\
\langle V_{RR,p=k/R}\rangle &= (q^{k/2}+q^{-k/2})\frac{\Gamma(\frac{1}{2}-|p|)}{\Gamma(\frac{1}{2}+|p|)} = 2\cos(\pi p)\frac{\Gamma(\frac{1}{2}-|p|)}{\Gamma(\frac{1}{2}+|p|)}
\end{aligned}
\tag{5.15}
$$

which reproduce up to a normalization factor the ZZ boundary state of super-Liouville theory (5.12). This is also a nontrivial check of the dictionary (3.84).

6. On Black Holes in Type 0 String Theory

We have shown that type 0A and 0B string theories have the integrable structure of Toda lattice hierarchy. This has enabled us to prove the T-duality between type 0A and type 0B MQM at least for perturbations by purely momentum modes or winding modes. The matrix model deformed by modes of winding number ± 1 is equivalent (in the $\mu \to 0$ limit) to the sine-Liouville theory ($\mathcal{N}=2$ Liouville theory in type 0 case), which is believed to be dual to string theory in the Euclidean 2D black hole [21,34,36]. The exact free energy of the deformed type 0A and 0B MQM can be computed by solving the Hirota differential equations.

Let us first consider the case of type 0B MQM. The winding mode perturbations, just as in $c=1$ string, generate the Toda integrable flow. The unperturbed free energy of type 0B MQM is perturbatively the same as that of $c=1$ string, up to a redefinition $\alpha' \to 2\alpha'$ and an overall factor of 2 (coming from doubling the fermi sea). The method used in [21] to compute the free energy perturbed by $\lambda_\pm \equiv t_{\pm 1}$ can be directly applied to type 0B case. The overall factor of 2 in the unperturbed free energy is very important, since the Hirota differential equations are nonlinear. In the limit of large λ and $\mu=0$, the free energy is now (in units $\alpha'=1/2$)

$$\mathcal{F}(\lambda,\mu=0,R) = -(1-R)^2(2R-1)^{\frac{R}{1-R}}\lambda^{\frac{2}{1-R}} - \frac{R+R^{-1}}{12(1-R)}\ln(\lambda\sqrt{2R-1}) + \cdots \tag{6.1}$$

where we have exhibited the genus 0 and genus 1 terms in the expansion. The asymptotic radius of the Euclidean black hole in type 0B string is $R=\sqrt{\alpha'/2}=1/2$. Expanding near this radius, we have[6]

$$\mathcal{F} = -2\pi(R-\frac{1}{2})M - \frac{5}{12}\ln M + \cdots \tag{6.2}$$

where $M \propto \lambda^{\frac{2}{1-R}}$. At $R=1/2$, M is expected to be the mass of the black hole. In fact, the effective string coupling in $\mathcal{N}=2$ Liouville theory is $g_s\lambda^{-2}$, which is the combination invariant under shifting the Liouville coordinate. The mass of the black hole goes like $M \sim 1/g_s^2$, therefore must be proportional to λ^4. This agrees with the expectation from the matrix model result (6.2).

The density of states in the black hole background typically has Hagedorn growth behavior

$$\rho(M) \sim M^{s_1}e^{\beta_H M} \tag{6.3}$$

[6] We are grateful to A. Adams for discussions on this point.

Comparing with the free energy near T_H (6.2), we find

$$s_1 + 1 = -\frac{R + R^{-1}}{24} = -\frac{5}{12} \tag{6.4}$$

This can be compared to the $c = 1$ string case, where the asymptotic radius of the 2D black hole is $R = 3/2$, and one finds $s_1 + 1 = -(R + R^{-1})/48 = -13/288$ [21].

One can study Euclidean black holes in type 0A theory in a similar way. The uncharged black holes should be described by a background with the lowest NS-NS winding modes condensed. As in (5.8), the perturbation of these winding modes are represented in type 0A MQM as deformations by the operators $\mathrm{Tr}\Omega + \mathrm{Tr}\tilde{\Omega}$ and $\mathrm{Tr}\Omega^{-1} + \mathrm{Tr}\tilde{\Omega}^{-1}$. Strictly speaking, the coefficients of these perturbations are not the same as the time variables that generate the integrable flow, but essentially related to them through nonlinear bosonization maps, as we have seen in section 3.4. It then appears much more complicated to solve the differential constraints on the free energy exactly.

However, if we are only interested in the perturbative expansion of the free energy, the calculation is greatly simplified. In the T-dual type 0B picture, perturbatively the two sides of the fermi sea decouple. The deformation by NS-NS winding modes can be treated as independent perturbations in the two decoupled sectors. Each sector is perturbatively the same as $c = 1$ string, up to a redefinition of α'. In the end, the free energy of type 0A theory deformed by NS-NS ± 1 winding modes is simply obtained from the solution of [21] for $c = 1$ string with a replacement $\alpha' \to \alpha'/2$, and a factor of 2 coming from the two sectors[7]. In units with $\alpha' = 2$, the answer is

$$\mathcal{F}(\lambda, \mu = 0, R) = -\frac{1}{2}(2 - R)^2 (R - 1)^{\frac{R}{2-R}} \lambda^{\frac{4}{2-R}} - \frac{R + R^{-1}}{6(2 - R)} \ln(\lambda\sqrt{R - 1}) + \cdots \tag{6.5}$$

Again, near the black hole asymptotic radius $R = \sqrt{\alpha'/2} = 1$, we have

$$\mathcal{F} = -2\pi(R - 1)M - \frac{1}{12} \ln M + \cdots \tag{6.6}$$

where $M \propto \lambda^{\frac{4}{2-R}} \simeq \lambda^4$. The growth of the density states is of the form (6.3) with the exponent s_1 given by

$$s_1 + 1 = -\frac{R + R^{-1}}{24} = -\frac{1}{12}. \tag{6.7}$$

[7] Unlike the case of type 0B theory, this factor of 2 trivially multiplies the answer of the perturbed free energy.

It would be interesting to reproduce the exponents (6.4), (6.7) from 1-loop calculations in $\mathcal{N} = 2$ Liouville theory. One can also consider the sectors with nonzero net D0-brane charge q perturbed by the lowest winding modes, which should lead to charged nonextremal black holes.

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