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γ-SETS AND OTHER SINGULAR SETS OF REAL NUMBERS

Fred GALVIN*
Department of Mathematics, University of Kansas, Lawrence, Kansas 66045, USA

Arnold W. MILLER**
Department of Mathematics, The University of Texas, Austin, Texas 78712, USA

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A family of $\mathcal{J}$ of open subsets of the real line is called an ω-cover of a set X iff every finite subset of X is contained in an element of $\mathcal{J}$. A set of reals X is a γ-set iff for every ω-cover $\mathcal{J}$ of X there exists $\langle D_n: n<\omega\rangle \in \mathcal{J}^\omega$ such that

$$X \subseteq \bigcup_n \bigcap_{m>n} D_m.$$

In this paper we show that assuming Martin's axiom there is a γ-set X of cardinality the continuum.

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γ-set   Martin's axiom
ω-cover   Fréchet space

In the papers of Gerlits–Nagy [4] and McCoy [6] a study is made of properties of a space X which imply or are equivalent to other properties of the space $C(X)$ (i.e. space of continuous real-valued functions on X with the topology of pointwise convergence). A family $\mathcal{J}$ of open subsets of X is an ω-cover of X iff every finite subset of X is contained in an element of $\mathcal{J}$. A space X has the γ-property iff for every ω-cover $\mathcal{J}$ of X there exists a sequence $\langle D_n: n<\omega\rangle \in \mathcal{J}^\omega$ such that

$$X \subseteq \bigcup_m \bigcap_{n>m} D_n.$$

In McCoy [6] and Gerlits–Nagy [4] it is shown that $C(X)$ is Fréchet iff X has the γ-property. (Actually a gap in the proof of McCoy [6], Theorem 1 was found by Galvin and a correct proof was found by Gerlits.)

A space X is a *Fréchet space* if whenever $x \in \bar{A} \subseteq X$, there exists a sequence in A which converges to x.

The main result of this paper is:

Theorem 1. *Assuming Martin's axiom, there exists a γ-set of reals X of cardinality the continuum.*

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Some set theoretic hypothesis is necessary since Gerlits–Nagy [4] show that γ-sets have Rothberger's property C'' (and hence have strong measure zero). A set of reals X has property C'' iff for every sequence $\langle \mathcal{J}^n : n < \omega \rangle$ of open covers of X there exists $D_n \in \mathcal{J}_n$ such that

$$X \subseteq \bigcup_{n<\omega} D_n.$$

They also note that the γ-property implies "always of first category". It is not hard to see that the continuous image of a γ-set is a γ-set. Therefore we have the following corollary.

Corollary. *Assuming* MA, *there is a set of reals X of cardinality the continuum such that every continuous image of X has property C'' and is always of the first category.*

This answers a question of Sierpiński [7].

It is not hard to see that "every subspace of the real line of cardinality less than c has property γ" is equivalent to the statement: "if $A \subseteq P(\omega)$, $|A| < c$, and $|\bigcap A_0| = \omega$ for every finite $A_0 \subseteq A$, then there is a set $X \in [\omega]^\omega$ such that $X - Y$ is finite for all $Y \in A$". What we show is in fact that if every subspace of the real line of cardinality $< c$ has property γ, then some subspace of the real line of cardinality c has property γ. This was pointed out to us by Alan D. Taylor.

Theorem 1 was also proved (independently) by Ryszard Frankiewicz.

To prove Theorem 1 we will need the following Lemma.

Lemma 1.1. (MA) *Suppose X is a set of reals of cardinality less than the continuum. Then X is a γ-set.*

Proof. This is due to Gerlits–Nagy [4]. □

We work in Cantor space 2^ω which is the countably infinite product of the two point discrete space. In fact, identify 2^ω with $P(\omega)$ (the set of all subsets of $\omega = \{0, 1, 2, \ldots\}$) via characteristic functions. Let $[\omega]^{<\omega}$ be the set of finite subsets of ω. For $Y \subseteq \omega$ define $Y^* = \{X \subseteq \omega \mid Y \backslash X \text{ is finite}\}$ and let $[Y]^\omega$ be the set of infinite subsets of Y.

