Ratio m_c/m_s with Wilson Fermions

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We determine the quark mass ratio m_c/m_s on the lattice, using Wilson-type fermions. Configurations with $N_f = 2$ dynamical clover-improved fermions by the QCDSF Collaboration are used, which were made available through the ILDG. In the valence sector we use a sophisticated, mass-independently $O(a)$-improved Wilson-type action with small cutoff effects even in the charm mass region. After an extrapolation to the physical pion mass, to zero lattice spacing and to infinite box volume, we find $m_c/m_s = 11.27(30)(26)$.

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Introduction.—Quark masses are among the fundamental parameters of the standard model of particle physics. As they cannot be measured directly, their determination involves a substantial amount of theory—for decades uncertainties have been hard to estimate and error bars were large [1]. In recent years lattice QCD has made enormous progress at pinning quark masses down with a few-percent accuracy; see, e.g., [2] for a summary. For *ratios* of quark masses the situation is even better, since in this case no lattice-to-continuum matching factor (whose accurate determination represents one of the most demanding steps in such a computation) is needed.

The charm-to-strange quark mass ratio m_c/m_s (which is scheme and scale independent) is of direct phenomenological relevance [3]. It has been determined by the HPQCD [4] and ETM Collaborations [5]. Both Collaborations use lattice formulations with small cutoff effects even in the charm quark mass region, albeit with isospin (or taste) symmetry breaking, i.e., the pions are nondegenerate, in spite of a single m_q being used, an effect which disappears $\propto a^2$ with a the lattice spacing. By contrast unimproved or $O(a)$-improved Wilson fermions avoid such effects, at the price of having comparatively larger cutoff effects (see Appendix A of [2] for a discussion).

In [6] we constructed a Brillouin-improved Wilson action which was claimed to show small cutoff effects without isospin breaking, thus allowing for a one-to-one identification between lattice and continuum flavor. The latter feature is important, as isospin breaking effects require a more involved analysis, rendering it less transparent. Here we test the smallness of the cutoff effects by calculating the ratio m_c/m_s in this formulation (with tree-level clover improvement and one step of link smearing) in the valence sector (for s and c). The lattices with 2 degenerate dynamical flavors (for u and d) are provided by the QCDSF Collaboration. The remainder of this article describes how we calculate the ratio on each ensemble, and how we remove the lattice artifacts to find the physical value of m_c/m_s. We end with an illustration of how this ratio may be used, together with a precise m_c input, to yield a robust estimate of m_s.

Strategy to compute m_c/m_s on each ensemble.—Our goal is to compute the quark mass ratio m_c/m_s with controlled systematics. We follow a two-step procedure. In the first step we tune, for each ensemble, the bare mass parameter κ of our action (see [6]) to the physical strange or charm quark mass and evaluate m_c/m_s on that ensemble. In the second step we eliminate the lattice artifacts by means of a global fit.

Our strategy to compute m_c/m_s on a given ensemble can be summarized as follows. (1) Tune κ_s and κ_c at the same time such that $M_{\eta_s}^2/(M_{D_s^*}^2 - M_{D_s}^2)$ and $(2M_{D_s}^2 - M_{\eta_s}^2)/(M_{D_s^*}^2 - M_{D_s}^2)$ take their physical values of 0.801 38 and 12.402, respectively [1]. These numbers build on $M_{\eta_s} = 0.6858(8)$ GeV for the quarkline connected state, which follows via $(2M_K^2 - M_\pi^2)^{1/2}$ with SU(2)-symmetric values of M_K, M_π from [2], or from a direct computation [4]. (2) Determine for either tuned κ the PCAC quark mass, and form the ratio $r = m_c^{\mathrm{PCAC}}/m_s^{\mathrm{PCAC}}$. In this step m_s^{PCAC} is determined from the connected $\bar{s}s$ correlator, while m_c^{PCAC} follows from the $\bar{c}s$ correlator together with the strange mass determined before (see below). As a theoretical caveat let us remark that in general with Wilson-type fermions the sea quarks affect the renormalization properties of the valence flavors. For a bare PCAC quark mass [7]

