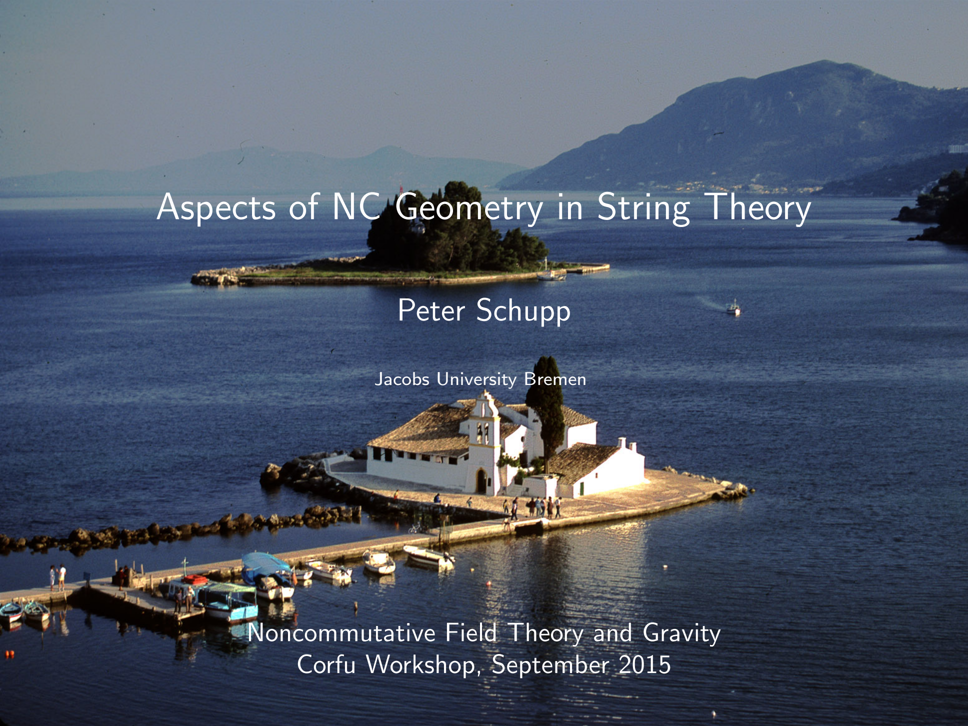


Aspects of NC Geometry in String Theory

Peter Schupp

Jacobs University Bremen

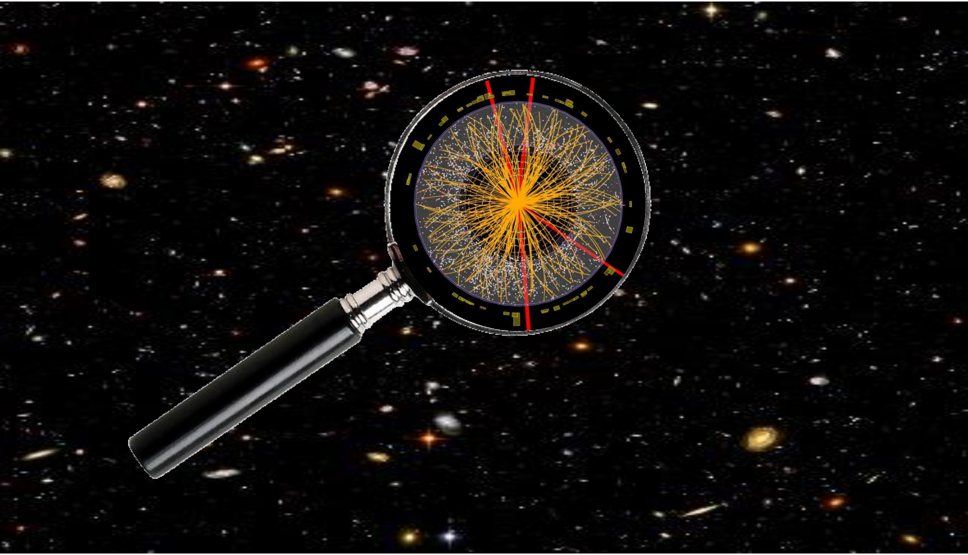
Noncommutative Field Theory and Gravity
Corfu Workshop, September 2015

Outline

- General aspects of quantization
- Strings and noncommutative geometry
- Strings and generalized geometry
- Nonassociativity and Quantum Physics

Introduction

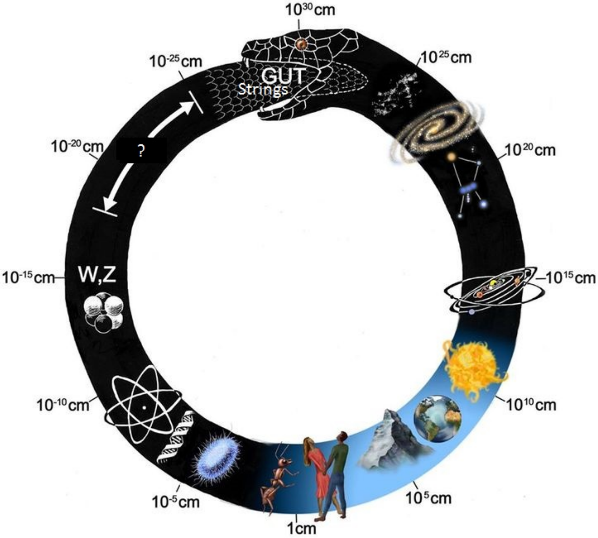
macrocosmos vs. microcosmos : general relativity vs. quantum field theory



see beyond the observable universe: mathematical structure of nature

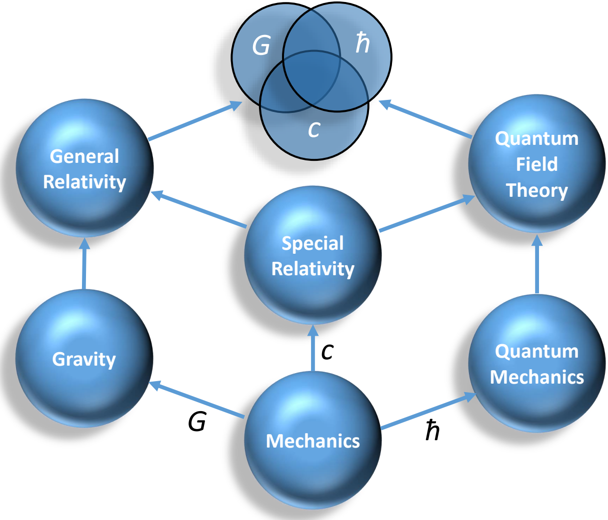
Introduction

Cosmic Ouroboros: large scale structures from small scale quantum fluctuations



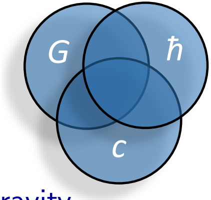
Introduction

“Geometry” $\longrightarrow$ Noncommutative/Generalized Geometry $\longleftarrow$ “Algebra”



deformation and unification $G, c, \hbar$ – plus Boltzmann's k

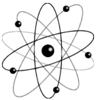
Introduction



Quantum theories of gravity

- String Theory/M-theory
  extended objects: strings, D-branes, M2/M5-branes,. . .
- Matrix-Theory; emergent gravity
- Loop Quantum Gravity, Group Field Theory, . . .

