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Time Dependent Parallel Viscosity and Relaxation Rate of Poloidal Rotation in the Banana Regime

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Abstract

Time dependent ion parallel viscous force in the banana regime with arbitrary inverse aspect ratio ϵ is calculated using the eigenfunction approach. The flux surface averaged viscosity is then used to study the relaxation process of the poloidal rotation which leads to oscillatory relaxation behavior. The relaxation rate ν_p is found approximately proportional to ν_{ii}/ϵ (where ν_{ii} is the ion collision frequency and ϵ is the inverse aspect ratio).

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I. Introduction

Parallel viscous force is one of the essential forces governing the poloidal momentum of plasma. In steady state, Hirshman and Sigmar[1] constructed a moment approach scheme to solve for the poloidal flows and calculate the neoclassical transport fluxes. However, in a dynamical system, one should include the time dependent feature of the viscosity. As will be shown, the time dependent contribution of the viscosity can actually be more important than the direct time derivative of momentum itself in the force balance equation. Therefore, employing a steady state viscosity in the evolution equation of momentum as in most existing numerical codes may give a correct steady state solution but very wrong dynamics. In particular, it has been pointed out[2] that the relaxation rate of the poloidal flow calculated from the static response of the distribution function[3] was incorrect. Recently, Shaing and Hirshman[4] have calculated the time dependent parallel viscosity yielding a relaxation rate ν_p of poloidal flow as $\nu_p \sim \nu_{ii}$. Taguchi[5] revisited this problem by partially adopting the eigenfunction approach given in Refs. 6-8 and yielded a different conclusion that $\nu_p \sim \frac{\nu_{ii}}{\epsilon^{1/2}}$.

In this work, a somewhat improved formalism for deriving the parallel viscosity is introduced to generally include the effects of large mass flow and arbitrary aspect ratio. The drift kinetic equation is solved using the eigenfunction approach as in Refs. 5-8. The asymptotic behavior of the resulting viscosity with respect to ν_p/ν_{ii} will be discussed and will lead to a Padé approximation form of viscosity having the correct behavior in both limits $\frac{\nu_p}{\nu_{ii}} \ll 1$ and $\frac{\nu_p}{\nu_{ii}} \gg 1$.

The relaxation process of poloidal flow is studied for arbitrary inverse aspect ratio ϵ using the averaged time dependent viscous force. For a large aspect ratio tokamak, the relaxation rate is found to be $\nu_p \sim \frac{\nu_{ii}}{\epsilon}$. Moreover, it is found that an finite frequency oscillation will inevitably occur during the damping of the poloidal rotation. For a large aspect ratio tokamak, the frequency of this oscillation is found to increase strongly with ϵ. For a wide range of aspect ratio, both the relaxation rate and oscillation frequency are calculated numerically as functions of ϵ.

II. Derivation of $\langle \mathbf{B} \cdot \nabla \cdot \mathbf{\Pi} \rangle$

Let us start with the lowest order ion drift kinetic equation in the rest frame[9]

$$\frac{\partial \bar{f}}{\partial t} + (v_\parallel \mathbf{b} + \mathbf{V}) \cdot \nabla \bar{f} - (\mathbf{V} \cdot \nabla \mu B + \mu B(\nabla \cdot \mathbf{V} - \mathbf{bb} : \nabla \mathbf{V})) \frac{\partial \bar{f}}{\partial \mu}$$
$$+ \left(v_\parallel \frac{\mathbf{b} \cdot \nabla \cdot \mathbf{P}}{mn} - \mu B \nabla \cdot \mathbf{V} - (v_\parallel^2 - \mu B)\mathbf{bb} : \nabla \mathbf{V} \right) \frac{\partial \bar{f}}{\partial w} = C(\bar{f}). \tag{1}$$

Here, $\mathbf{v}$ is the particle velocity in the ion rest frame, $w = v^2/2$, $\bar{f}$ is the gyro-averaged distribution function in the ion rest frame satisfying $\int d\mathbf{v}\ \mathbf{v}\bar{f} = 0$, μ is the magnetic moment, $\mathbf{V}$ is ion mass flow velocity,

$$\mathbf{P} \equiv \int d\mathbf{v}\ m\mathbf{v}\mathbf{v}\bar{f} \equiv nT\mathbf{I} + \mathbf{\Pi} \simeq nT\mathbf{I} + \frac{3}{2}\pi_\parallel(\mathbf{bb} - \frac{1}{3}\mathbf{I})$$

is the ion stress tensor, $\mathbf{\Pi}$ is the viscosity tensor, $\mathbf{I}$ is the unit tensor, and $C(\bar{f})$ the ion-ion collision operator.

As in the usual banana regime, $\Delta \equiv \frac{\nu_{ii}}{\omega_b} \ll 1$ is the small expansion parameter used to truncate the system. In addition, the time derivative is also assumed to be $O(\Delta)$, while the mass flow is assumed to be the zeroth order quantity. The continuity equation thus yields

$$\mathbf{V} = \frac{K(\psi)}{n}\mathbf{B} + \omega_\varphi R^2 \nabla \varphi \tag{2}$$

where to the lowest order[10], $\omega_\varphi(\psi) = -c\frac{\partial}{\partial \psi}\phi(\psi)$ corresponds to $\mathbf{E} \times \mathbf{B}$ drift and has to be a flux function, $K(\psi)$ corresponds to the poloidal flow.