Lemma 1.2. *Suppose $X \in [\omega]^\omega$ and $\mathcal{J}$ is an open ω-cover of $[\omega]^{<\omega}$. Then there exists $\langle D_n : n < \omega \rangle \in \mathcal{J}$ and $Y \in [X]^\omega$ such that*

$$Y^* \subseteq \bigcup_n \bigcap_{m>n} D_m.$$

Proof. Construct an increasing sequence $\langle k_n : n < \omega \rangle$ from X and a sequence $\langle D_n : n < \omega \rangle$ from $\mathcal{J}$ such that for each n and $A \subseteq \omega$ if $A \cap (k_n, k_{n+1}) = \emptyset$ then $A \in D_n$ ($(k_n, k_{n+1}) = \{l \in \omega : k_n < l < k_{n+1}\}$). To see that this can be done suppose that we already have $\langle D_1, D_2, \ldots, D_{n-1}\rangle$ and $\langle k_1, k_2, \ldots, k_n\rangle$. Let $Q=\{q_i: i<2^{k_n+1}\}$ be all elements of $[\omega]^{<\omega}$ contained in k_n+1. Since $\mathcal{J}$ is an ω-cover of $[\omega]^{<\omega}$ there exists $D_n \in \mathcal{J}$ which covers Q. Now since D_n is open there is some $k_{n+1} \in X$ such that for all $q \in Q$

$$\{A \subseteq \omega: A \cap k_{n+1} = q \cap k_{n+1}\} \subseteq D_n.$$

This does the construction. We claim that $Y=\{k_n: n<\omega\}$ has the required properties. To see this suppose $Z-Y \subseteq k_n$. But then for all $m>n$: $Z \cap (k_m, k_{m+1}) = \emptyset$ and hence $Z \in D_m$. □

Now we prove Theorem 1.

Proof of Theorem 1. Let $\mathcal{J}_\alpha$ for $\alpha<c$ be all of the countable families of open sets. Note that any open ω-cover can be refined to a countable open ω-cover consisting of clopen sets. Construct $X_\alpha \in [\omega]^\omega$ for $\alpha<c$ so that if $\alpha<\beta$, then $X_\beta \setminus X_\alpha$ is finite. At stage α use MA to get $X_\alpha \in [\omega]^\omega$ with $X_\alpha \setminus X_\beta$ finite for all $\beta<\alpha$. If $\mathcal{J}_\alpha$ is an ω-cover of

$$[\omega]^{<\omega} \cup \{X_\beta: \beta \leq \alpha\},$$

then use Lemma 1.1 to obtain $\langle D_n: n<\omega\rangle \in \mathcal{J}_\alpha^\omega$ such that

$$\{X_\beta: \beta \leq \alpha\} \cup [\omega]^{<\omega} \subseteq \bigcup_n \bigcap_{m>n} D_m.$$

Since $\{D_n: n<\omega\}$ is a ω-cover of $[\omega]^{<\omega}$ by Lemma 1.2 there exists k_n for $n<\omega$ and $X_{\alpha+1} \in [X_\alpha]^\omega$ such that

$$X_{\alpha+1}^* \subseteq \bigcup_n \bigcap_{m>n} D_{k_m}$$

and hence

$$\{X_\beta: \beta \leq \alpha\} \cup X_{\alpha+1}^* \subseteq \bigcup_n \bigcap_{m>n} D_{k_m}. \quad \square$$

Let $X=\{X_\alpha: \alpha<c\}$. The γ-set $[\omega]^{<\omega} \cup X$ is c-concentrated on $[\omega]^{<\omega}$, i.e. for any open $U \supseteq [\omega]^{<\omega}$, $X \setminus U$ has cardinality less than c. This implies that X is not a γ-set.

Theorem 2. *If X is any γ-set and G is a G_δ set containing X, then there exists an F_σ set F with $X \subseteq F \subseteq G$.*

Proof. Let $G=\bigcap_{n<\omega} O_n$ where O_n is open and $O_{n+1} \subseteq O_n$. For each $F \in [X]^{<\omega}$ let G_F be open with

$$F \subseteq G_F \subseteq \bar{G}_F \subseteq O_{|F|}.$$

Since X is a γ-set there would exist F_n such that

$$X \subseteq \bigcup_n \bigcap_{m>n} G_{F_m} \subseteq \bigcup_n \bigcap_{m>n} \bar{G}_{F_m}.$$

To make sure that $\bigcup_n \bigcap_{m>n} \bar{G}_{F_m}$ is contained in G it is enough to insure that $\{|F_n|: n<\omega\}$ is infinite. To get this let $\langle x_n: n<\omega\rangle$ be a sequence of distinct elements of X and instead choose for each n and $F \in [X\backslash\{x_n\}]^n$ an open set G_F with

$$F \subseteq G_F \subseteq \bar{G}_F \subseteq O_n$$

and $x_n \notin G_F$. Since x_n must be in all but finitely many G_{F_m} at most finitely many F_n have cardinality n. $\square$

It is true that the γ-property is hereditary for F_σ subsets of a γ-set.