quantum + *gravity* ⇒



Generalize geometry

- microscopic non-commutative/non-associative spacetime structures

Aspects of quantization

Noncommutative geometry considers the algebra of functions on a manifold and replaces it by a noncommutative algebra:

- Gelfand–Naimark:
  spacetime manifold $\rightsquigarrow$ noncommutative algebra
  "points" $\rightsquigarrow$ irreducible representations
- Serre–Swan:
  vector bundles $\rightsquigarrow$ projective modules
- Connes: noncommutative differential geometry
  (Dirac operator, spectral triple, ...)

  almost NC Standard Model: Higgs = gauge field in discrete direction

We shall concentrate on algebraic aspects in these lectures.

Aspects of quantization $\theta(x) \rightsquigarrow \star$

Deformation quantization of the point-wise product in the direction of a Poisson bracket $\{f, g\} = \theta^{ij}\partial_i f \cdot \partial_j g$:

$$f \star g = fg + \frac{i\hbar}{2}\{f, g\} + \hbar^2 B_2(f, g) + \hbar^3 B_3(f, g) + \dots \ ,$$

with suitable bi-differential operators B_n.

There is a natural gauge symmetry: "equivalent star products"

$$\star \mapsto \star' \ , \quad Df \star Dg = D(f \star' g) \ ,$$

with $Df = f + \hbar D_1 f + \hbar^2 D_2 f + \dots$

Weyl quantization associates operators to polynomial functions via symmetric ordering: $x^\mu \rightsquigarrow \hat{x}^\mu$, $x^\mu x^\nu \rightsquigarrow \frac{1}{2}(\hat{x}^\mu \hat{x}^\nu + x^\nu \hat{x}^\mu)$, etc.
extend to functions, define star product $\widehat{f_1 \star f_2} := \widehat{f_1}\,\widehat{f_2}$.

Aspects of quantization $\theta(x) \rightsquigarrow \star$

Deformation quantization of the point-wise product in the direction of a Poisson bracket $\{f, g\} = \theta^{ij}\partial_i f \cdot \partial_j g$:

$$f \star g = fg + \frac{i\hbar}{2}\{f, g\} + \hbar^2 B_2(f, g) + \hbar^3 B_3(f, g) + \dots \ ,$$

with suitable bi-differential operators B_n.

There is a natural gauge symmetry: "equivalent star products"

$$\star \mapsto \star' \ , \quad Df \star Dg = D(f \star' g) \ ,$$

with $Df = f + \hbar D_1 f + \hbar^2 D_2 f + \dots$

Weyl quantization associates operators to polynomial functions via symmetric ordering: $x^\mu \rightsquigarrow \hat{x}^\mu$, $x^\mu x^\nu \rightsquigarrow \frac{1}{2}(\hat{x}^\mu \hat{x}^\nu + x^\nu \hat{x}^\mu)$, etc.

extend to functions, define star product $\widehat{f_1 \star f_2} := \widehat{f_1}\,\widehat{f_2}$.

Aspects of quantization $\theta(x) \rightsquigarrow \star$

for $\theta =$ const.:

Moyal-Weyl star product

$$
\begin{aligned}
(f \star g)(x) &= \cdot \left[e^{\frac{i}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu} (f \otimes g) \right] \\
&\equiv \sum \frac{1}{m!} \left(\frac{i}{2}\right)^m \theta^{\mu_1\nu_1} \dots \theta^{\mu_m\nu_m} (\partial_{\mu_1} \dots \partial_{\mu_m} f)(\partial_{\nu_1} \dots \partial_{\nu_m} g) \\
&= f \cdot g + \frac{i}{2}\theta^{\mu\nu}\partial_\mu f \cdot \partial_\nu g + \dots
\end{aligned}
$$

partials commute, $[\partial_\mu, \partial_\nu] = 0 \;\Rightarrow$ star product $\star$ is associative

e.g. canonical commutation relations for $(X^I) = (x^1, \dots, x^d, p_1, \dots, p_d)$

$$[X^I, X^J]_\star = i\hbar\Theta^{IJ} \qquad \text{with } \Theta = \theta = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

starting point for phase-space formulation of QM

Aspects of quantization $\theta(x) \rightsquigarrow \star$

for $\theta = $ const.:

Moyal-Weyl star product

$$
\begin{aligned}
(f \star g)(x) &= \cdot \left[e^{\frac{i}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu} (f \otimes g) \right] \\
&\equiv \sum \frac{1}{m!} \left(\frac{i}{2}\right)^m \theta^{\mu_1\nu_1} \dots \theta^{\mu_m\nu_m} (\partial_{\mu_1} \dots \partial_{\mu_m} f)(\partial_{\nu_1} \dots \partial_{\nu_m} g) \\
&= f \cdot g + \frac{i}{2}\theta^{\mu\nu}\partial_\mu f \cdot \partial_\nu g + \dots
\end{aligned}
$$

partials commute, $[\partial_\mu, \partial_\nu] = 0 \;\Rightarrow$ star product $\star$ is associative

e.g. canonical commutation relations for $(X^I) = (x^1, \dots, x^d, p_1, \dots, p_d)$

$$
[X^I, X^J]_\star = i\hbar \Theta^{IJ} \qquad \text{with } \Theta = \theta = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}
$$

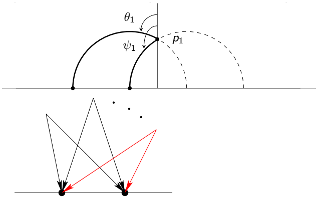
starting point for phase-space formulation of QM

Aspects of quantization $\theta(x) \rightsquigarrow \star$

Kontsevich formality and star product

U_n maps n k_i-multivector fields to a $(2 - 2n + \sum k_i)$-differential operator

$$U_n(\mathcal{X}_1, \ldots, \mathcal{X}_n) = \sum_{\Gamma \in G_n} w_\Gamma \, D_\Gamma(\mathcal{X}_1, \ldots, \mathcal{X}_n) \ .$$