$$m_j^{\mathrm{AWI}} = \frac{Z_A}{Z_P} m_j^{\mathrm{PCAC}}[1 + (b_A - b_P)am_j^{\mathrm{W}} + (\bar{b}_A - \bar{b}_P)a\,\mathrm{Tr}(M) + O(a^2)], \tag{1}$$

where m_j^{W} is the Wilson mass of flavor j, and M the quark mass matrix. The Z_J with $J \in \{A, P\}$ are lattice-to-continuum matching factors, while $b_J = 1 + O(\alpha_s)$, $\bar{b}_J = O(\alpha_s^2)$ denote improvement coefficients. As we follow a tree-level improvement strategy (with $c_{\mathrm{SW}} = 1$, see [6]) the ratio

$$\frac{m_j^{\rm AWI}}{m_k^{\rm AWI}} = \frac{m_j^{\rm PCAC}}{m_k^{\rm PCAC}}[1 + O(\alpha_s a) + O(a^2)] \qquad (2)$$

is found to carry two types of cutoff effects. As we shall see, the lack of knowledge of which type would numerically dominate creates a major source of systematic error on the final result.

Once $r = m_c^{\rm PCAC}/m_s^{\rm PCAC}$ is in hand for each ensemble, the final answer follows through three more steps (which, in practice, will be combined into a single global fit). (3) Correct, for each ensemble, the value of r for the effect of the finite spatial volume L^3. (4) Extrapolate, for each β, the result of step 3 to $M_\pi^{\rm phys} = 134.8$ MeV [2] in the sea. (5) Extrapolate the result of step 4 to the continuum, using an $O(\alpha_s a)$ or $O(a^2)$ *Ansatz*. To test how reliably the systematic uncertainties are assessed, we will repeat steps 3–5 for the control quantity $M_\phi^2/(M_{D_s^*}^2 - M_{D_s}^2)$, whose physical value is known.

Analysis details and final result for m_c/m_s.—We now give details of how we determine the ratio m_c/m_s on each ensemble, and how we eliminate the lattice artifacts by means of a global fit.

We use the $N_f = 2$ configurations by QCDSF [8–11] made available through the ILDG [12]. Since we measure dimensionless ratios, one might naively think that no scale determination is needed. However, in the extrapolation to the physical point a scale is required. We will use [11]

$$a[\text{fm}] = 0.076, 0.072, 0.060 \quad \text{at } \beta = 5.25, 5.29, 5.40 \qquad (3)$$

for this purpose, but apart from the extrapolation this scale is not used. Given the resources available to us, we select the 13 ensembles marked with a bullet or circle in Table I for analysis. They cover a wide range of pion masses and box volumes (both in fm and in $M_\pi L$ units), so that a controlled extrapolation to the physical pion mass and infinite volume should be possible.

On a given ensemble, for a few mass parameters $1/\kappa_s$, $1/\kappa_c$, we determine the correlators of four mesons (the pseudoscalar and vector with $\bar{c}s$ and $\bar{s}s$ flavor content). From these we form the observables $O_1 = M_{P\bar{s}s}^2/(M_{V\bar{c}s}^2 - M_{P\bar{c}s}^2)$, $O_2 = (2M_{P\bar{c}s}^2 - M_{P\bar{s}s}^2)/(M_{V\bar{c}s}^2 - M_{P\bar{c}s}^2)$, and $O_3 = (2m_{cs} - m_{ss})/m_{ss}$, where m_{ij} denotes the average of the PCAC masses with flavor i and j, based on the improved symmetric derivative $\bar{\partial}\phi(t) = [\phi(t-2) - 8\phi(t-1) + 8\phi(t+1) - \phi(t+2)]/12$. For each observable a spline interpolation in $1/\kappa_s$ and $1/\kappa_c$ is constructed. The target value $O_1 \equiv 0.801\,38$ defines a line in the $(1/\kappa_s,\ 1/\kappa_c)$ plane, and the same holds true for $O_2 \equiv 12.402$. The point where these two lines intersect defines the tuned set $(1/\kappa_s^*$, $1/\kappa_c^*)$, and the value of O_3 at this point is the desired ratio r on that ensemble. The spacing in $1/\kappa_s$ and $1/\kappa_c$ is chosen sufficiently narrow so that the uncertainty due to the interpolation is completely negligible. Since all of this is done inside a jackknife, the jitter of the crossing point is fully propagated into the statistical error of the tuned r, as listed in Table II. For an illustration, see [13].

