Regular Paper

ON STATE REPRESENTATIONS OF NONLINEAR IMPLICIT SYSTEMS

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(Received: April 3rd, 2009)

This work considers a semi-implicit system $\Delta$, that is, a pair $(S, y)$, where $S$ is an explicit system described by a state representation $\dot{x}(t) = f(t, x(t), u(t))$, where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$, which is subject to a set of algebraic constraints $y(t) = h(t, x(t), u(t)) = 0$, where $y(t) \in \mathbb{R}^l$. An input candidate is a set of functions $v = (v_1, \ldots, v_s)$, which may depend on time $t$, on $x$, and on $u$ and its derivatives up to a finite order. The problem of finding a (local) proper state representation $\dot{z} = g(t, z, v)$ with input $v$ for the implicit system $\Delta$ is studied in this paper. The main result shows necessary and sufficient conditions for the solution of this problem, under mild assumptions on the class of admissible state representations of $\Delta$. These solvability conditions rely on an integrability test that is computed from the explicit system $S$. The approach of this paper is the infinite-dimensional differential geometric setting of Fliess et al. (1999).

Keywords: Nonlinear systems; state representations; implicit systems; realization theory; DAE’s; differential geometric approach, diffeities.

1 Introduction


$$y^{(n)} = \phi(y, \dot{y}, \ldots, y^{(n-1)}, u, \dot{u}, \ldots, u^{(s)}) \quad (1)$$

where the highest derivative of $y$ appears linearly. A comparison between these works can be found in Kotta and Mullari (2005).

A proper realization of system (1) is an equivalent system of the form

$$\dot{z}(t) = g(z(t), u(t)) \quad (2)$$

where $z(t) \in \mathbb{R}^n$ is the state and $u(t) \in \mathbb{R}^m$ is the input of the realization.

Strongly related to the input-output realization problem is the question of elimination of input derivatives by generalized state transformation. Let $S$ be a nonlinear system with state $x \in \mathbb{R}^n$ and input $v \in \mathbb{R}^m$, given by

$$\dot{x}(t) = f(x(t), v(t), \ldots, v^{(s)}) \quad (3)$$

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ISSN: 0020-7179 print/ISSN 1366-5820 online
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DOI: 10.1080/0020717YYxxxxxxxx
http://www.informaworld.com
One may seek a generalized (local) state transformation \( z = \phi(x, v^{(0)}, \ldots, v^{(\gamma)}) \) such that the system (3) is transformed into (2), where \( z \) is a new state for the system, with \( \dim z = \dim x \). This transformation is invertible in the sense that one may locally write \( x = \psi(z, v^{(0)}, \ldots, v^{(\delta)}) \). The necessary and sufficient conditions for the existence of such a transformation are given in Delaleau and Respondek (1995). A state representation (3) is said to be generalized if \( f \) depends on \( v^{(\alpha)} \) for \( \alpha > 0 \), whereas the state representation (2) is said to be classical, or proper\(^1\).

Recall that in the behavioral approach of Willems (1992) the input and the output are not chosen a priori. The same point of view is shared by the approach of Fliess et al. (1999), and this fact is in accordance of what is found in physical systems. The results of Delaleau and Respondek (1995) can be generalized in Pereira da Silva and Batista (2009) for the case where there is freedom to redefine the input, that is, \( v \) it is not necessarily the original input of the system.

The work Pereira da Silva and Batista (2009) consider systems of the form

\[
\dot{x}(t) = f(t, x(t), u(t))
\]

where \( x(t) \in \mathbb{R}^n \) and \( u(t) \in \mathbb{R}^m \). A set of functions \( v = (v_1, \ldots, v_\gamma) \) is chosen, and it is called the input candidate. Note that each function \( v_i \) may depend on \( t, x, u^{(0)}, \ldots, u^{(\gamma)} \) where \( \gamma \in \mathbb{N} \). The main result of that paper solves the problem of checking if (4) admits an equivalent system

\[
\dot{z}(t) = g(t, z(t), v(t))
\]

where \( z(t) \in \mathbb{R}^q \) is a new state for the system obtained by an endogenous transformation, which is much more general than the transformation \( \phi \) considered Delaleau and Respondek (1995). In this case, the dimension of the state \( x \) is not necessarily equal to the one of the new state \( z \). To see this, consider the trivial example \( \dot{x}_1 = x_2, \dot{x}_2 = u \). If one chooses \( v = x_1 \), this system admits a state representation \( \dot{z}_1 = v \) with \( z_1 = x_1 \). If \( v = u^{(1)} \), the system admits a state representation \( \dot{z}_1 = z_2, \dot{z}_2 = z_3, \dot{z}_3 = v \), where \( z_1 = x_1, z_2 = x_2, z_3 = u_1 \). This last state representation is nothing less than a dynamic extension of the original state \( x \).

Here, the results of Pereira da Silva and Batista (2009) are generalized in order to obtain, under mild assumptions, necessary and sufficient conditions for the existence of a proper state representation of a nonlinear implicit systems. As in Pereira da Silva and Batista (2009), the conditions are constructive, and provide a test that decides whether a given set of functions \( v \) is an input of the implicit system that admits a proper state representation. Furthermore, \( v \) is not necessarily coincident with the “original” input of the implicit system. At this point we shall give a preliminary statement of the main problem to be considered\(^2\) in this work.