Equation (1) thus reduces to

$$\frac{\partial \bar{f}}{\partial t} + (v_\parallel + \frac{K}{n}B)\nabla_\parallel \bar{f} + v_\parallel \left(\frac{\mathbf{b} \cdot \nabla \cdot \mathbf{P}}{mn} - \nabla_\parallel \frac{v_\parallel K B}{n} \right) \frac{\partial \bar{f}}{\partial w} = C(\bar{f}). \tag{3}$$

One observes that $K(\psi)$ is the driving force for $\bar{f}$ to deviate from Maxwellian and give rise to the viscous tensor. In general, one can retain the $O(1)$ $\tilde{n}(\theta, t)$ and $\tilde{T}(\theta, t)$ in a Maxwellian $f_M(\omega, \theta, t)$, and an arbitrarily large mass flow $\frac{K}{n}B$. However, for simplicity we further assume $1 > \frac{K}{n}B/v_{thi} \gg \Delta$ and both $\tilde{n}(\theta, t)$, $\tilde{T}(\theta, t)$ are of order Δ as the viscous force. Let's define the relaxation rate $\nu_p \equiv -\frac{\partial}{\partial t}\ln \bar{f}$. Equation (3) can be reduced to

$$v_\parallel \nabla_\parallel \left(\tilde{f} + \frac{2KBv_\parallel}{nv_{th}^2} f_M \right) = \frac{\mathbf{b} \cdot \nabla \cdot \mathbf{P}}{p} v_\parallel f_M + \nu_p \tilde{f} + C(\tilde{f}) \tag{4}$$

where $\tilde{f} = \overline{f} - f_M$, $f_M \equiv \frac{n(\psi)}{\pi^{3/2} v_{th}^3(\psi)} e^{-\frac{mw}{T(\psi)}}$, and terms on the right-hand side are of order Δ. One should notice the appearance of the local stress force $\mathbf{b} \cdot \nabla \cdot \mathbf{P}$ (instead of the viscous force) as an explicit drive for the deviation of distribution function from Maxwellian. It is precisely this term that makes the direct derivation of viscous force possible.

To the zeroth order, Eq. (4) yields

$$\tilde{f}_0 = -\frac{2KBv_\parallel}{nv_{th}^2} f_M + \sigma g(w, \mu, \psi) \tag{5}$$

where $\sigma \equiv \frac{v_\parallel}{|v_\parallel|}$. Also, $O(\Delta)$ of Eq. (4) is

$$v_\parallel \nabla_\parallel \tilde{f}_1 = \frac{\mathbf{b} \cdot \nabla \cdot \mathbf{P}}{p} v_\parallel f_M + \nu_p \tilde{f}_0 + C(\tilde{f}_0). \tag{6}$$

$g(w, \psi, \lambda)$ can thus be solved by bounce average of Eq. (6)

$$\frac{\langle \mathbf{B} \cdot \nabla \cdot \mathbf{P} \rangle}{p} f_M + \nu_p \left[-\frac{2K\langle B^2 \rangle}{nv_{th}^2} f_M + \left\langle \frac{B}{|v_\parallel|} \right\rangle g \right] + \left\langle \frac{B}{|v_\parallel|} C(g) \right\rangle = 0, \tag{7}$$

where the angle bracket denotes the flux surface average. Note that the distribution function must vanish at the banana tip if it is odd in σ, because of the particle conservation law at the tip. Since $\sigma g(w, \mu, \psi)$ is independent of θ, $g(w, \psi, \lambda)$ has to be zero in the trapped region. Using its rotational symmetry property

$$C(P_\ell(\xi)\phi(v)) = P_\ell(\xi) C^\ell(\phi(v)) \tag{8}$$

the linearized collision operator can be expanded[12] as

$$C(f) = 2\nu(v) h |\xi| \frac{\partial}{\partial \lambda} \lambda |\xi| \frac{\partial}{\partial \lambda} f + \sum_{\ell=0}^{\infty} P_\ell(\xi) \left[C^\ell(\hat{f}_\ell) + \frac{\ell(\ell+1)}{2} \nu(v) \hat{f}_\ell \right]. \tag{9}$$

Here, $\lambda \equiv \frac{\mu B_0}{w}$, $\xi \equiv \frac{v_\parallel}{v}$, $h \equiv \frac{B_0}{B}$, $P_\ell(\xi)$ is the Legendre polynomial,

$$\hat{f}_\ell \equiv \frac{2\ell+1}{2} \int_{-1}^{1} d\xi \; P_\ell(\xi) f, \tag{10}$$

$$\nu(v) = \frac{3\sqrt{2\pi}}{4} \nu_{ii} \phi(x)/x^3, \tag{11}$$

$$\phi(x) \equiv \left(1 - \frac{1}{2x^2}\right) erf(x) - \frac{1}{\sqrt{\pi}} \frac{e^{-x^2}}{x},$$

$x \equiv \frac{v}{v_{th}}$ and $erf(x)$ is the error function. Equation (9) can be further simplified by keeping only $\ell = 0, 1, 2$ since $C^\ell \simeq -\frac{\ell(\ell+1)}{2}\nu(v)$ for $\ell \gg 1$.

Equation (7) can thus be reduced to

$$\frac{\langle \mathbf{B} \cdot \nabla \cdot \mathbf{P} \rangle}{p} f_M - \nu_p \left[\frac{2K\langle B^2 \rangle}{n v_{th}^2} f_M + \frac{2B_0}{v} \left(\frac{\partial}{\partial \lambda} \langle |\xi| \rangle \right) g \right]$$

$$+ \frac{2B_0}{v} \nu(v) \frac{\partial}{\partial \lambda} \langle |\xi| \rangle \frac{\partial}{\partial \lambda} g + \frac{\langle B^2 \rangle}{v B_0} \left(C^1(V) + \nu(v) V \right) = 0, \tag{12}$$

where

$$V(v, \psi) \equiv \frac{3}{2} \int_0^{\lambda_c} d\lambda \; g(v, \lambda, \psi) \tag{13}$$

and $\lambda_c = h_{min}$. Since the system is studied in the rest frame, $\int d\mathrm{v} v_\parallel \overline{f} = 0$ must be satisfied. Equations (5) and (13) thus yield an important constraint on $V(v, \psi)$ that

$$\int d\mathrm{v} \frac{v V(v, \psi)}{3} = K(\psi) B_0. \tag{14}$$

This also implies that $V(v, \psi)$ relates to the poloidal flow moments as

$$V = \left(\sum_{\ell=0}^{\infty} U_\ell L_\ell^{3/2}(x^2) \right) \frac{m}{nT} v f_M, \tag{15}$$

where $L_\ell^{3/2}$ is the Laguerre polynomial with argument $x^2 \equiv \frac{v^2}{v_{th}^2}$. Here,

$$U_\ell \equiv \frac{(\frac{3}{2})! \ell!}{(\ell + \frac{3}{2})!} \int d\mathrm{v} \; \frac{v_\parallel B_0}{B} \left(\overline{f} + \frac{2 v_\parallel B K}{n v_{th}^2} f_M \right) L_\ell^{3/2}(x^2).$$

In particular $U_0 = K B_0, U_1 = -\frac{2}{5} \frac{q_p B_0}{T B_p}$, q_p is the poloidal heat flux, and B_p is the poloidal magnetic field.