Finally, we wish to correct for the systematic effect that the finite lattice spacing ($a > 0$), the larger-than-physical

TABLE I. Details of the QCDSF $N_f = 2$ lattices made available through the ILDG, with aM_π from [10] (in one case inferred from [11]). The values of M_π, L in the same block are based on the scales (3). For comparison we add information on M_π, a, L from [9].

β	$\kappa_{\rm sea}$	Box size	aM_π [8,10]	M_π[MeV]	L[fm]	$M_\pi L$	M_π[MeV] [9]	a[fm] [9]	L[fm] [9]	Use
5.25	0.134 60	$16^3 \times 32$	0.4932(10)	1281	1.22	7.9	987(2)	0.099	1.6	•
	0.135 75	$24^3 \times 48$	0.2556(06)	664	1.82	6.1	597(1)	0.084	2.0	•
	0.136 00	$24^3 \times 48$	0.1849(—)	480	1.82	4.4				•
5.29	0.135 00	$16^3 \times 32$	0.4206(09)	1153	1.15	6.7	929(2)	0.097	1.4	
	0.135 50	$16^3 \times 32$	0.3325(13)	911	1.15	5.3				
	0.135 50	$24^3 \times 48$	0.3270(06)	896	1.73	7.8	769(2)	0.089	2.0	•
	0.135 90	$16^3 \times 32$	0.2518(15)	690	1.15	4.0				
	0.135 90	$24^3 \times 48$	0.2395(05)	656	1.73	5.7	591(2)	0.084	1.9	•
	0.136 20	$24^3 \times 48$	0.1552(06)	425	1.73	3.7	395(3)	0.080	1.9	•
	0.136 32	$24^3 \times 48$	0.1106(12)	303	1.73	2.7				○
	0.136 32	$32^3 \times 64$	0.1075(09)	295	2.30	3.4	337(3)	0.077	2.5	•
	0.136 32	$40^3 \times 64$	0.1034(08)	283	2.88	4.1				
	0.136 40	$40^3 \times 64$	0.0660(10)	181	2.88	2.6				
5.40	0.135 00	$24^3 \times 48$	0.4030(04)	1325	1.44	9.7	1037(1)	0.077	1.8	
	0.135 60	$24^3 \times 48$	0.3123(07)	1027	1.44	7.5	842(2)	0.073	1.8	•
	0.136 10	$24^3 \times 48$	0.2208(07)	726	1.44	5.3	626(2)	0.070	1.7	•
	0.136 25	$24^3 \times 48$	0.1902(06)	626	1.44	4.6				
	0.136 40	$24^3 \times 48$	0.1538(10)	506	1.44	3.7	432(3)	0.068	1.6	•
	0.136 40	$32^3 \times 64$	0.1504(04)	495	1.92	4.8				•
	0.136 60	$32^3 \times 64$	0.0867(11)	285	1.92	2.8				•

TABLE II. Tuned kappas of the Brillouin operator and final $O_{3,4}$ for each ensemble. Usually 500 configurations were downloaded; in most cases inversions were performed on more than one time slice.

β	$\kappa_{\rm sea}$	Box size	Configurations	$1/\kappa_s$	$1/\kappa_c$	O_3	O_4
5.25	0.134 60	$16^3\times 32$	2×500	7.8310(18)	8.793(19)	13.48(22)	2.220(54)
	0.135 75	$24^3\times 48$	2×500	7.8504(12)	8.612(15)	12.87(11)	2.081(38)
	0.136 00	$24^3\times 48$	2×500	7.8520(11)	8.548(11)	12.89(12)	2.114(36)
5.29	0.135 50	$24^3\times 48$	2×400	7.8601(13)	8.641(13)	12.69(16)	2.123(39)
	0.135 90	$24^3\times 48$	2×500	7.8632(15)	8.574(16)	12.68(16)	2.089(42)
	0.136 20	$24^3\times 48$	2×500	7.8635(09)	8.502(09)	12.85(14)	2.056(34)
	0.136 32	$24^3\times 48$	386	7.8615(17)	8.477(17)	13.41(25)	2.031(57)
	0.136 32	$32^3\times 64$	2×500	7.8648(08)	8.478(09)	12.57(08)	1.977(27)
5.40	0.135 60	$24^3\times 48$	2×500	7.8823(10)	8.503(10)	13.01(15)	2.163(46)
	0.136 10	$24^3\times 48$	2×500	7.8859(10)	8.463(09)	12.55(17)	2.073(37)
	0.136 40	$24^3\times 48$	2×500	7.8842(11)	8.403(12)	12.66(19)	2.041(51)
	0.136 40	$32^3\times 64$	2×500	7.8864(08)	8.417(08)	12.25(11)	1.962(28)
	0.136 60	$32^3\times 64$	2×500	7.8862(08)	8.397(08)	12.42(11)	1.955(29)