**State Representation Problem for Implicit Systems.** Let an implicit system \( \Delta \) be given by

\[
\Delta : \begin{cases}
\dot{x}(t) = f(t, x(t), u(t)) \\
y(t) = h(t, x(t), u(t)) = 0
\end{cases}
\]

where \( x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m \), and given a set of function \( v = (v_1, \ldots, v_\gamma), \) where \( v_i = \phi_i(t, x(t), u(t), \ldots, u^{(\alpha_i)}), \alpha_i \in \mathbb{N}, i \in [s], \) when there exists an equivalent system\(^3\) (5) with a given input \( v \)? \( \diamond \)

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1. See Appendix A for a definition of classical state representation in the context of Diffieties that holds for time-varying systems.
2. This choice may be regarded as a virtual input, as in the context of back-stepping, see Krstic et al. (1995).
3. The statement as it stands is imprecise in the sense that some technical assumptions are deferred to Section 4, and also because the word “equivalent” is yet to be defined (Def. 2.4).
In this paper it will be shown that the solution of this problem relies on the geometric properties of the explicit system $S$ given by

$$
S : \begin{cases}
\dot{x}(t) = f(t, x(t), u(t)) \\
y(t) = h(t, x(t), u(t))
\end{cases}
$$

(7)

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$. Note that $S$ is obtained from $\Delta$ by disregarding the algebraic constraints $y \equiv 0$ and considering them as outputs $y = h(t, x(t), u(t))$. In particular, \{t, x, u^{(0)}, u^{(1)}, \ldots\} are global coordinates for $S$, and so, those variables are independent for $S$ (but not for $\Delta$). Furthermore, it will be clear that it is not important whether an original input of the implicit system\(^1\) is available \textit{a priori} or not. The situation that is considered here is general as possible. In fact, the input candidate $v$ may be a function of system variables, and of its derivatives up to some finite order.

Our approach will follow the infinite dimensional geometric setting introduced in control theory by Fliess et al. (1993), Pomet (1995), Fliess et al. (1999), in combination with the ideas presented in Pereira da Silva and Corrêa Filho (2001), Pereira da Silva (2008) and also Conte et al. (2007).

We will use the standard notations of differential geometry in the finite and infinite dimensional case. A brief overview of the infinite dimensional approach of Fliess et al. (1999) is presented in Appendix A. Some notations and the definitions of Appendix A are used along the paper (e.g. the definition of system as a diffiety, and the definition of (classical) state representation as a local coordinate system). The survey Pereira da Silva (2008) presents the results about this approach that are considered here.

The field of real numbers will be denoted by $\mathbb{R}$. The matrix $M^T$ stands for the transpose of a matrix $M$. The set of natural numbers \{1, \ldots, k\} will be denoted by $[k]$. For simplicity, we abuse notation, letting $(z_1, z_2)$ stand for the column vector $(z_1^T, z_2^T)^T$, where $z_1$ and $z_2$ are also column vectors. Let $x = (x_1, \ldots, x_n)$ be a vector of functions (or a collection of functions). Then $\{dx\}$ stands for the set $\{dx_1, \ldots, dx_n\}$. Let $S$ be a system with Cartan field $\frac{d}{dt}$ (see appendix A). The Lie derivative $L_{\frac{d}{dt}} \eta$ of a function (or a form) $\eta$ will be denoted by $\dot{\eta}$ (or $\eta^{(1)}$) and the $k$-fold Lie derivative $L_{\frac{d}{dt}}^k \eta$ of $\eta$ will be denoted by $\eta^{(k)}$. If $\eta = (\eta_1, \ldots, \eta_m)$ is a set of functions (or forms), then $\eta^{(k)}$ stands for $\eta^{(k)} = (\eta_1^{(k)}, \ldots, \eta_m^{(k)})$. A codistribution $\Omega = \text{span} \{\omega_i, i \in A\}$ is said to be \textit{integrable} if the exterior derivatives of each $\omega_i$ can be expressed as $d\omega_i = \sum_{j \in F} \eta_j \wedge \omega_j$ for convenient one forms $\eta_j$, $j \in F$, and $F$ is a finite set\(^2\). Cartan’s version of the Frobenius theorem can be used in the context of diffeities for finite dimensional integrable distributions $\Gamma$, when $\Gamma$ is nonsingular (see Pomet (1995), Pereira da Silva (2008)).

The paper is organized as follows. Section 2 presents some preliminary results about implicit systems. Section 3 introduces some important results about derived flags. The main results are presented in section 4. Some worked examples are developed in section 5. Conclusions and a comparison with the results of Pereira da Silva and Batista (2009) are stated in section 6. Some auxiliary results and their proofs are presented in the Appendices A to E. Finally, Appendix F presents an algorithm that summarizes the main results from a computational viewpoint.

## 2 Some facts about implicit systems

Consider an implicit system $\Delta$ of the form (6), and suppose that also that all the functions defining (6) are smooth. One will call $x(t) \in \mathbb{R}^n$ the “pseudo-state” and $u(t) \in \mathbb{R}^m$ will be called the “pseudo-input”, a terminology that will be justified later. Recall from (7), that $S$ is obtained from $\Delta$ by disregarding the constraints $y \equiv 0$. Furthermore, the functions $y = h(t, x, u)$

\(^1\)The original input is not necessarily be coincident with $v$.