Following the eigenfunction approach as in Refs. 7 and 8, letting

$$g = \sum_{n=1}^{\infty} Y_n(v, \psi) \Lambda_n(\lambda, \psi)$$

with

$$\frac{\partial}{\partial \lambda} \lambda \langle |\xi| \rangle \frac{\partial}{\partial \lambda} \Lambda_n = \kappa_n \frac{\partial \langle |\xi| \rangle}{\partial \lambda} \Lambda_n,$$

and using the orthogonality condition for Λ_n

$$\int_0^{\lambda} d\lambda\ \Lambda_n \Lambda_m \frac{\partial\langle|\xi|\rangle}{\partial\lambda} = \delta_{nm} \int_0^{\lambda_c} d\lambda\ \Lambda_n^2 \frac{\partial\langle|\xi|\rangle}{\partial\lambda}.$$

Equation (12) yields

$$g = \frac{\langle B^2\rangle}{2B_0^2} \frac{F(v,\lambda,\nu_p)}{f_c(v,\nu_p)} V(v,\psi) \tag{16}$$

with $V(v,\psi)$ satisfying

$$\left(\frac{1}{f_c(v,\nu_p)} - 1\right)\nu V - C^1(V) = \left(\frac{B_0\langle \mathbf{B}\cdot\nabla\cdot\mathbf{P}\rangle}{\langle B^2\rangle} - m\nu_p K B_0\right)\frac{v}{nT} f_M. \tag{17}$$

Here,

$$F(v,\lambda,\nu_p) \equiv -\sum_{n=1}^{\infty} \frac{\sigma_n \Lambda_n}{\kappa_n - \nu_p/\nu}, \tag{18}$$

$$f_c(v,\nu_p) \equiv \frac{3}{4}\frac{\langle B^2\rangle}{B_0^2}\int_0^{\lambda_c} d\lambda\ F(v,\lambda,\nu_p) = \frac{\langle B^2\rangle}{B_0^2}\sum_{n=1}^{\infty}\frac{\gamma_n}{\kappa_n - \nu_p/\nu} \tag{19}$$

is the effective fraction of circulating particles, and σ_n, γ_n are defined in Refs. 7 and 8.

The velocity dependent function $V(v)$ in Eq. (19) can be solved for by using the variational principle[11,13] (see Appendix A). However the purpose here is to calculate the parallel viscosity $\langle \mathbf{B}\cdot\nabla\cdot \overleftrightarrow{\mathbf{\Pi}}\rangle$. Using the fact that

$$\int d\mathrm{v}\ \ vC^1(V(v)) = 0,$$

which corresponds to the momentum conservation of the collision operator, Eqs. (15) and (17) yield

$$\langle \mathbf{B}\cdot\nabla\cdot \overleftrightarrow{\mathbf{\Pi}}\rangle = m\frac{\langle B^2\rangle}{B_0}\int d\mathrm{v}\ \ \frac{f_t(x,\nu_p)}{f_c(x,\nu_p)}\frac{v\nu V}{3} = m\frac{\langle B^2\rangle}{B_0^2}\sum_{\ell=0}^{\infty}\mu_\ell(\nu_p,\psi)U_\ell(\psi). \tag{20}$$

Here,

$$\mu_\ell \equiv \left\{\frac{f_t(x,\nu_p)}{f_c(x,\nu_p)}\nu(v)L_\ell^{3/2}(x^2)\right\}, \tag{21}$$

$$f_t(x,\nu_p) \equiv 1 - \frac{\langle B^2\rangle}{B_0^2}\sum_{n=1}^{\infty}\frac{1-\nu_p/\nu}{\kappa_n - \nu_p/\nu}\gamma_n = 1 - (1-\frac{\nu_p}{\nu})f_c(x,\nu_p), \tag{22}$$

$$\{G\} \equiv \frac{8}{3\sqrt{\pi}}\int_0^{\infty} dx\ G(x)x^4 e^{-x^2},$$

and $\nu(v)$ is given in Eq. (11).

Furthermore, the local (poloidally unaveraged) time dependent viscous forces are of great importance for studying the dynamics of a toroidally confined plasma system. It is actually an essential drive for the evolution of the poloidal variation of plasma density. Using the formalism adopted in this work, one can study some property of this local force (see Appendix B). However, complete calculation of this local force requires solving the next order bounce averaged equation which is quite complicated and is out of the scope of this work.

III. Asymptotic Behavior and Padé Approximation

As shown in Ref. 8, a Padé approximation of the complete result from the eigenfunction approach is typically possible. This simple analytic approximation is always quite accurate and highly desirable for further analysis of the dynamics of the system. In order to derive it, one should study the detailed asymptotic behavior of f_c, f_t and $F(v, \lambda, \nu_p)$.

At $\nu_p/\nu \ll 1$, one finds $f_c \simeq f_c^p$ and

$$F(v, \lambda, \nu_p) \simeq -\sum_{n=1}^{\infty} \frac{\sigma_n \Lambda_n}{\kappa_n} = \int_{\lambda}^{\lambda_c} \frac{1}{\langle|\xi|\rangle},$$

which corresponds to the pitch angle dominant solution; and (ii) at $\nu_p/\nu \gg 1$, one finds $f_c \simeq -\frac{\nu}{\nu_p} f_c^d$ and

$$F(v, \lambda, \nu_p) \simeq \frac{\nu}{\nu_p} \sum_{n=1}^{\infty} \sigma_n \Lambda_n = \frac{-\nu}{\nu_p} \frac{1}{\frac{\partial \langle|\xi|\rangle}{\partial \lambda}},$$

which corresponds to the drag dominant solution, as expected. f_c^p and f_c^d are the effective fractions of circulating particles for pitch angle dominant and drag dominant cases, respectively, as defined in Ref. 8. Note that the reason the drag dominant solution will occur at $\nu_p \gg \nu$ is due to the mathematical similarity of mass flow damping and drag force in the force balance equation.