Theorem 3. *Suppose X is a γ-set and Y an F_σ-set. Then $X \cap Y$ is a γ-set.*

Proof. Let $Y = \bigcup_{n<\omega} Y_n$ with Y_n closed and $Y_n \subseteq Y_{n+1}$. Suppose $\mathcal{J}$ is an open ω-cover of $X \cap Y$. Consider

$$\mathcal{J}^* = \{D \cup (\sim Y_n): D \in \mathcal{J} \text{ and } n<\omega\}.$$

Then $\mathcal{J}^*$ is an open ω-cover of X hence there exists D_{k_n} such that

$$X \subseteq \bigcup_m \bigcap_{n>m} (D_{k_n} \cup (\sim Y_{k_n})).$$

To insure that $\{k_n: n<\omega\}$ is infinite the same trick as in Theorem 2 can be employed, i.e. $(D \cup (\sim Y_n))\backslash\{x_n\}$. $\square$

In answer to a question of ours S. Todorčević showed that it is consistent to have a γ-set of cardinality c all of whose subsets are a γ-set. With his permission we include this result here.

Theorem 4. (S. Todorčević). *Assuming $\diamondsuit_{\omega_1}$ there exists a γ-set of X of cardinality $\omega_1 = c$ all of whose subsets are also a γ-set.*

Proof. For p a perfect subtree of $2^{<\omega}$ let $[p]$ denote the infinite branches of p. Let $D(p)$ denote any canonical element of $[p]$ (i.e. for example the branch which is left most). The set X will be equal to $\{D(p): p \in T\}$ where T is some Aronszajn tree of perfect sets. For $n<\omega$ define $p \leqslant_n q$ iff $p \subseteq q$ and $p \cap 2^n = q \cap 2^n$. Inductively build the tree T so that for all $\alpha<\beta$, $n<\omega$, and $p \in T_\alpha$ there exists $q \in T_\beta$ such that $q \leqslant_n p$. First let us note how this hypothesis allows us to construct a Aronzajn tree of perfect subsets. Suppose λ is a limit $>\omega_1$ and T_α for $\alpha<\lambda$ has been constructed. Choose α_n for $n<\omega$ increasing and cofinal in λ. Suppose $p_0 \in T_{\alpha_0}$ and construct a sequence $p_{n+1} \in T_{\alpha_{n+1}}$ and an increasing sequence $k_n < \omega$ so that

$$p_{n+1} \leqslant_{k_n} p_n$$

and k_{n+1} has the property that for all $s \in p_{n+1} \cap 2^{k_n}$ there are two incompatible extensions t_0, t_1 of s in $p_{n+1} \cap 2^{k_{n+1}}$. Then by the standard fusion argument $\bigcap_{n<\omega} p_n$ is itself of perfect subtree of $2^{<\omega}$. This would be one of the nodes at level λ. By doing it for all $p \in T_{<\lambda}$ and $n < \omega$ we can preserve the inductive hypothesis. In order to make the set and all its subsets into a γ-set we will need to strengthen the inductive assumption. While the next Lemma will not be used to prove the Theorem, its proof will suggest what we do next.

Lemma 4.1. *Suppose p is a perfect tree, and $\mathcal{J}$ is an open ω-cover of $[p]$. Then there exists a perfect tree $q \subseteq p$ and $D_n \in \mathcal{J}$ such that*

$$[q] \subseteq \bigcap_{n<\omega} D_n.$$

Proof. Build a sequence of perfect trees p_n (with $p = p_0$), $D_n \in \mathcal{J}$, and increasing k_n such that:

(1) $p_{n+1} \leqslant_{k_n} p_n$ and every $s \in p_n \cap 2^{k_n}$ has two incompatible extensions in $p_{n+1} \cap 2^{k_{n+1}}$; and

(2) $[p_{n+1}] \subseteq D_{n+1}$.