\(^2\)This means that the differential ideal generated by $\Omega$ is differentially closed (see Warner (1971)).
are considered to be outputs of $S$. Throughout this paper, the system $\Delta$ is the implicit system defined by (6), and $S$ is the explicit system given by (7).

System $S$ can be viewed as a diffeity with Cartan field $\frac{d}{dt}$ and output $y = h(t, x, u)$, in the framework of Fliess et al. (1999) that is briefly summarized in the appendix A. Then $y^{(k)}$ stands for the function $L^k_y = \frac{d^k}{dt^k} y$ defined on $S$, which may depend on $x, u^{(0)}, u^{(1)}, \ldots$.

It must be stressed out that $u$ is not necessarily the input of the implicit system, since the constraints $y^{(k)} \equiv 0$ may induce relations among the components of $x, u^{(0)}, u^{(1)}, \ldots$. For instance, the implicit system $\dot{x}_1 = u_1, \dot{x}_2 = u_2, y = x_1 + x_2 = 0$ is equivalent to the explicit system $\dot{x}_1 = u_1$ (and the relations $\dot{x}_2 = -x_1, u_2 = -u_1$). A possible state of the implicit system is $x_1$ and a possible input is $u_1$. This explains why $x$ and $u$ are called respectively pseudo-input and pseudo-state of (6).

The following codistribution, defined on the system (diffiety) $S$ given by (7), will be used in the sequel

$$\mathcal{Y} = \text{span}\left\{dt, (dy^{(k)} : k \in \mathbb{N})\right\}. \quad (8)$$

**Definition 2.1:** A local output subsystem of $Y$ for the explicit system $S$ defined (7), with output $y$ defined by (7), is a diffeity $Y$ and a Lie-Bäcklund submersion $\pi : U \subset S \rightarrow Y$, where $U \subset S$ is an open subset, such that $\pi^*(T^*_\xi Y) = \mathcal{Y}|_\xi$ for all $\xi \in U$. A local state representation $((x_a, x_b), (u_a, u_b))$ of $S$ is said to be strongly adapted to the output subsystem $Y$ if

(A) The corresponding Lie-Bäcklund submersion $\pi$ is locally given by

$$\pi(t, x_a, x_b, (u^{(0)}_a, u^{(j)}_b : j \in \mathbb{N})) = (t, x_a, (u^{(j)}_a, j \in \mathbb{N})).$$

(B) The local state equations of $S$ are of the form

$$\dot{x}_a = f_a(t, x_a, u_a) \quad (9a)$$

$$\dot{x}_b = f_b(t, x_a, x_b, u_a, u_b) \quad (9b)$$

where (9a) are the local state equations for $Y$.

(C) $\text{span}\left\{dx_a, (du^{(k)}_a : k \in \mathbb{N})\right\} = \text{span}\left\{dy^{(k)} : k \in \mathbb{N}\right\}$.

(D) The set of functions $\{x_a, u^{(k)}_a : k \in \mathbb{N}\}$ is contained in the set $\{y^{(k)} : k \in \mathbb{N}\}$.

**Remark 2.2** It is important to point out that the components of the input $(u_a, u_b)$ are redefined, that is, they are not necessarily a reordering of the original input $u$ of $S$. The same remark applies to the components of $(x_a, x_b)$, with respect to the original state $x$. A common abuse of notation occurs in the definition above. The same name $x_a$ stands for sets of coordinate functions defined on $S$, and also on $Y$. It would be more precise to let $x_a$ be a subset of local coordinates of $Y$, and $\tilde{x}_a$ be a subset of local coordinates of $S$ such that $\tilde{x}_a = x_a \circ \pi$. The same remark applies to $u^{(k)}_a, k \in \mathbb{N}$.

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1. It must be pointed out again that $y = h(t, x, u)$ is regarded an output rather than a constraint.
2. The definition of state representation given in the appendix considers that $\{t, x, u^{(0)}, u^{(1)}, \ldots\}$ is a local coordinate system, and so the variables $t, x, u^{(0)}, u^{(1)}, \ldots$ must not be linked by any relation.
4. The weaker definition of adapted state equations considered in Theorem 4.3 of Pereira da Silva and Corrêa Filho (2001) is obtained if one replaces the assumptions (C) and (D) by the only assumption that $\mathcal{Y} = \text{span}\left\{dt, dx_a, (du^{(k)}_a : k \in \mathbb{N})\right\}$. This last theorem also shows that the output subsystem is locally unique up to local Lie-Bäcklund isomorphisms.
5. Using (C) and (D), one may show that $f_a$ does not depend on $t$. 

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Let $\tilde{\Delta}$ be the subset of the explicit system (diffiety) $S$ defined by the points of $S$ for which all the Lie derivatives $y^{(k)} = \frac{d^k}{dt^k} y$ vanish:

$$\tilde{\Delta} = \{ \xi \in S \mid y^{(k)}(\xi) = 0, k \in \mathbb{N} \}$$

**Definition 2.3:** An implicit system $\Delta$ (defined by (6)) is regular if

- $\tilde{\Delta} \neq \emptyset$.
- There exists a local output subsystem $Y$ for system $S$ with output $y$ around all $\xi \in \tilde{\Delta}$.
- Around all $\xi \in \Delta$, system $S$ admits local state equations that are strongly adapted to $Y$.