The star product for a given bivector θ is:

$$f \star g = \sum_{n=0}^{\infty} \frac{(\mathrm{i}\,\hbar)^n}{n!} \, U_n(\theta, \ldots, \theta)(f, g)$$

$$= f \cdot g + \frac{i}{2} \sum \theta^{ij} \, \partial_i f \cdot \partial_j g - \frac{\hbar^2}{4} \sum \theta^{ij} \theta^{kl} \, \partial_i \partial_k f \cdot \partial_j \partial_l g$$

$$- \frac{\hbar^2}{6} \left(\sum \theta^{ij} \partial_j \theta^{kl} \, (\partial_i \partial_k f \cdot \partial_l g - \partial_k f \cdot \partial_i \partial_l g) \right) + \ldots$$

Kontsevich (1997)

Aspects of quantization $\theta(x) \rightsquigarrow \star$

AKSZ construction: action functionals in BV formalism of sigma model QFT's in $n+1$ dimensions for symplectic Lie n-algebroids E

Alexandrov, Kontsevich, Schwarz, Zaboronsky (1995/97)

$n = 1$ (open string):

Poisson sigma model

2-dimensional topological field theory, $E = T^*M$

$$S^{(1)}_{\mathrm{AKSZ}} = \int_{\Sigma_2} \Big(\xi_i \wedge \mathrm{d}X^i + \frac{1}{2}\, \theta^{ij}(X)\, \xi_i \wedge \xi_j \Big) \ ,$$

with $\theta = \frac{1}{2}\, \theta^{ij}(x)\, \partial_i \wedge \partial_j$, $\xi = (\xi_i) \in \Omega^1(\Sigma_2, X^* T^* M)$

perturbative expansion $\Rightarrow$ Kontsevich formality maps

(valid on-shell ($[\theta, \theta]_S = 0$) as well as off-shell, e.g. twisted Poisson)

Cattaneo, Felder (2000)

Noncommutativity in electrodynamics and string theory

- electron in constant magnetic field $\vec{B} = B\hat{e}_z$:

$$\mathcal{L} = \frac{m}{2}\dot{\vec{x}}^2 - e\dot{\vec{x}} \cdot \vec{A} \qquad \text{with} \qquad A_i = -\frac{B}{2}\epsilon_{ij}x^j$$

$$\lim_{m\to 0} \mathcal{L} = e\frac{B}{2}\dot{x}^i\epsilon_{ij}x^j \qquad \Rightarrow \qquad [\hat{x}^i, \hat{x}^j] = \frac{2i}{eB}\epsilon^{ij}$$

- bosonic open strings in constant B-field

$$S_\Sigma = \frac{1}{4\pi\alpha'}\int_\Sigma \left(g_{ij}\partial_a x^i\partial^a x^j - 2\pi i\alpha' B_{ij}\epsilon^{ab}\partial_a x^i\partial_b x^j\right)$$

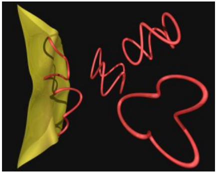
in low energy limit $g_{ij} \sim (\alpha')^2 \to 0$:

$$S_{\partial\Sigma} = -\frac{i}{2}\int_{\partial\Sigma} B_{ij}x^i\dot{x}^j \qquad \Rightarrow \qquad [\hat{x}^i, \hat{x}^j] = \left(\frac{i}{B}\right)^{ij}$$

C-S Chu, P-M Ho (1998); V Schomerus (1999); Seiberg, Witten

Strings and NC geometry

Open strings on D-branes in B-field background



$$\langle [x^i(\tau), x^j(\tau')] \rangle = i\theta^{ij}$$

non-commutative string endpoints with $\star$-product depending on θ via

$$\frac{1}{g+B} = \frac{1}{G+\Phi} + \theta \qquad (closed - openstringrelations)$$

add fluctuations $B \rightsquigarrow B + F$; depending on regularization scheme:

$$\rightarrow \quad \begin{cases} \text{non-commutative gauge theory} & \text{(e.g. point-splitting)} \\ \text{ordinary gauge theory} & \text{(e.g. Pauli-Villars)} \end{cases}$$

$\Rightarrow$ SW map: commutative $\leftrightarrow$ noncommutative theory (duality)

Strings and NC geometry

A SW map (according to Seiberg & Witten) is a field redefinition

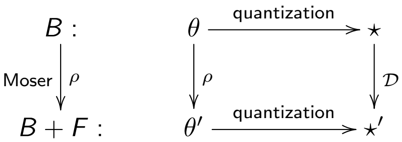
$$\widehat{A}_\mu[A,\theta] = A_\mu + \frac{1}{4}\theta^{\xi\nu}\left\{A_\nu, \partial_\xi A_\mu + F_{\xi\mu}\right\} + \dots\ ,$$

such that $\quad \delta A_\mu = \partial_\mu \Lambda \quad \Leftrightarrow \quad \delta\hat{A}_\mu = \partial_\mu\hat{\Lambda} + i[\hat{\Lambda} \stackrel{\star}{,} \hat{A}_\mu]\ .$

Introduce covariant coordinates

$$X^\nu = \mathcal{D}(x^\nu) = x^\nu + \theta^{\nu\mu}\hat{A}_\mu[A,\theta] \qquad \text{with} \qquad \mathcal{D}(f \star' g) = \mathcal{D}f \star \mathcal{D}g\ .$$

$\Rightarrow$ a SW map is really a covariantizing change of coordinates.



Jurčo, PS, Wess (2001)

Example: QM with 3-cocycle

$\boxed{\theta \to \theta'}$

charged particle in a magnetic field

$$\omega = dp_i \wedge dx^i \mapsto \omega' = \omega + eF \qquad F_{ij} = \partial_i A_j - \partial_j A_i = \epsilon_{ijk} B_k$$

$$\theta \mapsto \theta' = \theta - e\,\theta \cdot F \cdot \theta + e^2\,\theta \cdot F \cdot \theta \cdot F \cdot \theta - \ldots = \begin{pmatrix} 0 & I \\ -I & eF \end{pmatrix}$$

quantize θ and θ', determine SW map ...

$$\star \mapsto \star' = \mathcal{D}^{-1} \circ \star \circ (\mathcal{D} \otimes \mathcal{D})$$

$$\mathcal{D}(x^i) = x^i \qquad \mathcal{D}(p_i) = p_i - eA_i \quad \text{(exact result!)}$$