Suppose we have k_n and p_n. Let $X \subseteq [p_n]$ be finite and such that $p_n \cap 2^{k_n} = \{x \restriction k_n : x \in X\}$. Since $\mathcal{J}$ is an ω-cover there exists $D_{n+1} \in \mathcal{J}$ with $X \subseteq D_{n+1}$. Now choose $k_{n+1} > k_n$ large enough so that (1) is satisfied and also

$$\{y \in 2^\omega : \exists x \in X \quad x \restriction k_{n+1} = y \restriction k_{n+1}\} \subseteq D_{n+1}.$$

Now let

$$p_{n+1} = \{t \in p_n : \exists x \in X \quad t \text{ and } x \restriction k_{n+1} \text{ are comparable}\}.$$

Then letting $q = \bigcap_{n<\omega} p_n$ we see that by (1) q is a perfect tree and by (2) it has the property that

$$[q] \subseteq \bigcap_{n<\omega} D_n. \quad \square$$

Define for $R \subseteq q \cap 2^m$ and p and q perfect trees, $p \leqslant_R q$ iff $p \cap 2^m = R$ and $p \subseteq q$. In order to do the above argument as we inductively construct our Aronzajn tree of perfect subsets we will demand that for any $\alpha < \beta$, $q \in T_\alpha$, $m < \omega$, and $R \subseteq q \cap 2^m$, there exists $p \in T_\beta$ such that $p \leqslant_R q$. Now suppose we build our tree T and defined $X = \{D(p) : p \in T\}$. And also $Y \subseteq X$ and we are trying to show Y is a γ-set. So suppose $\mathcal{J}$ is an open countable ω-cover of Y and for each $p \in T$ let

$$C_p = \text{closure of } Y \cap [p].$$

What $\diamond_{\omega_1}$ allows us to assume is that there are sequences $\mathcal{J}_\alpha$, $\langle C_p^\alpha : p \in Q_\alpha \rangle$, and B^α for $\alpha < \omega_1$ such that for any $Y \subseteq X$ and $\mathcal{J}$ there are stationarily many $\alpha < \omega_1$ such that

$$Q_\alpha = T_{<\alpha};$$

$$\mathcal{J}_\alpha = \mathcal{J};$$

$$C_p^\alpha = C_p \text{ for all } p \in T_{<\alpha}; \text{ and}$$

$$B^\alpha = Y \cap \{D(p): p \in T_{<\alpha}\}.$$

This is possible because B^α is a countable set, $\mathcal{J}$ is a countable set of open sets and so it can be coded by a subset of ω, and C_p^α is a closed set for each $p \in T_{<\alpha}$.

Let us assume that α is a limit ordinal for which our $\diamond_{\omega_1}$ sequence has caught Y and $\mathcal{J}$ and for notational convenience drop the sub and superscript α. It is thus necessary to build T_α and $D_n \in \mathcal{J}$ so that

$$B \subseteq \bigcup_m \bigcap_{n>m} D_n;$$

and for each $p \in T_\alpha$

$$Y \cap [p] \subseteq \bigcup_m \bigcap_{n>m} D_n;$$

as well as preserving our inductive hypothesis on T. First note that since $\mathcal{J}$ is an ω-cover of Y for any $F \in [B]^{<\omega}$ there exists $D \in \mathcal{J}$ such that $F \subseteq D$. Also for any $\beta_0 < \beta_1 < \alpha$, $q_0 \in T_{\beta_0}$, and $R \subseteq q \cap 2^n$, there exists $q_1 \in T_{\beta_1}$ and $D \in \mathcal{J}$ such that $q_1 \leq_R q_0$ and $C_{q_1} \subseteq D$. To see how to do this let

$$R' = \{s \in R: C(q_0) \cap [s] \neq \emptyset\}.$$

Since $C(q_0) = \mathrm{cl}([q_0] \cap Y)$ it must be true that for some $D \in \mathcal{J}$ for all $s \in R'$, $C(q_0) \cap [s] \cap D \neq \emptyset$. Now find $m > n$ and $T \subseteq 2^m \cap q_0$ such that for all $s \in R$ there exists a unique $t \in T$ such that $s \subseteq t$ and if $s \in R'$, then $[t] \subseteq D$. Then if $q_1 \leq_T q_0$ (whose existence is guaranteed by our inductive hypothesis), we have that $C_{q_1} \subseteq D$.