Sufficient conditions for showing that a given implicit system is regular are given in Pereira da Silva and Corrêa Filho (2001), Pereira da Silva (2008). They are essentially linked to the notion of zero dynamics that appears in the decoupling theory. For completeness, these results are restated in Appendix E.

The following definition regards an implicit system as an immersed submanifold.

**Definition 2.4:** Consider an implicit system $\Delta$ defined by (6) and the explicit system $S$ defined by (7). A diffiety $\Gamma$ is said to be equivalent to the implicit system $\Delta$ if:

- There exists a Lie-Bäcklund immersion $\iota : \Gamma \rightarrow S$.
- For every solution $\sigma(t)$ of $\Delta$, there exists a solution $\nu(t)$ of $\Gamma$ such that $\sigma(t) = \iota \circ \nu(t)$.

The equivalent system $\Gamma$ is said to be canonical if $\Gamma$ is an immersed submanifold. It can be shown that a regular implicit system defined by (6) is equivalent to an immersed submanifold. It is shown that a regular implicit system defined by (6) is equivalent to an immersed submanifold. This result is the Proposition 2.5 below.

**Proposition 2.5:** (Pereira da Silva and Corrêa Filho (2001), Pereira da Silva (2008)) Let $\Delta$ be a regular implicit system defined by (6) and let $S$ be the explicit system associated to (7). Then the subset $\tilde{\Delta} \subset S$ defined by (10) has a canonical structure of immersed (embedded) submanifold of $S$ such that the canonical insertion $\iota : \tilde{\Delta} \rightarrow S$ is a Lie-Bäcklund immersion. Furthermore, $\Delta$ admits a local classical state representation around every point $\xi \in \tilde{\Delta}$. In particular, $\tilde{\Delta}$ is canonically equivalent to $\Delta$.

The idea of the proof of Proposition 2.5 is to consider the local state representation (9) that is strongly adapted to the output subsystem. It is shown that $\{t, x_a, x_b, U_a, U_b\} \cup \{t, x_b, U_b\}$ are respectively local coordinates for $S$ and $\Delta$, where $U_a = \{u_a^{(j)} : j \in \mathbb{N}\}$ and $U_b = \{u_b^{(j)} : j \in \mathbb{N}\}$. In these coordinates $\iota(t, x_b, U_b) = (t, 0, x_b, 0, U_b)$.

It must be pointed out that the proof of proposition 2.5 shows also that the local state equations for $\Delta$ are given by $\dot{x}_b = f_b(t, 0, x_b, 0, u_b)$. In particular, the implicit system is equivalent to the a kind of “zero dynamics”, as pointed out in Byrnes and Isidori (1991), Krishnan and McClamroch (1994).

### 3 Adapted projections and derived flags

The main results of this paper are based on derived flags, defined on the explicit system $S$ given by (7), considering the quotient with respect the codistribution $\mathcal{Y}$ defined in (8). Such kind of
Proposition 3.1:

Let \( \theta \) be any adapted projection defined on an open set \( U \subset S \). Let \( \Omega_0 \) be a construction is called relative derived flag, and it plays an important role in the theory of implicit systems (see Pereira da Silva and Corrêa Filho (2001)).

Given any state representation \((\tilde{x}, \tilde{u})\) defined on \( U \subset S \), with \( \tilde{x} = (x_a, x_b) \) and \( \tilde{u} = (u_a, u_b) \) of \( S \) that is adapted to the output subsystem \( Y \), its clear that the \( C^\infty(U) \)-module \( U = \text{span} \{ dt, dx_a, dx_b, (du_a^{(k)}, du_b^{(k)} : k \in \mathbb{N}) \} \) is locally decomposed as \( U = \mathcal{B} \oplus \mathcal{Y} \), where \( \mathcal{Y} = \text{span} \{ dt, dx_a, (du_a^{(j)} : j \in \mathbb{N}) \} \) and \( \mathcal{B} = \text{span} \{ dx_b, (du_b^{(j)} : j \in \mathbb{N}) \} \).

One may define locally the projection \( \theta : \mathcal{B} \oplus \mathcal{Y} \rightarrow \mathcal{B} \), called adapted projection, which associates a one-form \( \omega \) to its projection \( \theta(\omega) = \sum_{i=1}^{n_a} \beta ix_a + \sum_{k=0}^{\infty} \sum_{j=0}^{m_b} \epsilon_{jk} du_b^{(k)} \).