Similarly, one finds that (i) $f_t(x, \nu_p) \simeq f_t^p(1 + \frac{\nu_p}{\nu})$, for $\frac{\nu_p}{\nu} \ll 1$; and (ii) $f_t(x, \nu_p) \simeq f_t^d - \frac{\nu}{\nu_p} f_t^p$, for $\frac{\nu_p}{\nu} \gg 1$. This asymptotic behavior then leads to Padé approximation forms for $f_t(x,\nu_p)$ and $f_t(x,\nu_p)/f_c(x,\nu_p)$, i.e.

$$f_t(x,\nu_p) \simeq \frac{f_t^p - \frac{\nu_p}{\nu} f_t^d}{1 - \frac{\nu_p}{\nu}},$$

$$\frac{f_t(x,\nu_p)}{f_c(x,\nu_p)} \simeq (1 - \frac{\nu_p}{\nu}) \frac{f_t^p - \frac{\nu_p}{\nu} f_t^d}{f_c^p - \frac{\nu_p}{\nu} f_c^d}. \tag{23}$$

This simple analytic form is physically insightful and clearly leads to correct asymptotic behavior. In the next section, this expression will be used for studying the relaxation of poloidal flow.

IV. Relaxation of Poloidal Flow

Concerning the relaxation of the poloidal flow, the parallel momentum balance equation is used to obtain (also cf. Refs. 2,4,5),

$$\left(2 + \frac{1}{q^2}\right) \epsilon^2 \nu_p = \mu_0(\nu_p, \psi) \tag{24}$$

where $q(\psi)$ is the safety factor. Here, the term on the right-hand side corresponds to parallel viscosity; the term on the left-hand side corresponds to direct time derivative of the poloidal flow. Also note that terms $\mu_\ell U_\ell$ with $\ell \geq 1$ have been neglected for simplicity. Typically, inclusion of $\ell \geq 1$ terms can give about a 20% correction and will not change the qualitative behavior of the result.

The relaxation rate ν_p can be solved from Eqs. (21) and (24). To understand the qualitative behavior of the relaxation process, it is useful to utilize the Padé approximation form given in Eq. (23) and rewrite Eq. (21) as

$$\mu_0 \equiv \frac{f_t^p}{f_c^p} \{\nu\} - \frac{f_t^d}{f_c^d} \nu_p + \frac{(f_t^p - f_t^d)^2}{f_c^p f_c^d} \left\{ \frac{\nu}{f_c^p \nu - f_c^d \nu_p} \right\} \nu_p. \tag{25}$$

Here, the first term on the right-hand side refers to the standard pitch angle dominant result for static system; the second term refers to $\nu_p \gg \nu_{ii}$ result from the drag dominant solution, and the third term is due to coupling between the two effects.

For a large aspect ratio tokamak ($\epsilon \ll 1$) and assuming $\nu_p \gg \nu_{ii}$, one finds

$$\left\{ \frac{\nu}{f_c^p \nu - f_c^d \nu_p} \right\} \simeq -\frac{\{\nu\}}{\nu_p}, \tag{26}$$

where the third term is negligible compared with the first term. Using the fact that[1]

$$\{\nu\} = \left(2 - \sqrt{2}\ln(1+\sqrt{2})\right) \nu_{ii},$$

Eqs. (24)-(26) yield

$$\nu_p \simeq \frac{0.687}{\epsilon} \nu_{ii}. \tag{27}$$

One notices that the ν_p dependent term in $\langle B \cdot \nabla \overleftrightarrow{\mathbf{\Pi}} \rangle$ is more important than the term from the direct time derivative of poloidal flow. On the other hand, by assuming $\frac{\nu_p}{\nu_{ii}} \ll 1$, one finds $\mu_0 \simeq f_t^p(\{\nu\} + f_t^p \nu_p)$ which results in $\nu_p \simeq \frac{0.51}{\sqrt{\epsilon}} \nu_{ii}$. Nonetheless, this result contradicts the underlying assumption.

To compare our result with the result in Ref. 5, we remark that Eq. (20) agrees with Eq. (15) of Ref. 5. However, special care must be taken when studying the large aspect ratio limiting behavior using the eigenfunction approach. For instance, as pointed out in Refs. 7,8, $\gamma_1 = 1 + O(\epsilon)$, $\gamma_n = O(\epsilon)$ for $n > 1$, but $\sum_{n=1}^{\infty} \gamma_n = 1 + O(\epsilon^{3/2})$. In addition, since the result $\frac{\nu_p}{\nu_{ii}}$ has a strong dependence on ϵ, it is inconsistent to perform the large aspect ratio expansion while keeping $\frac{\nu_p}{\nu_{ii}}$ as an independent quantity as done in Ref. 5.

Furthermore, positive ν_p from Eq. (27) implies that the integrand in Eq. (25) has resonant behavior which gives rise to an imaginary contribution to μ_ℓ. This imaginary part then leads to ν_{pI} which corresponds to an oscillation during the relaxation. Namely, the poloidal flow

$$U_p = K B_p \propto e^{-(\nu_{pr} + i\nu_{pI})t}. \tag{28}$$

For a large aspect ratio tokamak, by assuming $\nu_{pI} \ll \nu_{pr} \simeq 0.687\nu_{ii}/\epsilon$, one finds that the integrand in Eq. (25) is singular at

$$x \simeq \left(\frac{\sqrt{2}\nu_{ii}}{\nu_{pr}} f_c^p\right)^{1/2} \ll 1$$

which leads to

$$\mathrm{Im}\left\{ \frac{\nu}{f_c^p \nu - f_c^d \nu_p} \right\} \simeq \frac{4\sqrt{\pi}}{3} (f_c^p)^{3/2} \left(\frac{\sqrt{2}\nu_{ii}}{\nu_{pr}} \right)^{5/2}. \tag{29}$$

It is thus straightforward to obtain

$$\nu_{pI} \simeq 13.15\epsilon\nu_{ii}. \tag{30}$$

One would expect that ν_{pI} becomes comparable with the damping rate for finite aspect ratio.