pion mass ($M_\pi > M_\pi^{\rm phys}$), and the finite spatial volume ($L^3<\infty$) have on the measured m_c/m_s, by means of a global fit to our data set. For each artifact, we invoke an extrapolation formula which is consistent with both theoretical expectations and the data. We shall consider several (reasonable) options for each effect, and treat the spread of these as the systematic error of the final result. The dominant cutoff effects may be proportional to $\alpha_s a$ (what theory suggests) or proportional to a^2 (what empirical evidence seems to prefer [6]). In the range of interest the dependence on $m_{u,d}^{\rm sea}$ may be a quadratic or cubic function of M_π. Finite volume effects may be proportional to $K_1(M_\pi L)/(M_\pi L)\sim \exp(-M_\pi L)/(M_\pi L)^{3/2}$ times $M_\pi^2/(4\pi F_\pi)^2$ [as in the case of $M_\pi(L)/M_\pi - 1$], or just proportional to $1/L^3$ (as frequently used in the old lattice literature). By combining these forms we arrive at the 8 *Ansätze*

$$r^{(i,j,k)}(a,M_\pi,L) = b[1 + c^{(i)}f^{(i)}(a) + d^{(j)}g^{(j)}(M_\pi) + e^{(k)}h^{(k)}(M_\pi,L)] \quad (4)$$

with $i,j,k\in\{1,2\}$, where $f^{(1)}=\alpha_s a$, $f^{(2)}=a^2$, $g^{(1)}=M_\pi^2$, $g^{(2)}=M_\pi^3$, $h^{(1)}=\sqrt{M_\pi/L^3}\exp(-M_\pi L)$, $h^{(2)}=1/L^3$. Note that this is the first time that we make use of the auxiliary scales (3); here we need to invoke them, since the coefficients $c^{(i)}$, $d^{(j)}$, $e^{(k)}$ are dimensionful quantities.

The last point to be discussed is which ensembles are included in the fit. It turns out that the ensemble $5.29_0.13\,632^3_{24}\times 48$ cannot be described by any of the *Ansätze*; once we drop it all versions of (4) yield consistently $\chi^2/{\rm d.o.f}\simeq 1$. For $r^{(i,j,k)}(0,M_\pi^{\rm phys},\infty)$ one finds 11.01(36), 11.02(32), 11.24(33), 11.05(35), 11.39(26), 11.41(24), 11.60(25), 11.43(26), respectively, where the errors are purely statistical. To avoid underestimating the effect of the extrapolation, we need to include the spread among these 8 results as a source of systematic uncertainty. This yields $m_c/m_s = 11.27(30)(22)$, where the standard deviation of the distribution is used as the systematic error. Since some of the ensembles feature large pion masses and small volumes we use the cuts $M_\pi < 670, 900$ MeV and/or $L>1.4$, 1.7 fm to check for any additional systematic uncertainties. This yields six independent options for our data set. The center of this enlarged distribution (from the $8\times 6 = 48$ analyses) is lower than the central value mentioned above, amounting to an additional systematic uncertainty of 0.14 which we add in quadrature to the previous one. This yields our final result

$$m_c/m_s = 11.27(30)(26) \quad (5)$$

in the continuum, at the physical mass point, and in infinite volume.