This projection is clearly a module morphism (or a linear map between vector spaces, when one works pointwise). Let \( \Gamma = \text{span} \{ \omega_i, i \in \Lambda \} \) be a given codistribution, and define \( \theta \Gamma = \text{span} \{ \theta(\omega_i), i \in \Lambda \} \). Then, by construction

\[ \Gamma + \mathcal{Y} = \theta \Gamma \oplus \mathcal{Y} \]

Let \( \tilde{\theta} : \tilde{\mathcal{B}} \oplus \tilde{\mathcal{Y}} \rightarrow \tilde{\mathcal{B}} \) be another projection (locally) constructed from other adapted state representation that is also defined on \( U \). Since \( \theta(\omega) - \omega \in \mathcal{Y} \) and \( \tilde{\theta}(\omega) - \omega \in \mathcal{Y} \), it follows that \( \theta(\omega) - \theta(\omega) \in \mathcal{Y} \). Let \( \pi : \mathcal{U} \rightarrow \mathcal{U}/\mathcal{Y} \) be the canonical projection, i.e., the map \( \omega \mapsto \omega \mod \mathcal{Y} \). Then

\[ \pi \circ \tilde{\theta} = \pi \circ \theta \]

Note that \( \mathcal{B} = \theta(\mathcal{U}) \) and \( \tilde{\mathcal{B}} = \tilde{\theta}(\mathcal{U}) \). Furthermore, \( \pi|\mathcal{B} \rightarrow \mathcal{U}/\mathcal{Y} \) and \( \pi|\tilde{\mathcal{B}} \rightarrow \mathcal{U}/\mathcal{Y} \) are isomorphisms\(^1\) such that \( \pi(\theta \omega) = \pi(\tilde{\theta} \omega) \). In particular, if \( \theta \Gamma \) is a nonsingular finite dimensional codistribution defined on \( S \), then

\[ \dim \tilde{\theta}(\Gamma) = \dim \theta(\Gamma) \]

\(^1\)Isomorphisms of modules, or isomorphism of vector spaces, depending on the case.
codistribution defined on $U$ and let $\Gamma_0 = \Theta \Omega_0$. Let $\tilde{\Gamma}_0 = \Omega_0 + \mathcal{Y}$. Define the relative derived flags$^2$

$$\Gamma_k = \text{span} \{ \omega \in \Gamma_{k-1} \mid \dot{\omega} \in \Gamma_{k-1} + \mathcal{Y} \}, \ k \in \mathbb{N}$$

and

$$\tilde{\Gamma}_k = \text{span} \{ \omega \in \tilde{\Gamma}_{k-1} \mid \dot{\omega} \in \tilde{\Gamma}_{k-1} \}, \ k \in \mathbb{N}$$

Then, one has $\Theta \tilde{\Gamma}_k = \Gamma_k$, and $\Gamma_k \oplus \mathcal{Y} = \tilde{\Gamma}_k$, for all $k \in \mathbb{N}$. Furthermore$^1$:

$$\Gamma_k = \{ \omega \in \Gamma_{k-1} \mid \Theta \dot{\omega} \in \Gamma_{k-1} \} \quad (16)$$

**Proof** It is clear that $\Gamma_{k+1} \subset \Gamma_0$, for $k = 0, 1, \ldots$. As $\Gamma_0 \cap \mathcal{Y} = \{0\}$ it follows that $\Gamma_k \cap \mathcal{Y} = \{0\}$ for $k = 0, 1, \ldots$. By definition, from (13), the properties hold for $k = 0$. By induction, assume that $\Theta \tilde{\Gamma}_k = \Gamma_k$ and $\Gamma_k \oplus \mathcal{Y} = \tilde{\Gamma}_k$. Let $\omega \in \Gamma_{k+1}$. Then it will be shown that $\Theta(\omega) = \omega$ and $\omega \in \tilde{\Gamma}_{k+1}$. In particular, one concludes that $\Gamma_{k+1} \subset \Theta \tilde{\Gamma}_{k+1}$. In fact, the first statement is a consequence of the fact that $\Theta \mathcal{B}$ is the identity map, where $\mathcal{B} = \text{im} \Theta$ and $\Gamma_{k+1} \subset \Gamma_0 \subset \Theta \mathcal{B}$. Now let $\omega \in \Gamma_{k+1}$. It follows that $\dot{\omega} \in \Gamma_k + \mathcal{Y} = \tilde{\Gamma}_k$. Then $\omega \in \tilde{\Gamma}_{k+1}$.

Now let $\omega \in \tilde{\Gamma}_{k+1}$. Then, $\omega \in \tilde{\Gamma}_k$. Since $\Theta \omega = \omega + \eta$ for some $\eta \in \mathcal{Y}$, then $\frac{d}{dt} (\Theta \omega) = \dot{\omega} + \dot{\eta}$. As $\dot{\eta} \in \mathcal{Y}$, and $\Gamma_k \oplus \mathcal{Y} = \tilde{\Gamma}_k$, then $\frac{d}{dt} (\Theta \omega) \in \tilde{\Gamma}_k + \mathcal{Y}$. Hence $\Theta \omega \in \Gamma_{k+1}$ and so, $\Gamma_{k+1} \subset \Theta \tilde{\Gamma}_{k+1}$. Now, as $\Gamma_{k+1} = \Theta \tilde{\Gamma}_{k+1}$, by (13), it follows that $\Theta \tilde{\Gamma}_{k+1} + \mathcal{Y} = \Gamma_{k+1} \oplus \mathcal{Y} = \tilde{\Gamma}_{k+1} + \mathcal{Y}$. Since $\Gamma_0 \supset \mathcal{Y}$, it is easy to see that $\tilde{\Gamma}_k \supset \mathcal{Y}$ for $k \in \mathbb{N}$. In fact, this follows from the fact that $\omega \in \mathcal{Y}$ implies that $\dot{\omega} \in \mathcal{Y}$. Hence, $\Gamma_{k+1} \oplus \mathcal{Y} = \tilde{\Gamma}_{k+1}$. Equation (16) can be proved in a similar way, and is left to the reader. \qed