For a finite aspect ratio tokamak, Eq. (24) is solved numerically for $\nu_p = \nu_{pr} + i\nu_{pI}$. Numerically, it is well known that the integrand will be singular when the imaginary part becomes very small compared with the real part. It is worth mentioning here that a method introduced as follows can avoid this numerical singularity. Using the Laplace transformation for complex variable[14]

$$\frac{1}{f_c^p\nu - f_c^d\nu_p} = i\sigma_I \int_0^\infty dy\ e^{-i\sigma_I(f_c^p\nu - f_c^d\nu_p)y},$$

one obtains

$$\mathrm{Re}\left\{\frac{\nu}{f_c^p\nu - f_c^d\nu_p}\right\} = \int_0^\infty dy\ e^{-|\nu_{pI}|f_c^d y}\left\{\nu\sin\left(f_c^p\nu - f_c^d\nu_{pr}\right)\right\},$$

$$\mathrm{Im}\left\{\frac{\nu}{f_c^p\nu - f_c^d\nu_p}\right\} = \sigma_I \int_0^\infty dy\ e^{-|\nu_{pI}|f_c^d y}\left\{\nu\cos\left(f_c^p\nu - f_c^d\nu_{pr}\right)\right\}.$$

Here "{}" is defined after Eq. (22), and $\sigma_I \equiv \nu_{pI}/|\nu_{pI}|$. The results of ν_{pr}, ν_{pI} versus ϵ are given in Fig. 1. One notices that

$$\nu_{pr} \simeq 0.5\nu_{ii}/\epsilon \tag{31a}$$

$$\nu_{pI} \simeq \nu_{ii} \tag{31b}$$

except for very small ϵ. In addition, ν_{pI} and ν_{pr} are of the same order for finite ϵ.

However, the behavior of ν_{pr} for small ϵ in Fig. 1 does not appear to agree well with the large aspect ratio result in Eq. (27) until $\epsilon \leq 10^{-3}$. This is due to the fact that the expansion parameter in the banana region is $\sqrt{\epsilon}$. In order to get a better analytic description for poloidal flow evolution in a large aspect ratio tokamak, one includes the $\sqrt{\epsilon}$ correction in Eqs. (24)-(25) and obtains

$$\nu_{pr} \simeq \frac{0.687}{\epsilon(1 + 1.406\sqrt{\epsilon})}\nu_{ii}, \tag{32a}$$

$$\nu_{pI} \simeq \frac{13.15\epsilon}{(1+1.487\sqrt{\epsilon})}\nu_{ii}. \tag{32b}$$

For $\epsilon \leq 0.2$, the numerical and analytical results (from Eqs. (32a) and (32b)) of ν_{pr}, ν_{pI} are given in Figs. 2 and 3; both figures show good agreement.

V. Conclusions

The time dependent parallel viscous force has been calculated by using the eigenfunction approach. The result is given in Eqs. (20)-(22). Meanwhile, the asymptotic behavior of this result is then studied and leads to a simple and physically insightful analytic approximated expression as given in Eq. (23). In addition, some comments on the feature and required calculation for deriving the poloidally unaveraged viscous forces have been made in Appendix B. In particular, the heat stress force has been found to be proportional to B^2 times the averaged heat viscous force as given in Eq. (B9). On the other hand, the calculation of momentum viscous force is more subtle and requires the next order solution for distribution function.

By calculating the time dependent parallel viscosity, the oscillatory behavior in addition to the relaxation processes of the poloidal flow have been studied for arbitrary aspect ratio. It is concluded that neglecting the time dependent term in the parallel viscous force will lead to a totally incorrect description of the relaxation process of the poloidal flow. The numerical results of the oscillation frequency ν_{pI} and relaxation rate ν_{pr} are shown in Fig. 1. Also, approximated analytic results of ν_{pI} and ν_{pr} are given in Eqs. (31) for moderate to small aspect ratio and Eqs. (32) for large aspect ratio, respectively. In particular, for very large aspect ratio, the relaxation rate is found to be $\nu_{pr} \sim \frac{\nu_{ii}}{\epsilon}$ in contrast to the previous results in Refs. 3,4,5.

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Appendix A
Variational Principle for Solving $V(v)$

Let us define

$$Q \equiv \left(\frac{B_0 \langle \mathbf{B} \cdot \nabla \cdot \overleftrightarrow{\mathbf{\Pi}} \rangle}{\langle B^2 \rangle} - m\nu_p K B_0 \right) / nT \tag{A1}$$

and

$$\nu_{eff} \equiv \left(\frac{1}{f_c(\nu, \nu_p)} - 1 \right) \nu.$$

Equqation (17) can thus be written as

$$\hat{\nu}_{eff} V - C^1(V) = Q v f_M, \tag{A2}$$

and V has to satisfy the constraint of Eq. (14). Using Eq. (8) and self-adjoint property of the collision operator, a variational function[11,13] can be constructed

$$R[F] \equiv \frac{P^2[F]}{S[F,F]} \tag{A3}$$

where

$$P[F] \equiv Q \int d\mathbf{v} \frac{vF}{3} \tag{A4}$$

$$S[F_1, F_2] \equiv \int d\mathbf{v} \left(\frac{\hat{\nu}_{eff}}{3} \frac{F_1 F_2}{f_M} - \xi \frac{F_1}{f_M} C(\xi F_2) \right), \tag{A5}$$

and the trial function F has to satisfy the constraints

$$\int d\mathbf{v} \frac{vF}{3} = K B_0 \tag{A6}$$

$$\delta R[F] = 0. \tag{A7}$$

Also note that for $F = V$,

$$R[V] = P[V] = S[V, V]. \tag{A8}$$

As usual, this variational function corresponds to the entropy production due to relaxation of poloidal flow and solution $\delta R = 0$ relates to the minimum entropy production. That is,

$$R[V] = Q K B_0 = \left(\frac{K B_0^2 \langle \mathbf{B} \cdot \nabla \cdot \overleftrightarrow{\mathbf{\Pi}} \rangle}{\langle B^2 \rangle} - m\nu_p (K B_0)^2 \right) / nT .$$