SW map = change of coordinates in phase-space = minimal substitution

Example: QM with 3-cocycle

$\boxed{\theta \to \theta'}$

alternatively: deformed canonical commutation relations

$$[x^i, x^j]' = 0 \; , \; [x^i, p_j]' = i\hbar \; , \; [p_i, p_j]' = i\hbar e F_{ij} \quad (\text{where } F_{ij} = \epsilon_{ijk} B_k)$$

Let $\mathbf{p} = p_i \sigma^i$ and $H = \dfrac{\mathbf{p}^2}{2m} \quad \Rightarrow \quad$ Pauli Hamiltonian:

$$H = \frac{1}{2m}\left(\frac{1}{4}\{\sigma^i, \sigma^j\}\{p_i, p_j\}' + \frac{1}{4}[\sigma^i, \sigma^j][p_i, p_j]'\right) = \frac{\vec{p}^{\,2}}{2m} - \frac{\hbar e}{2m}\vec{\sigma} \cdot \vec{B}$$

Lorentz-Heisenberg equations of motion:

$$\frac{d\vec{p}}{dt} = \frac{i}{\hbar}[H, \vec{p}\,]' = \frac{e}{2m}\left(\vec{p} \times \vec{B} - \vec{B} \times \vec{p}\right) \; , \quad \frac{d\vec{r}}{dt} = \frac{i}{\hbar}[H, \vec{r}\,]' = \frac{\vec{p}}{m}$$

in this formalism $\nabla \cdot B \neq 0$ is allowed (magnetic sources)

Example: QM with 3-cocycle

Jacobi identity:

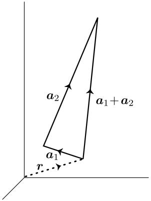
$$[p_1,[p_2,p_3]']' + [p_2,[p_3,p_1]']' + [p_3,[p_1,p_2]']' = \hbar^2 e \nabla \cdot \vec{B} = \hbar^2 e \mu_o \rho_m$$

For $\rho_m \neq 0$: non-associativity, $\nexists$ linear operator $\vec{p} = -i\hbar\nabla - e\vec{A}$

Translations are generated by $T(\vec{a}) = \exp(\frac{i}{\hbar}\vec{a}\cdot\vec{p})$:

$$T(\vec{a}_1)T(\vec{a}_2) = e^{\frac{ie}{\hbar}\Phi_{12}}\, T(\vec{a}_1+\vec{a}_2)$$

$$[T(\vec{a}_1)T(\vec{a}_2)]\,T(\vec{a}_3) = e^{\frac{ie}{\hbar}\Phi_{123}}\, T(\vec{a}_1)[T(\vec{a}_2)T(\vec{a}_3)]$$

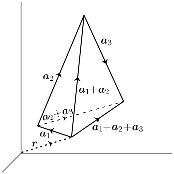


Φ_{12} = flux through triangle $(\vec{a}_1,\vec{a}_2)$
Φ_{123} = flux out of tetrahedron $(\vec{a}_1,\vec{a}_2,\vec{a}_3) = \mu_0 q_m$

Associativity of translations is restored for:

$$\boxed{\frac{\mu_0 e q_m}{\hbar} \in 2\pi\mathbb{Z}}$$

(Dirac charge-quantization)



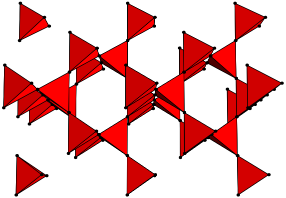
Jackiw '85,'02

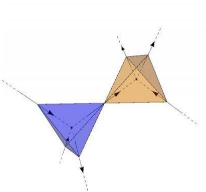
point-like magnetic monopoles ... else: need NAQM

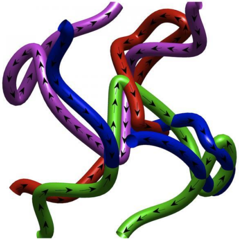
Example: QM with 3-cocycle

Magnetic monopoles in the lab

spin ice pyrochlore, Dirac strings and monopoles







Castelnovo, Moessner, Sondhi (2008)
Fennell; Morris; Hall, . . . (2009)
Lieb, Schupp (1999)

Strings and NC geometry: effective actions

Massless bosonic modes

- open strings: A_μ, ϕ^i $\rightarrow$ gauge and scalar fields on D-branes

Open string effective action

$$\mathcal{S}_{\mathrm{DBI}} = \int d^n x \det{}^{\frac{1}{2}}(g + B + F) = \int d^n x \det{}^{\frac{1}{2}}(\hat{G} + \hat{\Phi} + \hat{F}) = \mathcal{S}_{\mathrm{DBI}}^{NC}$$

commutative $\leftrightarrow$ non-commutative duality
generalized symmetry fixes action

Expand to first non-trivial order $\Rightarrow$

$$\mathcal{S}_{\mathrm{DBI}} = \int d^n x \frac{|-g|^{\frac{1}{2}}}{4 g_s} g^{ij} g^{kl} (B+F)_{ik} (B+F)_{jl} \qquad \text{(Maxwell/Yang-Mills)}$$

$$\mathcal{S}_{\mathrm{DBI}}^{NC} = \int d^n x \frac{|\theta|^{-\frac{1}{2}}}{4 \hat{g}_s} \hat{g}_{ij} \hat{g}_{kl} \{\hat{X}^i, \hat{X}^k\}\{\hat{X}^j, \hat{X}^l\} \qquad \text{(Matrix Model)}$$

Strings and NC geometry: effective actions

Nambu-Dirac-Born-Infeld action

commutative $\leftrightarrow$ non-commutative duality implies

$$S_{p\text{-DBI}} = \int d^n x \frac{1}{g_m} \det{}^{\frac{p}{2(p+1)}} [g] \det{}^{\frac{1}{2(p+1)}} \left[g + (C+F)\tilde{g}^{-1}(C+F)^T\right]$$

$$= \int d^n x \frac{1}{G_m} \det{}^{\frac{p}{2(p+1)}} \left[\widehat{G}\right] \det{}^{\frac{1}{2(p+1)}} \left[\hat{G} + (\hat{\Phi} + \hat{F})\widehat{\tilde{G}}^{-1}(\hat{\Phi} + \hat{F})^T\right]$$

This action interpolates between early proposals based on supersymmetry and more recent work featuring higher geometric structures.

expand and quantize $\rightsquigarrow$ Nambu matrix-model:

$$\frac{1}{2(p+1)\widehat{g}_m} \mathrm{Tr}\left(\hat{g}_{i_0 j_0} \cdots \hat{g}_{i_p j_p} \left[\widehat{X}^{j_0}, \ldots, \widehat{X}^{j_p}\right] \left[\widehat{X}^{i_0}, \ldots, \widehat{X}^{i_p}\right]\right)$$