Given a control system $S$, assume that $\Gamma = \text{span} \{ \omega_1, \ldots, \omega_s \}$ is a smooth codistribution defined on an open set $U \subset S$. One may define the first term of the derived flag in two different forms. The first one, when one regards $\Gamma$ as a $C^\infty(U)$-submodule:

$$\Omega_1 = \text{span} \{ \omega \in \Gamma \mid \dot{\omega} \in \Gamma \} \quad (17)$$

The second one is when one regards things pointwise. At a point $\nu \in U$, $\Gamma|_\nu$ is a $\mathbb{R}$-subspace of $T_\nu U$ and then:

$$\Gamma_1|_\nu = \text{span} \{ \omega|_\nu \in \Gamma \mid \dot{\omega}|_\nu \in \Gamma|_\nu \} \quad (18)$$

The following proposition states that, under some regularity assumptions, the pointwise definition (17) coincides (locally) with (17). The technique that is used in its proof is useful for the computation of derived flags.

**Proposition 3.2:** Define $\Gamma_1|_\nu$ for $\nu \in U$ by (18). Assume that the family $\{ \omega_1, \ldots, \omega_s \}$ is pointwise independent on $U$. Let $\xi \in U$ be a regular point of $\Gamma_1$. Then, $\Gamma_1$ is locally smooth around $\xi$ and the two definitions (18) and (17) are locally equivalent.

**Proof** The proof is deferred to Appendix D. \qed

Now let $\Theta : \mathcal{B} \oplus \mathcal{Y} \to \mathcal{B}$ be an adapted projection. Let $\Gamma = \text{span} \{ \omega_1, \ldots, \omega_s \} \subset \mathcal{B}$. Then, analogously to the definitions above, one may define the relative derived flags in two different forms.

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$^2$In the context of exterior differential systems, derived flags are defined by $\tilde{\Gamma}_k = \{ \omega \in \tilde{\Gamma}_k \mid d\omega \in (\tilde{\Gamma}_k) \}$. Such derived flags are considered in Pereira da Silva and Corrêa Filho (2001). Under the assumption of the integrability of the members of the derived flag, equation A.4 of that paper shows the equivalence between the last definition and the one considered in this work. See also Pereira da Silva (2008) for a similar situation.

$^1$The equation (16) is a suitable form for computations, since it refers only to finite dimensional objects.
ways. The first one regards codistributions as $C^\infty(U)$-modules:

$$\Omega_1 = \text{span}\{\omega \in \Gamma \mid \dot{\omega} \in \Gamma \oplus \mathcal{Y}\}$$  \hspace{1cm} (19)

The second one is the pointwise definition. At a point $\nu \in U$, $\Gamma|\nu$ is a $\mathbb{R}$-subspace of $T^*_\nu U$ and then:

$$\Gamma_1|\nu = \text{span}\{\omega|\nu \in \Gamma \mid \dot{\omega}|\nu \in \Gamma|\nu \oplus \mathcal{Y}|\nu\}$$  \hspace{1cm} (20)

The following proposition generalizes Proposition 3.2 to relative derived flags. As the proof is essentially the same, it furnishes an algorithm for computing relative derived flags.

**Proposition 3.3:** Define $\Gamma_1|\nu$ for $\nu \in U$ by (20). Assume that the family $\{\omega_1, \ldots, \omega_s\}$ is pointwise independent on $U$. Let $\xi \in U$ be a regular point of $\Gamma_1$. Then, $\Gamma_1$ is locally smooth around $\xi$ and the two definitions (20) and (19) are locally equivalent.

**Proof** It is not difficult to show from (16) that the proof of the proposition can be obtained from the proof of Proposition 3.2 if one replaces $\dot{\omega}|\nu$ by $\theta \dot{\omega}|\nu$ in that proof. \hfill \Box

## 4 Classical state representations of implicit systems

The state representation problem studied in this paper is now stated in a precise manner.

**Definition 4.1:** (State Representation Problem for Implicit Systems.) Assume that an implicit system (6) is regular. Consider the explicit system $S$ with output $y$ defined by (7). Let $v = (v_1, \ldots, v_s)$ be a set of functions defined around a point $\nu \in \Delta \subset S$, where $\Delta$ is defined by (10). The set $v$ is the input candidate of the implicit system. The State Representation Problem for an Implicit system with Input $v$ around some point $\nu \in \hat{\Delta}$ is the problem of finding a proper state representation $((z_a, z), (v_a, v))$ for the explicit system $S$, defined around $\nu$, that is strongly adapted to subsystem $\mathcal{Y}$, if one exists.

**Remark 4.2** Roughly speaking, $v$ is the data of the problem, and the question is to verify the existence of such $z$. By Definition 2.3, system $S$ admits (locally) a state representation $((x_a, x_b), (u_a, u_b))$ with state representation (9). If $u_b$ coincides with $v$ in (9), then the problem is solved. However, this is not necessarily the case.