Note by the same argument we can show that given $F \in [B]^{<\omega}$ and finitely many $q_0^i \in T_{\beta_0^i}$, $R_i \subseteq q_0^i \cap 2^{n_i}$, and β_1^i for $i < N$ we can find $D \in \mathcal{J}$ and $q_1^i \in T_{\beta_1^i}$ such that

$$F \cup \bigcup_{i<N} C(q_1^i) \subseteq D.$$

Thus by dovetailing all we want to do into ω many steps, we construct T_α with the required properties. □

For $X \subseteq [0, 1]$ let

$$X + 1 = \{x + 1: x \in X\}.$$

Theorem 5. *Suppose $A \subseteq X \subseteq [0, 1]$ and $(X \backslash A) \cup (A + 1)$ is a γ-set. Then A is G_δ and F_σ in X.*

Proof. For each $F\in[(X\backslash A)\cup(A+1)]^{<\omega}$ let C_F, $D_F\subseteq[0,1]$ be open sets with disjoint closures such that

$$F\subseteq C_F\cup(D_F+1).$$

By the γ-property there exists F_n for $n<\omega$ such that

$$(X\backslash A)\cup(A+1)\subseteq\bigcup_n\bigcap_{m>n}(C_{F_m}\cup(D_{F_m}+1)).$$

Since $\bar{C}_{F_n}$ and $\bar{D}_{F_n}$ are disjoint

$$\bigcup_n\bigcap_{m>n}\bar{C}_{F_m}\text{ and }\bigcup_n\bigcap_{m>n}\bar{D}_{F_m}$$

are disjoint, and they show that $X\backslash A$ and A are F_σ in X. □

Thus as a corollary to Todorčevic's result we have that it is consistent that there are γ-sets X and Y such that neither $X\times Y$ nor $X\cup Y$ is a γ-set. (Note that $X\cup(Y+1)$ is homeomorphic to the closed subset of $X\times Y$, $(X\times\{y_0\})\cup(\{x_0\}\times Y)$.) To see this, let X be a γ-set of cardinality c all of whose subsets are γ-sets. Let $A\subseteq X$ be neither G_δ nor F_σ in X. Then by Theorem 5, $(X\backslash A)\cup(A+1)$ is not a γ-set while both $X\backslash A$ and $A+1$ are γ-sets.

Our method of construction is very similar to that employed by Friedman–Talagrand [2] and Erdös–Kunen–Mauldin [1]. In these two papers it is shown that assuming Martin's axiom there is a set of reals X of cardinality the continuum with the property that for every Y of measure zero the set $X+Y$ has measure zero and for every Y of first category the set $X+Y$ has first category.

Next we show that if X is a γ-set, then for every Y of first category the set $X+Y$ has first category.

Theorem 6. *Suppose X is a γ-set. Then for every Y first category, $X+Y$ has first category.*

Proof. Let I and J always denote intervals and C a finite union of intervals. R is the real line.

Lemma 6.1. *Suppose P is a compact nowhere dense set, $F\in[R]^{<\omega}$ and I_i for $i<n$ arbitrary. Then there exists $C\supseteq F$ and $J_i\subseteq I_i$ for $i<n$ with*

$$\bar{J}_i\cap(\bar{C}+P)=\emptyset.$$

Proof. Let C_i for $i<\omega$ be decreasing with $F\subseteq C_i$ and $\bigcap_{i<\omega}\bar{C}_i=F$. Since $F+P$ is closed nowhere dense there exists $J_i\subseteq I_i$ for $i<n$ with $\bar{J}_i\cap(F+P)=\emptyset$. Since

$$\left(\bigcap_{m<\omega}\bar{C}_m\right)+P=\bigcap_{m<\omega}(\bar{C}_m+P)$$

by compactness there exists m for every $i<n$

$$\bar{J}_i \cap (\bar{C}_m + P) = \emptyset. \quad \square$$

Clearly it is enough to prove Theorem 6 for $Y=P$ a compact nowhere dense set. Let $\{I_n: n<\omega\}$ list all intervals with rational end points. Let $\mathcal{J}_n$ be a family of open sets such that for all $C \in \mathcal{J}_n$ there exists $J_m \subseteq I_m$ for $m<n$ such that

$$\bar{J}_m \cap (\bar{C} + P) = \emptyset,$$

and $\mathcal{J}_n$ covers the n element subsets of X. Let $\{x_n: n<\omega\}$ be distinct elements of X and let