To illustrate the procedure we present one of the 8 global fits—the $(i,j,k)=(2,1,2)$ variety with $O(a^2)$, $O(M_\pi^2)$, and $O(1/L^3)$ terms—in Fig. 1. The data have been shifted by the effect of those terms which are not on display. For instance, in the continuum extrapolation panel

$$\text{plotdata}\,(a) = \text{data}(a,M_\pi,L) - \text{fit}(a,M_\pi,L) + \text{fit}(a,M_\pi^{\rm phys},\infty) \quad (6)$$

is shown as a function of a^2, while in the pion mass extrapolation panel the last term reads "fit$(0,M_\pi,\infty)$," and in the infinite volume extrapolation panel it is "fit$(0,M_\pi^{\rm phys},L)$." Note that this affects only the presentation, not the final result (5).

To test whether our assessment of systematic uncertainties is true and fair, we apply the same analysis procedure to the observable $O_4 = M_\phi^2/(M_{D_s^*}^2 - M_{D_s}^2)$. This gives 1.79 (08)(12), which agrees perfectly with the physical value 1.7707 [1]. This supports the view that our analysis procedure yields reliable estimates of the uncertainties in (5).

From quark mass ratios to individual masses.—To give the reader an idea of what can be done with our result (5), we combine it with an aggregate value of m_c to obtain

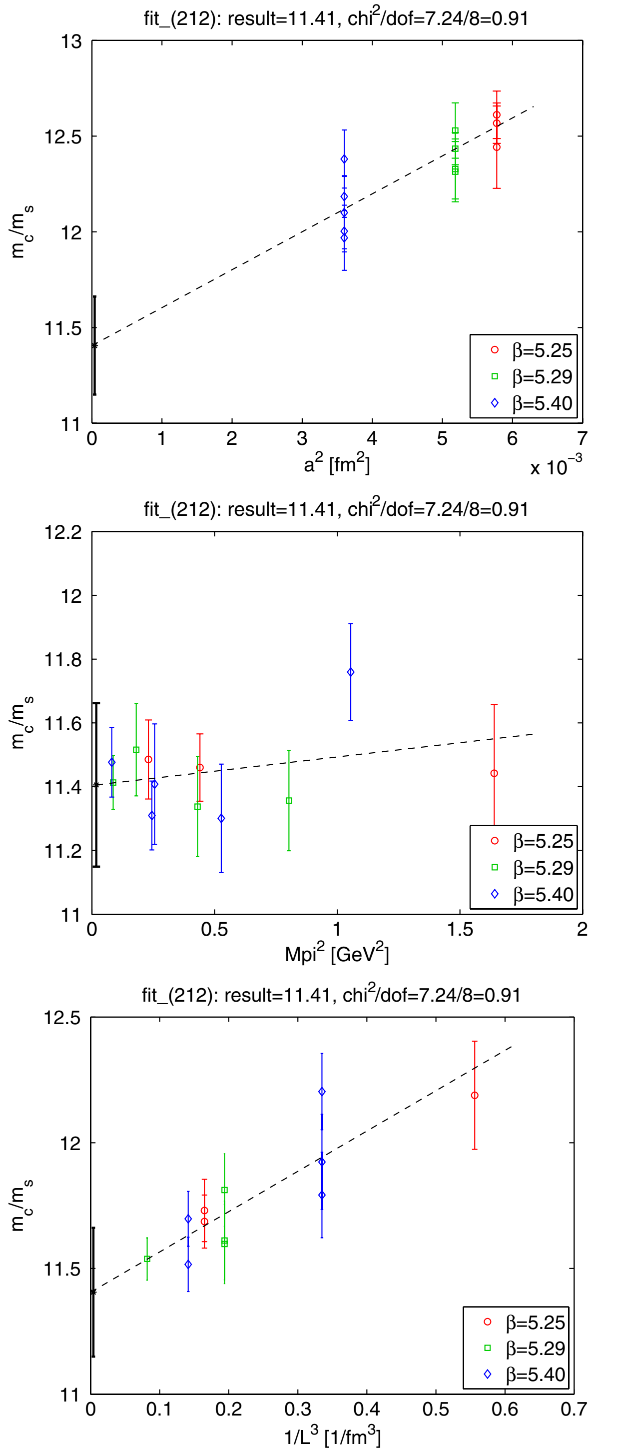

FIG. 1 (color online). One of the 8 global fits, namely, $r^{(2,1,2)}$ with $O(a^2)$, $O(M_\pi^2)$, $O(1/L^3)$ correction terms, for the joint extrapolation to zero lattice spacing, physical pion mass, and infinite volume.