In fact, the problem is locally solvable around some $\nu \in \hat{\Delta}$, according to Definition 2.1, if and only if $S$ admits local state equations that are strongly adapted to $\mathcal{Y}$, given by

$$\dot{z}_a = f_a(t, z_a, v_a)$$  \hspace{1cm} (21a)

$$\dot{z} = f_b(t, z_a, z, v_a, v)$$  \hspace{1cm} (21b)

Furthermore, it follows by the proof of proposition 2.5, that $\hat{\Delta}$ is locally equivalent to the implicit system (6), $(z, v)$ is a local state representation of $\hat{\Delta}$, and the implicit system admits local proper state equations of the form

$$\dot{z} = f_b(t, 0, z, 0, v)$$  \hspace{1cm} (22)

The following theorem gives necessary and sufficient conditions for the solution of the State Representation Problem for Regular Implicit Systems.

**Theorem 4.3:** Let $((x_a, x_b), (u_a, u_b))$ be a local proper state representation of system $S$ defined by (7) that is adapted to $\mathcal{Y}$. Let $\theta$ be the associated adapted projection (see section 3). Let $\mathcal{Y}$ be

---

1Recall that the components of $v$ may depend on $t, x, u, u^{(1)}, \ldots$
the codistribution, defined on \( S \) by (8). Let \( \gamma \) be the least nonnegative integer\(^2\) such that one may locally write that \( \text{span} \{ dv \} \subset \text{span} \{ dx_b, du_b^{(0)}, \ldots, du_b^{(\gamma)} \} + \mathcal{Y} \).

Then the State Representation Problem for the Implicit System (6) with input \( v \) is solvable around some \( v \in S \) if and only if there exists a non-negative integer \( \delta \) such that, for the codistribution \( \Gamma_0 \) defined by

\[
\Gamma_0 = \text{span} \left\{ dx_b, du_b^{(0)}, \ldots, du_b^{(\gamma)}, \theta \left( dv^{(0)} \right), \ldots, \theta \left( dv^{(\delta)} \right) \right\}
\]

and for the codistributions \( \Gamma_k \) defined by

\[
\Gamma_k = \text{span} \left\{ \omega \in \Gamma_{k-1} | \omega \in \Gamma_{k-1} + \mathcal{Y} \right\}
\]

one has:

(i) \( \Gamma_k \) is finite dimensional and nonsingular and \( \dim \Gamma_{k-1} - \dim \Gamma_k = \dim v \) for \( k = 1, \ldots, \delta + 1 \). 

(ii) \( \Gamma_{\delta+1} + \mathcal{Y} \) is integrable.

(iii) \( \Gamma_0 = \Gamma_1 \oplus \text{span} \left\{ \theta \left( dv^{(0)} \right) \right\} \).

(iv) The set \( \mathcal{V}^{(k)} = \{ \theta \left( dv^{(k)} \right) \} \) is locally linearly independent for \( k = 0, \ldots, \delta \).

The proof of Theorem 4.3 is based on the following auxiliary results.

**Lemma 4.4:** Let \( S \) be the system (7). Let \( \mathcal{Y} \) be the codistribution defined by (8). Let \( (\tilde{z}, \tilde{v}) = ((z_a, z), (v_a, v)) \) and \( (\tilde{x}, \tilde{u}) = ((x_a, x_b), (u_a, u_b)) \) be two local proper state representations of \( S \) that are strongly adapted to subsystem \( Y \) around some point \( \xi \in S \).

1. If one has\(^1\) \( \text{span} \{ dv \} \subset \text{span} \left\{ dx_b, du_b^{(\alpha)}, \ldots, du_b^{(\alpha)} \right\} \oplus \mathcal{Y} \) for \( \alpha \in \mathbb{N} \) then \( \text{span} \{ dz \} \subset \text{span} \left\{ dx, dz, du_b^{(\alpha)}, \ldots, du_b^{(\alpha)} \right\} \oplus \mathcal{Y} \).

2. Let \( \beta \in \mathbb{N} \) be the smallest integer for which one may locally write, \( \text{span} \{ dv \} \subset \text{span} \left\{ dt, \tilde{dv}^{(0)}, \ldots, \tilde{dv}^{(\beta)} \right\} \). If \( \text{span} \{ dt, \tilde{dv}, \tilde{dv}^{(0)}, \ldots, \tilde{dv}^{(n)} \} \subset \text{span} \left\{ dt, dx, du, \ldots, du^{(\alpha)} \right\} \) then \( \beta \leq n + m \gamma \), where \( n = \text{card} \tilde{x} \) and \( m = \text{card} \tilde{u} \).

**Proof** See appendix B.

The following corollary\(^2\) depends on the technical conditions for the existence of strongly adapted state equations given in Theorem E.2 of Appendix E.

**Corollary 4.5:** Let \( S \) be the explicit system with state representation \( (x, u) \) defined by (7). Assume that the conditions of Theorem E.2 holds. Let \( x = (x_a, x_b) \) and \( u = (u_a, u_b) \) be the strongly adapted state representation constructed by Theorem E.2. Let \( (z_a, z), (v_a, v) \) be a local proper state representation around \( \xi \in S \) that is also strongly adapted to the output subsystem \( \mathcal{Y} \). Let \( \alpha \in \mathbb{N} \) be an integer such that \( \text{span} \{ dv \} \subset \text{span} \left\{ dt, dx, du, \ldots, du^{(\alpha)} \right\} \oplus \mathcal{Y} \). Then \( \text{span} \{ dz \} \subset \text{span} \left\{ dt, dx, du, \ldots, du^{(\alpha)} \right\} \oplus \mathcal{Y} \).