Jurčo, PS, Vysoky (2012-14)

Strings and NC geometry: effective actions

Massless bosonic modes

- closed strings: $g_{\mu\nu}$, $B_{\mu\nu}$, Φ $\to$ background geometry, gravity

Closed string effective action

Weyl invariance (at 1 loop) requires vanishing beta functions:

$$\beta_{\mu\nu}(g) = \beta_{\mu\nu}(B) = \beta(\Phi) = 0$$

$$\Downarrow$$

equations of motion for $g_{\mu\nu}$, $B_{\mu\nu}$, Φ

$$\Uparrow$$

closed string effective action

$$\int d^D x |-g|^{\frac{1}{2}} \left(R - \frac{1}{12} e^{-\Phi/3} H_{\mu\nu\lambda} H^{\mu\nu\lambda} - \frac{1}{6} \partial_\mu \Phi \partial^\mu \Phi + \dots \right)$$

NC/generalized geometry appears to fix also this action

Strings and generalized geometry: non-geometric fluxes

Non-geometric flux backgrounds

T-dualizing a 3-torus with 3-form H-flux gives rise to geometric and

non-geometric fluxes $\quad H_{ijk} \xrightarrow{T_k} f_{ij}{}^{k} \xrightarrow{T_j} Q_i{}^{jk} \xrightarrow{T_i} R^{ijk}$

Hellermann, McGreevy, Williams (2004)
Hull (2005), Shelton, Taylor, Wecht (2005)
Lüst (2010), Blumenhagen, Plauschinn (2010)

Generalized (doubled) geometry ($O(d,d)$ isometry, g, B,...)

Non-geometry geometrized in membrane model
quantization $\Rightarrow$ non-associative $\star$-product

$$f \star g = \cdot \exp\left(\frac{i\hbar}{2}\left[R^{ijk} p_k \partial_i \otimes \partial_j + \partial_i \otimes \tilde{\partial}^i - \tilde{\partial}^i \otimes \partial_i\right]\right)$$

(nonassociative) quantum mechanics with a 3-cocyle

Mylonas, PS, Szabo (2012-2013)

Strings and generalized geometry: non-geometric fluxes

H_{ijk}	3-form background flux
${f_{ij}}^k$	geometric flux, $[e_i, e_j]_L = {f_{ij}}^k e_k$
Q_i^{jk}	globally non-geometric, T-fold
R^{ijk}	locally non-geometric, non-associative

structure constants of a generalized bracket:

$$
\begin{aligned}
[e_i, e_j]_C &= {f_{ij}}^k e_k + H_{ijk} e^k \\
[e_i, e^j]_C &= Q_i^{jk} e_k - f_i{}^j{}_k e^k \\
[e^i, e^j]_C &= R^{ijk} e_k + Q^{ij}{}_k e^k
\end{aligned}
$$

twisted Courant/Dorfman/Roytenberg bracket on $\Gamma(TM \oplus T^*M)$
governs worldsheet current and charge algebras

Alekseev, Strobl; Halmagyi; Bouwknegt; . . .

Dorfman bracket

Generalizes the Lie bracket of vector fields $X \in \Gamma(TM)$ to
$V = X + \xi \in \Gamma(TM \oplus T^*M)$:

$$[X + \xi, Y + \eta]_D = [X, Y] + \mathcal{L}_X \eta - \iota_Y d\xi \quad (+\text{twisting terms})$$

$E = TM \oplus T^*M$ is called "generalized tangent bundle"

E with the Dorfman bracket, the natural pairing $\langle -, - \rangle$ of TM and T^*M and the projection $h : E \to TM$ (anchor) forms a Courant algebroid.

"twisting terms" can involve H, R, ...

Courant bracket: $[V, W]_C = \frac{1}{2}([V, W]_D - [W, V]_D)$

Generalized geometry

Courant algebroid

vector bundle $E \xrightarrow{\pi} M$, anchor $h \in \mathrm{Hom}(E, TM)$,
$\mathbb{R}$-bilinear bracket $[-,-]$ and fiber-wise metric $\langle -,- \rangle$ on $\Gamma E \times \Gamma E$,
s.t. for $e, e', e'' \in E$:

$$[e,[e',e'']] = [[e,e'],e''] + [e',[e,e'']] \tag{1}$$

$$h(e)\langle e',e' \rangle = 2\langle e',[e,e'] \rangle = 2\langle e,[e',e'] \rangle \tag{2}$$

Consequences:

$$[e, fe'] = h(e).f\, e' + f[e,e'] \qquad\qquad f \in C^\infty(M) \tag{3}$$

$$h([e,e']) = [h(e), h(e')]_L \tag{4}$$

note: both axioms (2) can be polarized
(1) and (3) are the axioms of a Leibniz algebroid

Generalized geometry

Exact Courant algebroid

$$0 \to T^*M \to E \to TM \to 0 \qquad \Rightarrow \qquad E \cong TM \oplus T^*M$$

Symmetries of pairing $\langle\ ,\ \rangle$: $O(d,d)$ $\rightarrow$ next slide

Symmetries of Dorfman bracket [,]:

e.g. $e^B(V+\xi) = V + \xi + i_V B$ preserves bracket up to $i_V i_W dB$
$\Rightarrow$ symmetries of bracket: $\mathrm{Diff}(M) \ltimes \Omega^2_{\text{closed}}(M)$.

twisted Dorfman bracket $[\ ,\ ]_H = [\ ,\ ] + i_V i_W H$ for $H \in \Omega^3_{\text{closed}}(M)$,
then: $e^B : [\ ,\ ]_H \mapsto [\ ,\ ]_{H+dB}$; twisted differential: $d_H = d + H\wedge$.

Generalized geometry

$\underline{E = TM \oplus T^*M}$

$\langle V + \xi, W + \eta \rangle = i_V \eta + i_W \xi$ $\qquad \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$

signature (n, n) $\Rightarrow$ symmetries: $O(n, n)$, e.g.:

- B-transform: $e^B(V + \xi) = V + \xi + B(V)$ $\qquad \begin{pmatrix} I & 0 \\ B & I \end{pmatrix}$
- θ-transform: $e^\theta(V + \xi) = V + \xi + \theta(\xi)$ $\qquad \begin{pmatrix} I & \theta \\ 0 & I \end{pmatrix}$

  commutative $\leftrightarrow$ non-commutative symmetry
- $O_N(V + \xi) = N(V) + N^{-T}(\xi)$, smooth $\qquad \begin{pmatrix} N & 0 \\ 0 & N^{-T} \end{pmatrix}$

any $\mathcal{O} \in O(n, n)$ can be written as $\mathcal{O} = e^{-B} O_N e^{-\theta}$