an estimate of m_s. For m_c several precise results are available, which use either sum rule techniques or perturbative estimates of moments of current correlators. By contrast, computing m_s directly on the lattice involves renormalization factors like the factor Z_A/Z_P in (1) whose nonperturbative determination is technically quite demanding. Therefore, computing m_s via (5) from m_c offers the possibility to check the current best calculations of m_s (see [2] for an overview) without recurrence to Z factors [4].

We now collect the current best estimates for the charm mass, which have a 1%–2% error. The first result $m_c(3\ \text{GeV}) = 0.986(6)$ GeV [14] is based on the current correlator method on the lattice. The remaining ones are based on sum rules and experimental electron-positron annihilation cross section data, namely $m_c(3\ \text{GeV}) = 0.986(13)$ GeV [15], $m_c(m_c) = 1.277(26)$ GeV [16], and $m_c(3\ \text{GeV}) = 0.987(09)$ GeV [17], respectively (for an examination of the uncertainties involved, see, in particular, [16]). Through standard 4-loop $\overline{\text{MS}}$ running, these results can be evolved to the common scale $\mu = 2$ GeV, where they read $m_c(2\ \text{GeV}) = 1.092(7)$, 1.092(14), 1.096(22), 1.093(10) GeV, respectively. A straight mean of the central values and of the systematic uncertainties yields the conservative average $m_c(\overline{\text{MS}}, 2\ \text{GeV}) = 1.093(13)$ GeV [14–17].

Upon combining this input value with our result (5) we arrive at the estimate

$$m_s(\overline{\text{MS}}, 2\ \text{GeV}) = 97.0(2.6)(2.5)\ \text{MeV}, \qquad (7)$$

which does not build on a renormalization factor. At this point we may continue by using the ratios $m_s/m_{ud} = 27.53(20)(08)$ and $(m_d - m_u)/(m_d + m_u) = 0.381(05)(27)$ by the Budapest-Marseille-Wuppertal Collaboration [18,19], where $m_{ud} \equiv (m_u + m_d)/2$, to end up with

$$\begin{aligned} m_{ud} &= 3.52(10)(09)\ \text{MeV}, \\ m_u &= 2.18(06)(11)\ \text{MeV}, \\ m_d &= 4.87(14)(16)\ \text{MeV}. \end{aligned} \qquad (8)$$

Still, the precision reached is competitive in view of the global averages given in [2].

This concludes our illustration how the light quark masses can be obtained without recurrence to renormalization factors, at the price of including perturbative information.

Summary.—The goal of this note has been to calculate the ratio m_c/m_s, using our relativistic fermion action [6] in the valence sector, with a controlled extrapolation to zero lattice spacing, to physical sea pion mass and infinite box volume. The only systematic effect which is not controlled is the quenching of the strange and/or charm quark, but this is the case in other state-of-the-art calculations [4,5], too, and there are good reasons to believe that the effect is negligible on the scale of the error in (5) (cf. the discussion in [2]).

Our result (5) is consistent with the values $m_c/m_s = 11.85(16)$ by the HPQCD Collaboration [4] and 12.0(3) by the ETM Collaboration [5] (note that the spread among the entries in their Table 7 has not been propagated into their final error), with a slight tension at the level of 1.36σ and 1.47σ, respectively. Though nominally less precise, our result serves as an important benchmark, since our formulation bears the unique feature that it is free of any lattice-induced isospin (or taste) breaking. The relatively mild slope in $\alpha_s a$ or a^2 as determined by our global fits and the small overall spread among the entries in the $O_3 = m_c/m_s$ column of Table II support the view that the formulation [6] entails small cutoff effects up to the region of the physical charm quark mass.

For illustration, we combine our ratio (5) with an average of m_c from [14–17] to obtain the value (7) of m_s. While there are results on m_s with a higher claimed precision (see, e.g., [2] for a review), our computation is the only one which avoids both Z factors and unphysical isospin breaking effects, and this renders the result particularly robust and reliable.

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