A PROOF OF THE CROSSING NUMBER OF $K_{3,n}$ IN A SURFACE

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Abstract

In this note we give a simple proof of a result of Richter and Siran by basic counting method, which says that the crossing number of $K_{3,n}$ in a surface with Euler genus $\varepsilon$ is

$$\left\lfloor \frac{n}{2\varepsilon + 2} \right\rfloor \{n - (\varepsilon + 1)(1 + \left\lfloor \frac{n}{2\varepsilon + 2} \right\rfloor)\}.$$  \(1\)

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1. Introduction

In [1], Guy and Jenkyns showed that the crossing number of $K_{3,n}$ in the torus is $\left\lfloor \frac{(n - 3)^2}{12} \right\rfloor$. In [2], Richter and Siran generalized their result and showed the following:

Theorem 1.1. If the surface $\Sigma$ has Euler genus $\varepsilon$, then the crossing number of $K_{3,n}$ in $\Sigma$ is given by

$$cr_{\Sigma}(K_{3,n}) = \left\lfloor \frac{n}{2\varepsilon + 2} \right\rfloor \{n - (\varepsilon + 1)(1 + \left\lfloor \frac{n}{2\varepsilon + 2} \right\rfloor)\}.$$  \(1\)

(The Euler genus of a surface $\Sigma$ is $2h$ if $\Sigma$ is the sphere with $h$ handles and $k$ if $\Sigma$ is the sphere with $k$ crosscaps.) In this note, we give a simple proof of Theorem 1.1 by using basic counting method. In the following, we will denote the right hand side of (1) by $f(\varepsilon, n)$. 
2. Proof of Theorem 1.1

To prove that \( cr_\Sigma(K_{3,n}) \leq f(\varepsilon, n) \), one can refer to [2] for the drawings. To complete the proof, it suffices to show that

\[
(2) \quad cr_\Sigma(K_{3,n}) \geq f(\varepsilon, n).
\]

We will prove (2) by induction. For \( n \leq 2\varepsilon + 2 \), from [3] and [4], we know that \( K_{3,n} \) can be embedded in \( \Sigma \). Therefore, \( cr_\Sigma(K_{3,n}) = 0 = f(\varepsilon, n) \), which shows that (2) is true for \( n \leq 2\varepsilon + 2 \).

Therefore we may assume that \( n > 2\varepsilon + 2 \). Let \( n = (2\varepsilon + 2)q + r \) where \( 0 \leq r \leq 2\varepsilon + 1 \). Then

\[
(3) \quad f(\varepsilon, n) = (\varepsilon + 1)(q^2 - q) + qr.
\]

Note that, in a crossing-free drawing of a (connected) subgraph of \( K_{3,n} \) in \( \Sigma \), every face has even degree. Let \( t_j \) be the number of regions with \( j \) bounding arcs; and \( F, E, V \) be the number of faces, arcs, vertices, respectively. Then \( t_j = 0 \) if \( j \) is odd, \( F = t_4 + t_6 + t_8 + \ldots \), and \( 2E = 4t_4 + 6t_6 + 8t_8 + \ldots \), and by the Euler’s formula for \( \Sigma \),

\[
(4) \quad V \geq 2 - \varepsilon + E - F,
\]

\[
(5) \quad V \geq 2 - \varepsilon + t_4 + 2t_6 + 3t_8 + \ldots \geq 2 - \varepsilon + F.
\]

Suppose we have an optimal drawing of \( K_{3,n} \) in \( \Sigma \), i.e., one with \( cr_\Sigma(K_{3,n}) \) crossings, and that by removing \( cr_\Sigma(K_{3,n}) \) edges, a crossing-free drawing is produced. Then (4) and (5) give \( E - V = (3n - cr_\Sigma(K_{3,n})) - (3 + n) \leq F + \varepsilon - 2 \leq V + 2\varepsilon - 4 = 3 + n + 2\varepsilon - 4 \), so

\[
(6) \quad cr_\Sigma(K_{3,n}) \geq n - 2 - 2\varepsilon.
\]

If \( q = 1 \), then \( n = (2\varepsilon + 2) + r \). Then by (3) and (6), we have

\[
cr_\Sigma(K_{3,(2\varepsilon+2)+r}) \geq r = f(\varepsilon, (2\varepsilon + 2) + r).
\]

This implies that (2) holds for \( q = 1 \).

Therefore we may assume that \( q \geq 2 \). Since \( K_{3,n} \) contains \( n \) different \( K_{3,n-1} \) and each of \( K_{3,n-1} \) contains at least \( f(\varepsilon, n - 1) \) crossings by induction hypothesis. Note that a crossing in a drawing of \( K_{3,n} \) appears in \( n - 2 \)
different drawings of $K_{3,n-1}$. Hence

$$\text{cr}_\Sigma(K_{3,n}) \geq \frac{n}{n-2} \text{cr}_\Sigma(K_{3,n-1}) = \frac{n}{n-2} f(\varepsilon, n-1). \tag{7}$$

From (3) and (7), we have

$$\text{cr}_\Sigma(K_{3,n}) \geq \begin{cases} (\varepsilon + 1)(q^2 - q) + qr - 1 + \frac{qr + r - 2}{n-2}, & \text{if } 1 \leq r \leq 2\varepsilon + 1; \\ (\varepsilon + 1)(q^2 - q), & \text{if } r = 0. \end{cases} \tag{8}$$

Note that $q \geq 2$ and $1 \leq r \leq 2\varepsilon + 1$ imply that $\frac{qr + r - 2}{n-2} > 0$. Hence (3), (8) and the fact that the crossing number is an integer imply that (2) holds for $q \geq 2$. This completes the proof of Theorem 1.1.

References


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