Generalized geometry

consider an idempotent linear map $\tau : \Gamma(E) \to \Gamma(E)$, $\tau^2 = 1$

eigenvalues $\pm 1 \rightsquigarrow$ splitting $E = V_+ \oplus V_-$ with eigenbundle:

$$V_+ = \{V + A(V) \mid V \in TM\} = \{A^{-1}(\xi) + \xi \mid \xi \in T^*M\} \qquad A = g + B$$

$$V_- = \{V + \tilde{A}(V) \mid V \in TM\} = \{\tilde{A}^{-1}(\xi) + \xi \mid \xi \in T^*M\} \qquad \tilde{A} = -g + B$$

in matrix form: $\tau \begin{pmatrix} V \\ \xi \end{pmatrix} = \begin{pmatrix} -g^{-1}B & g^{-1} \\ g - Bg^{-1}B & Bg^{-1} \end{pmatrix} \begin{pmatrix} V \\ \xi \end{pmatrix}$

positive definite metric via τ: $(e_1, e_2)_\tau := \langle \tau e_1, e_2 \rangle = \langle e_1, \tau e_2 \rangle$
$\Rightarrow$ generalized metric

$$\mathbb{G} = \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix}$$

Dorfman bracket as a derived bracket

recall: the Lie-bracket of vector fields is a derived bracket:

Cartan relations

$X, Y \in \Gamma(TM)$: vector fields

$$\iota_X \iota_Y + \iota_Y \iota_X = 0$$

$$d\,\iota_X + \iota_X\, d = \mathcal{L}_X$$

$$d\,\mathcal{L}_X - \mathcal{L}_X\, d = 0$$

$$\mathcal{L}_X \iota_Y - \iota_Y \mathcal{L}_X = [\{\iota_X, d\}, \iota_Y] = \iota_{[X,Y]} \qquad \text{Lie-bracket}$$

$$\mathcal{L}_X \mathcal{L}_Y - \mathcal{L}_Y \mathcal{L}_X = \mathcal{L}_{[X,Y]}$$

Generalized geometry: derived brackets

generalized vector field: $X + \xi \in \Gamma(TM \oplus T^*M)$

Clifford module $\Omega^\bullet(M)$

$$\gamma_{(X+\xi)} \cdot \omega = \iota_X \omega + \xi \wedge \omega$$

de-Rham differential

$$d : \Omega^k(M) \to \Omega^{k+1}(M)$$

can be twisted by a (closed) 3-form H:

$$d_H \omega = d\omega + H \wedge \omega$$

generalized Lie derivative

$$\mathcal{L}_{X+\xi}\omega = \mathcal{L}_X \omega + (d\xi - \iota_X H) \wedge \omega$$

Generalized geometry: derived brackets

Clifford-Cartan relations

$V, W \in \Gamma(TM \oplus T^*M)$, $\gamma_V \equiv V^\alpha(x)\gamma_\alpha$

$$\gamma_V\gamma_W + \gamma_W\gamma_V = \langle V, W\rangle \qquad \gamma_\alpha\gamma_\beta + \gamma_\beta\gamma_\alpha = G_{\alpha\beta}$$

$$d\,\gamma_V + \gamma_V\, d = \mathcal{L}_V$$

$$d\,\mathcal{L}_V - \mathcal{L}_V\, d = 0$$

$$\mathcal{L}_V\gamma_W - \gamma_W\mathcal{L}_V = [\{\gamma_V, d\}, \gamma_W] = \gamma_{[V,W]_D} \qquad \text{Dorfman-bracket}$$

$$\mathcal{L}_V\mathcal{L}_W - \mathcal{L}_W\mathcal{L}_V = \mathcal{L}_{[V,W]_D}$$

$\Rightarrow$ (twisted) Dorfman bracket

$$[X+\xi, Y+\eta]_D = [X,Y] + \mathcal{L}_X\eta - \iota_Y d\xi + \iota_X\iota_Y H$$

Geometrized non-geometry: membrane sigma model

extended objects in background fields

	•		
object:	point particle	closed string	...
algebraic structure:	non-commutative	non-associative	...
AKSZ-model:	Poisson-sigma (open string)	Courant-sigma (open membrane)	...

Geometrized non-geometry: membrane sigma model

Courant sigma model

TFT with 3-dimensional membrane world volume Σ_3

$$S^{(2)}_{\rm AKSZ} = \int_{\Sigma_3} \Big(\phi_i \wedge {\rm d}X^i + \frac{1}{2}\, G_{IJ}\, \alpha^I \wedge {\rm d}\alpha^J - h_I{}^i(X)\, \phi_i \wedge \alpha^I + \frac{1}{6}\, T_{IJK}(X)\, \alpha^I \wedge \alpha^J \wedge \alpha^K\Big)$$

embedding maps $X : \Sigma_3 \to M$, 1-form α, aux. 2-form ϕ, fiber metric G, anchor h, 3-form T (e.g. H-flux, f-flux, Q-flux, R-flux).

AKSZ construction: action functionals in BV formalism of sigma model QFT's for symplectic Lie n-algebroids E

Alexandrov, Kontsevich, Schwarz, Zaboronsky (1995/97)

Geometrized non-geometry: membrane sigma model

R-space Courant sigma-model AKSZ membrane action

$$S_R^{(2)} = \int_{\Sigma_3} \Big(d\xi_i \wedge \mathrm{d}X^i + \frac{1}{6}\, R^{ijk}(X)\, \xi_i \wedge \xi_j \wedge \xi_k \Big)$$

for constant backgrounds, using Stokes leads to boundary action

$$S_R^{(2)} = \int_{\Sigma_2} \Big(\eta_I \wedge \mathrm{d}X^I + \frac{1}{2}\, \Theta^{IJ}(X)\, \eta_I \wedge \eta_J \Big) \ :$$

Poisson sigma-model with auxiliary fields η_I and

$$\Theta = \big(\Theta^{IJ}\big) = \begin{pmatrix} R^{ijk}\, p_k & \delta^i{}_j \\ -\delta_i{}^j & 0 \end{pmatrix} \quad \longrightarrow \quad \star \quad \text{(non-associative!)}$$

doubled target space $\sim$ phase space, $X = (x^1, \dots, x^d, p_1, \dots, p_d)$

Non-associative product

$$\boxed{f \star g = \cdot \exp\left(\tfrac{i\hbar}{2}\left[R^{ijk} p_k \partial_i \otimes \partial_j + \partial_i \otimes \tilde{\partial}^i - \tilde{\partial}^i \otimes \partial_i\right]\right)}$$