Explicitly, by letting $F = (a + \alpha x^2)\frac{m}{nT} v f_M$, Eq. (A6) requires that $a = KB_0 - 5/2\alpha$, e.g.,

$$F = \left((x^2 - \frac{5}{2})\alpha + KB_0 \right) \frac{m}{nT} v f_M.$$

Here, α, a variational parameter to be determined, corresponds to the poloidal heat flux. Using this variational procedure, one can actually derive the poloidal heat flux in terms of poloidal mass flow without calculating the parallel heat viscosity.[1]

Appendix B

Some Comments on Local Viscous Forces

To study the local (poloidally unaveraged) viscous force, one needs to understand more about the first order distribution function in Eq. (6) which can be rewritten as

$$B\nabla_{\|}\tilde{f}_1 = \frac{\mathbf{B}\cdot\nabla\cdot\mathbf{P}}{p}f_M - \nu_p\left[\frac{2KB^2}{nv_{th}^2}f_M + \frac{2B_0}{v}\left(\frac{\partial}{\partial\lambda}|\xi|\right)g\right]$$

$$+\frac{2B_0}{v}\nu(v)\frac{\partial}{\partial\lambda}|\xi|\frac{\partial}{\partial\lambda}g + \frac{B^2}{vB_0}\left(C^1(V)+\nu(v)V\right). \tag{B1}$$

Equations (B1) and (17) then yield

$$B\nabla_{\|}\tilde{f}_1 = \left(\mathbf{B}\cdot\nabla\cdot\mathbf{P} - \frac{B^2}{\langle B^2\rangle}\langle\mathbf{B}\cdot\nabla\cdot\mathbf{P}\rangle\right)\frac{f_M}{p}$$

$$+\frac{2B_0}{v}\left[\nu\frac{\partial}{\partial\lambda}|\xi|\frac{\partial}{\partial\lambda}g - \nu_p\left(\frac{\partial}{\partial\lambda}|\xi|\right)g + \frac{B^2}{B_0^2}\frac{\nu V}{2f_c}\right]. \tag{B2}$$

Note that

$$\mathbf{B}\cdot\nabla\cdot\mathbf{P} = \int d\mathbf{v}\ mv_{\|}v_{\|}B\nabla_{\|}\tilde{f}_1. \tag{B3}$$

Let us decompose $\tilde{f}_1$ into

$$\tilde{f}_1 = \alpha + H + G_1(v,\lambda,\psi) \tag{B4}$$

where

$$B\nabla_{\|}\alpha = \left(\mathbf{B}\cdot\nabla\cdot\mathbf{P} - \frac{B^2}{\langle B^2\rangle}\langle\mathbf{B}\cdot\nabla\cdot\mathbf{P}\rangle\right)\frac{f_M}{p}, \tag{B5}$$

$$B\nabla_{\|}H = \frac{2B_0}{v}\left[\nu\frac{\partial}{\partial\lambda}|\xi|\frac{\partial}{\partial\lambda}g - \nu_p\left(\frac{\partial}{\partial\lambda}|\xi|\right)g + \frac{B^2}{B_0^2}\frac{\nu V}{2f_c}\right]. \tag{B6}$$

After some straightforward manipulations, Eqs. (13), (20), (21) and (B6) yield

$$\int d\mathbf{v}\ \ mv_{\|}v_{\|}B\nabla_{\|}H = \frac{B^2}{\langle B^2\rangle}\langle\mathbf{B}\cdot\nabla\cdot\mathbf{P}\rangle. \tag{B7}$$

Equations (B3)-(B7) thus lead to

$$B\nabla_{\|}\alpha = \left(\int d\mathbf{v}\ mv_{\|}v_{\|}B\nabla_{\|}\alpha\right)\frac{f_M}{p},$$

which also implies that

$$\alpha = A(\theta)\frac{f_M}{p}.$$

Here $A(\theta)$ is an arbitrary periodic function which relates to $\tilde{n}(\theta)$ and can only be determined from the continuity equation and the momentum equation, but not from the drift kinetic equation. That is,

$$\mathbf{B}\cdot\nabla\cdot\mathbf{P} = \nabla_{\|}A(\theta) + \frac{B^2}{\langle B^2\rangle}\langle\mathbf{B}\cdot\nabla\cdot\mathbf{P}\rangle. \tag{B8}$$

On the other hand, the heat stress force (= heat viscous force + $\frac{5p}{2m}\nabla_{\|}T$) can be directly derived without knowledge of $A(\theta)$. E.g.,

$$\mathbf{B}\cdot\nabla\cdot\Theta + \frac{5p}{2m}\nabla_{\|}T = \int d\mathrm{v}\ mv_{\|}v_{\|}L_1^{3/2}B\nabla_{\|}\tilde{f}_1 = \frac{B^2}{\langle B^2\rangle}\langle\mathbf{B}\cdot\nabla\cdot\Theta\rangle. \tag{B9}$$

It is worth mentioning here that the heat stress force in Eq. (B8) alone can be used in heat balance equation for the derivation of poloidal heat flux. That is, one does not need to calculate the heat viscous force. On the other hand, the momentum stress tensor in Eq. (B8) is not known until $A(\theta)$ is calculated. However, $A(\theta)$ can only be determined when $\tilde{n}(\theta)$ is known, while the derivation of $\tilde{n}(\theta)$ requires the knowledge of the momentum stress tensor. This implies that one needs to calculate the local momentum viscous force.

To calculate the local momentum viscous force, instead of solving for $A(\theta)$, one needs to calculate the parallel pressure gradient forces induced by H and G_1 which correspond to the zeroth order Legendre component of H and G_1. This requires one to solve for G_1 from the $O(\Delta^2)$ bounced average equation which does not vanish in the trapped region. This calculation is possible by extending the approach given in this work and keeping only C^0 and C^2 in Eq. (9); however, it is expected to be very complicated and is out of the scope of this paper. Finally, we remark that the discussions made in this Appendix are valid for both dynamical and static systems.

Figure Captions

1. Real and imaginary components of $\hat{\nu}_p = \epsilon \frac{\nu_p}{\nu_{ii}}$ versus inverse aspect ratio $\epsilon \equiv r/R_0$. Notice that $\hat{\nu}_{pr} \sim 0.5$ while $\hat{\nu}_{pI} \sim \epsilon$ for $\epsilon \geq 0.1$.