$$\mathcal{J} = \bigcup_n \{C \setminus \{x_n\}: C \in \mathcal{J}_n\}.$$

$\mathcal{J}$ is an open ω-cover of X and thus there exists C_n

$$X \subseteq \bigcup_n \bigcap_{m>n} C_m$$

and we may assume $C_m \in \mathcal{J}_{k_m}$ where the k_m are distinct. But by construction, for all $n<\omega$

$$\bigcap_{m>n} \bar{C}_m + P$$

is nowhere dense. $\square$

We are unable to prove Theorem 6 with measure zero in place of first category. But we are able to if the γ-property is replaced by the strong γ-property. We say that X is a strong γ-set iff there exists an increasing sequence $\langle k_n: n<\omega\rangle$ such that for any sequence $\langle \mathcal{J}_n: n<\omega\rangle$ where $\mathcal{J}_n$ is an open cover of $[X]^{k_n}$ there exists $\langle C_n: n<\omega\rangle$ with $C_n \in \mathcal{J}_n$ and

$$X \subseteq \bigcup_n \bigcap_{m>n} C_m.$$

Theorem 7. *If X is a strong γ-set, then for any measure zero Y, $X+Y$ has measure zero.*

Proof. Let C, D stand for sets which are finite unions of intervals and let I, J stand for intervals. μ is Lebesgue measure and we assume $X \subseteq [0,1]$.

Lemma 7.1. *For every $n<\omega$ and $F \in [0,1]^n$, D, and $\varepsilon>0$ there exists $C \supseteq F$ such that*

$$\mu(C+D) \leq n\mu(D) + \varepsilon.$$

Proof. Suppose $D = \bigcup_{k<N} J_k$, where the J_k are disjoint intervals, so $\mu(D) =$

$\sum_{k<N} \mu(J_k)$. Since $\mu(I+J)=\mu(I)+\mu(J)$,

$$\mu\left(I+\bigcup_{k<N} J_k\right) \leqslant \sum_{k<N} (\mu(I)+\mu(J_k)) \leqslant N\mu(I)+\mu(D).$$

Let $I_1, I_2, \ldots, I_n$ cover F and

$$\sum_{1\leqslant i\leqslant n} N\mu(I_i) < \varepsilon.$$

Then

$$\mu\left(\bigcup_{1\leqslant i\leqslant n} I_i + \bigcup_{k<N} J_k\right) \leqslant \sum_{1\leqslant i\leqslant n} \mu\left(I_i + \bigcup_{k<N} J_k\right)$$

$$\leqslant \sum_{1\leqslant i\leqslant n} N\mu(I_i) + n\mu(D). \quad \square$$

Lemma 7.2. *If $\mu(Y)=0$, then for any sequence $\varepsilon_n>0$ for $n<\omega$ there exists D_n such that $\mu(D_n)<\varepsilon_n$ and $Y\subseteq \bigcap_n \bigcup_{m>n} D_m$.*

Proof. Well known. $\square$

We now show that $\mu(X+Y)=0$. For any sequence $\varepsilon_n>0$ let D_n be as from Lemma 7.2. For each n let $\mathcal{J}_n$ be a cover of $[X]^{k_n}$ such that for all $C_n\in\mathcal{J}_n$

$$\mu(C_n+D_n)\leqslant k_n\mu(D_n)+\varepsilon_n.$$

By the strong γ-property there exists $C_n\in\mathcal{J}_n$ such that

$$X\subseteq \bigcup_n \bigcap_{m>n} C_m.$$

Then

$$X+Y\subseteq \bigcup_n \bigcap_{m>n} C_m + \bigcap_n \bigcup_{m>n} D_m \subseteq \bigcup_n (C_n+D_n).$$

Since $\sum_n (k_n+1)\varepsilon_n$ can be made arbitrarily small $X+Y$ has measure zero. $\square$

The definition of strong γ-set was completely motivated by Theorem 7. We now indicate how to modify the proof of Theorem 1 to get large strong γ-sets.

Theorem 8. (MA). *There exists a set of reals X of cardinality c which is a strong γ-set.*

Proof. Let $k_{n+1}\geqslant 2\cdot k_n + n\cdot 2^n$.