- 2-cyclicity

$$\int \mathrm{d}^{2d}x \; f \star g = \int \mathrm{d}^{2d}x \; g \star f = \int \mathrm{d}^{2d}x \; f \cdot g$$

- 3-cyclicity

$$\int \mathrm{d}^{2d}x \; f \star (g \star h) = \int \mathrm{d}^{2d}x \; (f \star g) \star h$$

- inequivalent quartic expressions

$$\int f_1 \star \big(f_2 \star (f_3 \star f_4)\big) = \int (f_1 \star f_2) \star (f_3 \star f_4) = \int \big((f_1 \star f_2) \star f_3\big) \star f_4$$

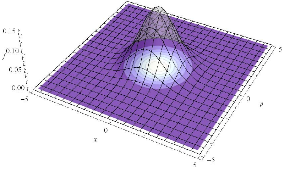
$$\int f_1 \star \big((f_2 \star f_3) \star f_4\big) = \int \big(f_1 \star (f_2 \star f_3)\big) \star f_4$$

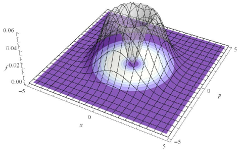
Phase-space formulation of QM

Similar to the density operator formulation of quantum mechanics.

- Operators and states are functions on phase space.
- Algebraic structure introduced with the help of a star product, traces by integration.

Popular choices of star products:
Moyal-Weyl (symmetric ordering, Wigner quasi-probability function)
Wick-Voros (normal ordering, coherent state quantization)





(QHO states in Wick-Voros formulation)

Nonassociative quantum mechanics

Phase-space formulation of QM, suitably generalized:

A *state* ρ is an expression of the form

$$\rho = \sum_{\alpha=1}^{n} \lambda_\alpha \, \psi_\alpha \otimes \psi_\alpha^* \qquad \text{with} \qquad \int |\psi_\alpha|^2 = 1$$

λ_α are probabilities and ψ_α are *phase space wave functions*:

Expectation value:

$$\langle A \rangle = \sum_\alpha \lambda_\alpha \int \psi_\alpha^* \star (A \star \psi_\alpha) = \int A \cdot S_\rho \ ,$$

with *state function*

$$S_\rho = \sum_\alpha \lambda_\alpha \psi_\alpha \star \psi_\alpha^* \ , \qquad \int S_\rho = 1 \ .$$

Nonassociative quantum mechanics

- *Operators*: complex-valued functions on phase-space – the star product severs as operator product
- *Observables*: real-valued functions on phase-space
- *Dynamics*: Heisenberg-type time evolution equations

$$\frac{\partial A}{\partial t} = \frac{i}{\hbar}[H, A]_\star$$

these are in general not derivations of the star product!

Nonassociative quantum mechanics

Eigenfunctions and eigenstates

"star-genvalue equation"

$$A \star f = \lambda f \qquad \text{with } \lambda \in \mathbb{C}$$

complex conjugation implies $f^* \star A^* = \lambda^* f^*$

- real functions have real eigenvalues

$$f^* \star (A \star f) - (f^* \star A) \star f = (\lambda - \lambda^*)(f^* \star f)$$

$$(\lambda - \lambda^*) \int f^* \star f = (\lambda - \lambda^*) \int |f|^2 = 0 \ .$$

- eigenfunctions with different eigenvalues are orthogonal

Nonassociative quantum mechanics

Associator and common eigen states

if $X^I \star S = \lambda^I S$ and $X^J \star S = \lambda^J S$ and $X^K \star S = \lambda^K S$ then

$$\int [(X^I \star X^J) \star X^K] \star S = \int (X^I \star X^J) \star (X^K \star S)$$

$$= \lambda^K \int (X^I \star X^J) \star S = \lambda^K \int X^I \star (X^J \star S) = \lambda^K \lambda^J \lambda^I$$

likewise $\int [X^I \star (X^J \star X^K)] \star S = \lambda^I \lambda^K \lambda^J$.

taking the difference implies

$$[[X^I, X^J, X^K]]_\star = \lambda^K \lambda^J \lambda^I - \lambda^I \lambda^K \lambda^J = 0$$

$\Rightarrow$ Nonassociating observables do not have common eigen states

$\rightsquigarrow$ spacetime coarse graining

Nonassociative quantum mechanics

Positivity

$$\langle A^* \circ A \rangle = \sum_\alpha \lambda_\alpha \int \psi_\alpha^* \star [A^* \star (A \star \psi_\alpha)] = \sum_\alpha \lambda_\alpha \int (\psi_\alpha^* \star A^*) \star (A \star \psi_\alpha)$$

$$= \sum_\alpha \lambda_\alpha \int (A \star \psi_\alpha)^* \cdot (A \star \psi_\alpha) = \sum_\alpha \lambda_\alpha \int |A \star \psi_\alpha|^2 \geq 0$$

$\rightsquigarrow$ semi-definite, sesquilinear form

$$(A, B) := \langle A^* \circ B \rangle = \sum_\alpha \lambda_\alpha \int (A \star \psi_\alpha)^* \cdot (B \star \psi_\alpha)$$

$\Rightarrow$ Cauchy-Schwarz inequality

$$|(A, B)|^2 \leq (A, A)(B, B) \ .$$

$\rightsquigarrow$ uncertainty relations

Nonassociative quantum mechanics

Uncertainty relations

uncertainty in terms of shifted coordinates $\widetilde{X}^I = X^I - \langle X^I \rangle$

$$(\Delta X^I)^2 = (\widetilde{X}^I, \widetilde{X}^I)$$

Cauchy-Schwarz

$$(\Delta X^I)^2 (\Delta X^J)^2 \geq |(\widetilde{X}^I, \widetilde{X}^J)|^2 = \frac{1}{4} |\langle [X^I, X^J]_\circ \rangle|^2 + \frac{1}{4} |\langle \{\widetilde{X}^I, \widetilde{X}^J\}_\circ \rangle|^2$$

$\Rightarrow$ Born-Jordan-Heisenberg-type uncertainty relation

$$\Delta X^I \cdot \Delta X^J \geq \frac{1}{2} \big| \langle [X^I, X^J]_\circ \rangle \big|$$

recall: $[x^i, x^j] = i\hbar R^{ijk} p_k$, $[x^i, p_j] = i\hbar \delta_j$, $[p_i, p_j] = 0 \;\Rightarrow$

$$\Delta p_i \cdot \Delta p_j \geq 0 \qquad \Delta x^i \cdot \Delta p_j \geq \frac{\hbar}{2} \delta^i_j \qquad \Delta x^i \cdot \Delta x^j \geq \frac{\hbar}{2} \big| R^{ijk} \langle p_k \rangle' \big|$$