2. Normalized relaxation rate $\epsilon \frac{\nu_{pr}}{\nu_{ii}}$ versus inverse aspect ratio $\epsilon \equiv r/R_0$ for $\epsilon \leq 0.2$. Analytic results are obtained from Eq. (32a).

3. Normalized oscillation frequency $\frac{\nu_{pI}}{\epsilon \nu_{ii}}$ versus inverse aspect ratio $\epsilon \equiv r/R_0$ for $\epsilon \leq 0.2$. Analytic results are obtained from Eq. (32b).

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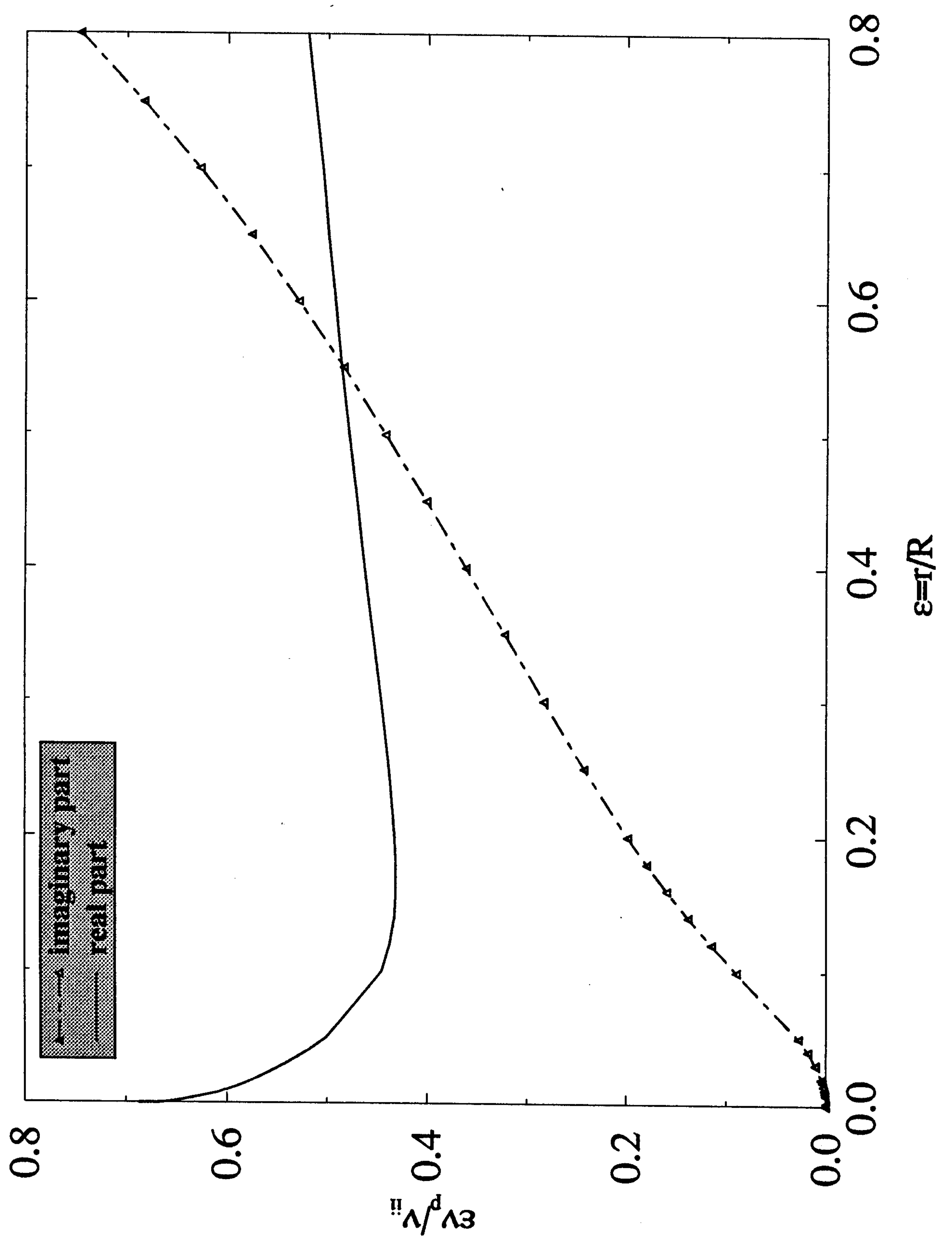