Lemma 8.1. (MA). *Suppose F is a set of reals of cardinality $< c$, $X\in[\omega]^\omega$, and $\langle \mathcal{J}_n: n<\omega\rangle$ is a sequence such that $\mathcal{J}_n$ is an open cover $[F\cup[\omega]^{<\omega}]^{k_n}$. Then there exists $Y\in[X]^\omega$ and $D_n\in\mathcal{J}_n$ such that*

$$F\cup Y^*\subseteq \bigcup_n \bigcap_{m>n} D_m.$$

Proof. Let F_n for $n<\omega$ list $[\omega]^{<\omega}$ with infinite repetitions and $F_n \subseteq \{0,1,2,\ldots,n-1\}$. We can assume each $\mathscr{J}_n$ is countable. Consider the partial order $\boldsymbol{P}$ consisting of the set of all triples (s, D, G) satisfying:

(1) $s=\langle s_1, s_2, \ldots, s_n\rangle$ is an increasing sequence from X;
(2) $D=\langle D_1, D_2, \ldots, D_n\rangle$ where $D_i \in \mathscr{J}_i$;
(3) $G\in[F]^{\leqslant k_n}$; and
(4) if $i<j\leqslant n$ and $A\backslash\{s_1, s_2, \ldots, s_n\}=F_i$, then $A\in D_j$.

We define $(s', D', G')\leqslant(s, D, G)$ iff

(1) s' extends s;
(2) D' extends D;
(3) $G'\supseteq G$; and
(4) $G\subseteq\bigcap_{n<i\leqslant n'} D'_i$.

First we note that every condition $\{s, D, G\rangle$ can be extended in the $\langle s, D\rangle$ part. Suppose $s=\langle s_1, s_2, \ldots, s_n\rangle$ and $D=\langle D_1, D_2, \ldots, D_n\rangle$. Let $Q=\{q\in[\omega]^{<\omega}: q\backslash\{s_1, s_2, \ldots, s_n\}=F_i$ for some $i\geqslant n\}$, then $|Q|=n\cdot 2^n$. Then $k_{n+1}\geqslant|G\cup Q|$, so choose $D_{n+1}\in\mathscr{J}_{n+1}$ which covers $G\cup Q$. Since D_{n+1} is open and covers Q, there exists $s_{n+1}>s_n$ so that for any $q\in Q$,

$$\{x\subseteq\omega: x\cap\{0,1,2,\ldots,s_{n+1}\}=q\cap\{0,1,2,\ldots,s_{n+1}\}\}\subseteq D_{n+1}.$$

Thus (4) is satisfied for this s_{n+1}. If $s'=\langle s_1, s_2, \ldots, s_{n+1}\rangle$ and $D'=\langle D_1, D_2, \ldots, D_{n+1}\rangle$, then $\langle s', D', G\rangle\leqslant\langle s, D, G\rangle$. This shows that for each $n<\omega$

$$\{(s, D, G): \text{length of } s \text{ and } D \text{ is at least } n\}$$

is dense in $\boldsymbol{P}$. Similarly for any $x\in F$

$$\{(s, D, G): x\in G\}$$

is dense in $\boldsymbol{P}$, since

$$(s, D, G\cup\{x\})\leqslant(s, D, G)$$

as long as $|G\cup\{x\}|\leqslant k_n$. To see that $\boldsymbol{P}$ has the countable chain condition consider any family of ω_1 conditions. Suppose each (s, D) part has length n and $G\in[F]^{k_n}$. Extend each (s, D) part to have length $n+1$, without increasing G. Since there are only countably many possible (s, D) parts, there must be two conditions (s, D, G) and (s, D, G'). Since $k_{n+1}\geqslant 2k_n$ we have that $(s, D, G\cup G')$ is a condition extending them both. By Martin's axiom we can find a generic filter meeting our dense sets. Let $Y=\{s_n: 1\leqslant n<\omega\}$ and $\langle D_n: 1\leqslant n<\omega\rangle$ be given by it. Clearly by the definition of extension

$$F\subseteq\bigcup_n\bigcap_{m>n} D_m.$$

Suppose $A\in Y^*$ (i.e. $Y\backslash A$ is finite). Then there exists infinitely many i so that $A\cap Y=F_i$. But then for all j and n, $i<j\leqslant n$, $A\cap\{s_1, s_2, \ldots, s_n\}=F_i$ and so $A\in D_j$.

Hence

$$Y^* \subseteq \bigcup_n \bigcap_{m>n} D_m.$$

This proves the Lemma. The standard transfinite induction completes the proof of the Theorem.

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