Nonassociative quantum mechanics

Area and volume operators

$$iA^{IJ} = [\widetilde{X}^I, \widetilde{X}^J]_\star \quad \text{and} \quad V^{IJK} = \frac{1}{2}[[\widetilde{X}^I, \widetilde{X}^J, \widetilde{X}^K]]_\star$$

expectation values of these (oriented) area and volume operators:

$$\langle A^{IJ} \rangle = \hbar\Theta^{IJ}(\langle p \rangle) \quad \text{and} \quad \langle V^{IJK} \rangle = \frac{3}{2}\hbar^2 R^{IJK}$$

with three interesting special cases

$$\langle A^{(x^i, p_j)} \rangle = \hbar\delta^i_j \ , \quad \langle A^{ij} \rangle = \hbar R^{ijk} \langle p_k \rangle \ , \quad \langle V^{ijk} \rangle = \frac{3}{2}\hbar^2 R^{ijk}$$

$\Rightarrow$ coarse-grained spacetime with quantum of volume $\frac{3}{2}\hbar^2 R^{ijk}$

Remark on Nambu-Poisson 3-brackets

Nambu-Poisson structures

- Appear in effective membrane actions
- Nambu mechanics: multi-Hamiltonian dynamics with generalized Poisson brackets; e.g. Euler's equations for the spinning top :

$$\frac{d}{dt}L_i = \{L_i, \frac{\vec{L}^2}{2}, T\} \qquad \text{with} \quad \{f,g,h\} \propto \epsilon^{ijk}\,\partial_i f\,\partial_j g\,\partial_k h$$

- more generally

$$\begin{aligned}\{\{f_0,\cdots,f_p\},h_1,\cdots,h_p\} &= \{\{f_0,h_1,\cdots,h_p\},f_1,\cdots,f_p\}+\ldots \\ &\quad \ldots+\{f_0,\ldots,f_{p-1},\{f_p,h_1,\cdots,h_p\}\}\end{aligned}$$

- The nonassociative $\star$-product quantizes these brackets:

$$\underbrace{[[x^i,x^j,x^k]]_\star}_{\text{Jacobiator}} = i\hbar\sum_l \left(R^{ijl}[p_l,x^k]_\star + \text{ cycl.}\right) = 3\hbar^2 R^{ijk}$$

Remark on Nambu-Poisson 3-brackets

Nambu-Poisson structures

- Appear in effective membrane actions
- Nambu mechanics: multi-Hamiltonian dynamics with generalized Poisson brackets; e.g. Euler's equations for the spinning top :

$$\frac{d}{dt}L_i = \{L_i, \frac{\vec{L}^2}{2}, T\} \qquad \text{with} \quad \{f, g, h\} \propto \epsilon^{ijk}\,\partial_i f\,\partial_j g\,\partial_k h$$

- more generally

$$\begin{aligned}\{\{f_0, \cdots, f_p\}, h_1, \cdots, h_p\} \quad &= \quad \{\{f_0, h_1, \cdots, h_p\}, f_1, \cdots, f_p\} + \ldots \\ &\quad \ldots + \{f_0, \ldots, f_{p-1}, \{f_p, h_1, \cdots, h_p\}\}\end{aligned}$$

- The nonassociative $\star$-product quantizes these brackets:

$$\underbrace{[[x^i, x^j, x^k]]_\star}_{\text{Jacobiator}} = i\hbar \sum_l \left(R^{ijl}[p_l, x^k]_\star + \text{ cycl.}\right) = 3\hbar^2 R^{ijk}$$

Remark on (non-associative) Jordan Algebras

"Noncommutative" Jordan Algebras

(1) $x(yx) = (xy)x$ "flexible"

(2) $x^2(yx) = (x^2y)x$ implies: $x^m(yx^n) = (x^my)x^n$

P. Jordan (1933), A.A. Albert (1946), R.D. Schafer (1955)

Question: Are we dealing with a Jordan algebra?

$x^I \star (x^K \star x^I) = (x^I \star x^K) \star x^I$ ✓

$(x^I)^{\star 2} \star (x^K \star x^I) = ((x^I)^{\star 2} \star x^K) \star x^I$ ✓

but $\vec{x}^2 \star (\vec{x}^2 \star \vec{x}^2) - (\vec{x}^2 \star \vec{x}^2) \star \vec{x}^2 = 2iR^2 \vec{p} \cdot \vec{x} \neq 0$

(with $R^{ijk} \equiv R\epsilon^{ijk}$) $\Rightarrow$ It's not a Jordan algebra Alexander Held, PS

Remark on (non-associative) Jordan Algebras

"Noncommutative" Jordan Algebras

(1) $x(yx) = (xy)x$ "flexible"

(2) $x^2(yx) = (x^2y)x$ implies: $x^m(yx^n) = (x^my)x^n$

P. Jordan (1933), A.A. Albert (1946), R.D. Schafer (1955)

Question: Are we dealing with a Jordan algebra?

$$x^I \star (x^K \star x^I) = (x^I \star x^K) \star x^I \quad \checkmark$$

$$(x^I)^{\star 2} \star (x^K \star x^I) = ((x^I)^{\star 2} \star x^K) \star x^I \quad \checkmark$$

but $\vec{x}^2 \star (\vec{x}^2 \star \vec{x}^2) - (\vec{x}^2 \star \vec{x}^2) \star \vec{x}^2 = 2iR^2 \vec{p} \cdot \vec{x} \neq 0$

(with $R^{ijk} \equiv R\epsilon^{ijk}$) $\Rightarrow$ It's not a Jordan algebra Alexander Held, PS

Remark on (non-associative) Jordan Algebras

"Noncommutative" Jordan Algebras

(1) $x(yx) = (xy)x$ "flexible"

(2) $x^2(yx) = (x^2y)x$ implies: $x^m(yx^n) = (x^my)x^n$

P. Jordan (1933), A.A. Albert (1946), R.D. Schafer (1955)

Question: Are we dealing with a Jordan algebra?

$$x^I \star (x^K \star x^I) = (x^I \star x^K) \star x^I \qquad \checkmark$$

$$(x^I)^{\star 2} \star (x^K \star x^I) = ((x^I)^{\star 2} \star x^K) \star x^I \qquad \checkmark$$

but $\vec{x}^2 \star (\vec{x}^2 \star \vec{x}^2) - (\vec{x}^2 \star \vec{x}^2) \star \vec{x}^2 = 2iR^2 \vec{p} \cdot \vec{x} \neq 0$

(with $R^{ijk} \equiv R\epsilon^{ijk}$) $\Rightarrow$ It's not a Jordan algebra — Alexander Held, PS


Thanks for listening! Questions?