Fig. 1

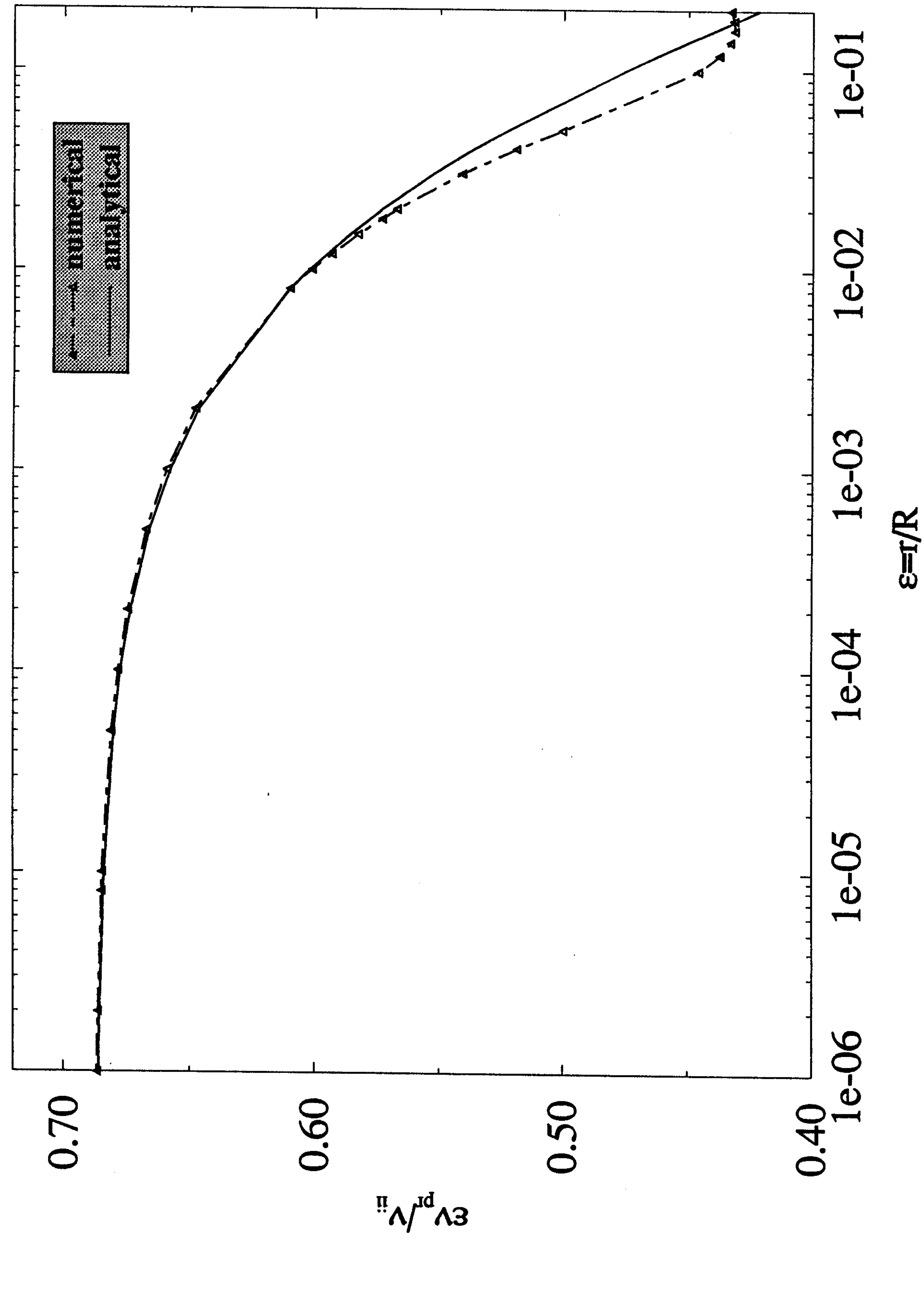
numerical
analytical
ε=r/R
1e-06
1e-05
1e-04
1e-03
1e-02
1e-01
0.70
0.60
0.50
0.40

Fig. 2

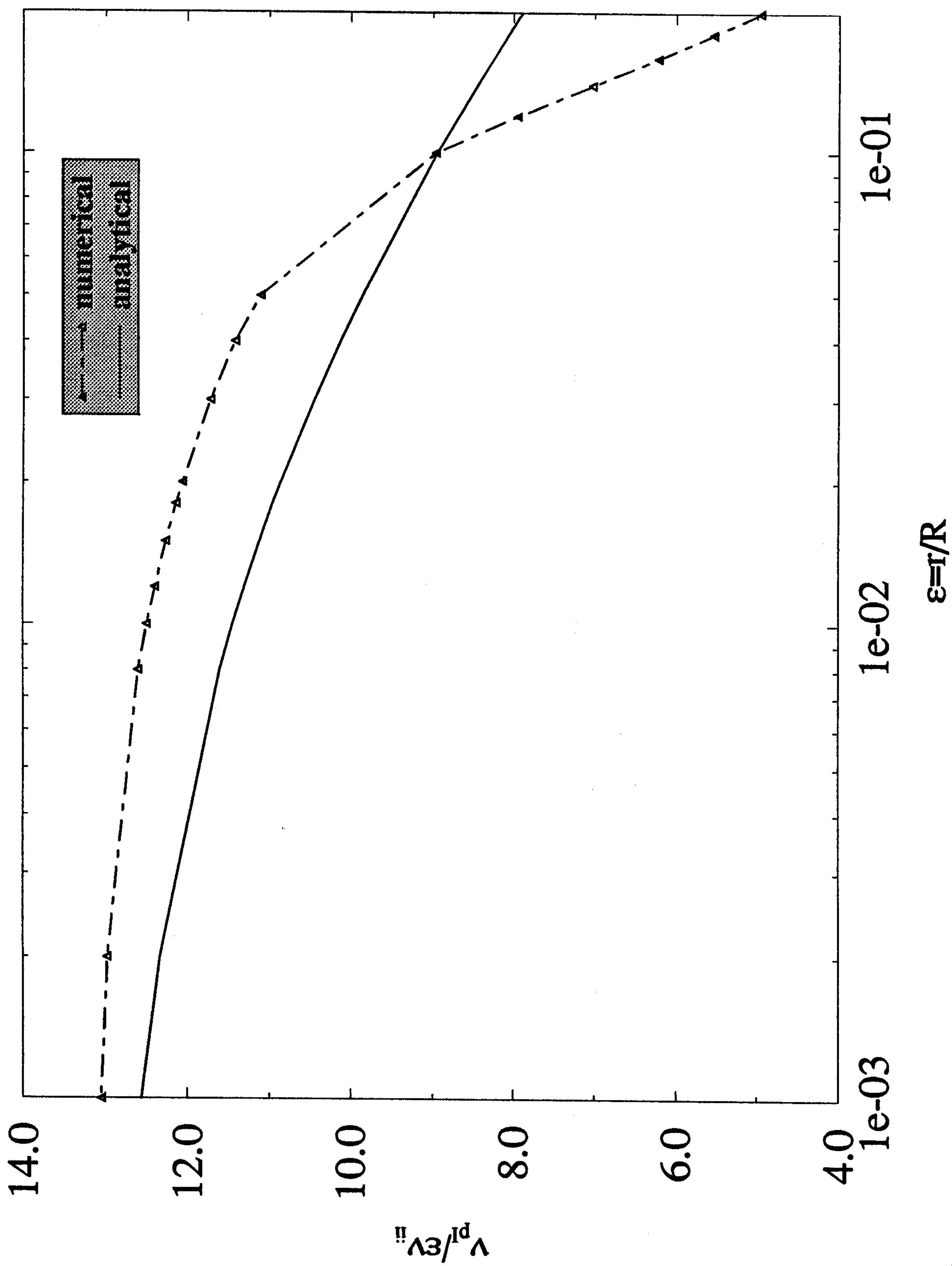

Fig. 3