**Proof** By Theorem E.2, \( \text{span} \{ dx \} + \mathcal{Y} = \text{span} \{ dx_b \} \oplus \mathcal{Y} \) and \( \text{span} \{ dx, du \} + \mathcal{Y} = \text{span} \{ dx_b, du_b \} \oplus \mathcal{Y} \). By differentiation of the last condition, it follows that \( \text{span} \left\{ dx, du, \ldots, du^{(\alpha)} \right\} + \mathcal{Y} = \text{span} \left\{ dx_b, du_b, \ldots, du_b^{(\alpha)} \right\} \oplus \mathcal{Y} \) for \( \alpha \in \mathbb{N} \). The corollary then follows from Lemma 4.4. Note that the (local) existence of such \( \alpha \) is implied by the fact that \( \{ t, x_a, x_b, (v_a^{(k)})^{(k)}, (v_b^{(k)})^{(k)} : k \in \mathbb{N} \} \) is a local coordinate system for which \( \mathcal{Y} = \text{span} \left\{ t, x_a, (v_a^{(k)})^{(k)} : k \in \mathbb{N} \right\} \).

\(^2\)The existence of the integer \( \gamma \) is assured by the fact that a state representation is a local coordinate system.

\(^1\)If \( \alpha = 0 \), then \( \text{span} \left\{ dx_b, du_b, \ldots, du_b^{(\alpha-1)} \right\} \) stands for \( \text{span} \{ dx_b \} \).

\(^2\)This corollary is used only in Theorem 4.9.
Lemma 4.6: Assume that the local state representation \((z, z_a), (v, v_a)\) of \(S\) is a solution of the state representation problem for a given implicit system \((S, y)\). Let \(\delta \in \mathbb{N}\), and let \(\mathcal{Y}\) be given by (8). Define \(\bar{\Gamma}_0 = \text{span}\{dz, dv^{(0)}, \ldots, dv^{(\delta)}\} \oplus \mathcal{Y}\) and \(\bar{\Gamma}_k = \{\omega \in \bar{\Gamma}_k \mid \dot{\omega} \in \bar{\Gamma}_k\}\), for \(k \in \mathbb{N}\). Then,

\[
\bar{\Gamma}_k = \text{span}\left\{dz, dv^{(0)}, \ldots, dv^{(\delta-k)}\right\} \oplus \mathcal{Y}, \text{ for } 0 \leq k \leq \delta. \tag{25a}
\]

\[
\bar{\Gamma}_{k+1} = \text{span}\{dz\} \oplus \mathcal{Y}. \tag{25b}
\]

Now, let \(\theta\) be an arbitrary adapted projection (see section 3). Let \(\Gamma_k = \bar{\Theta}\Gamma_k\), and \(L_k = \{\theta(dz), \theta(dv^{(0)}), \ldots, \theta(dv^{(\delta-k)})\}\). Then \(L_k\) is linearly independent, \(\Gamma_k = \text{span}\{L_k\}\), and \(\bar{\Gamma}_k = \Gamma_k \oplus \mathcal{Y}\) for \(k = 0, 1, \ldots, \delta + 1\).

Proof Some calculations show (25a) and (25b). Let \(\bar{\Theta}\) be the adapted projection associated to the adapted state representation \((z, z_a), (v, v_a)\). Then \(\theta(dz) = dz\) and \(\theta(dv^{(k)}) = dv^{(k)}\) for \(k \in \mathbb{N}\).

In particular, the set \(L_k = \{\theta(dz), \theta(dv^{(0)}), \ldots, \theta(dv^{(\delta-k)})\}\) is linearly independent. The proof may be completed using (15) and Proposition 3.1. \(\square\)

Lemma 4.7: Let \(\theta : \mathcal{B} \oplus \mathcal{Y} \to \mathcal{B}\) be an adapted projection associated to some (strongly) adapted state representation \((x_a, x_b), (u_a, u_b)\) that is defined in some open neighborhood \(U\) of \(\xi\). Assume that \(\Gamma \subset \mathcal{B}\) is nonsingular and finite dimensional and suppose that \(\Gamma \oplus \mathcal{Y}\) is integrable. Then there exists a set \(z = \{z_1, \ldots, z_s\}\) of \(s\) smooth functions that are locally defined on some open neighborhood \(V \subset U\) of \(\xi\) such that, on \(V\) one has \(\Gamma = \theta(\text{span}\{dz\})\), \(\Gamma \oplus \mathcal{Y} = \text{span}\{dz\} \oplus \mathcal{Y}\), and the set \(\{dz_1, \ldots, \theta dz_s\}\) is linearly independent pointwise.

Proof Since \(\Gamma\) is nonsingular and finite dimensional, one locally has \(\Gamma = \text{span}\{\omega_1, \ldots, \omega_s\}\) for convenient one-forms \(\omega_i\) defined on \(U\). To say that \(\Gamma \oplus \mathcal{Y}\) is integrable is equivalent to writing

\[
d\omega_i = \sum_{i=1}^s \eta_i \wedge \omega_i + \sum_{j=1}^{n_a} \epsilon_j \wedge dx_{a_j} + \sum_{k=1}^{m_a} \sum_{l=1}^\beta \gamma_{kl} \wedge du^{(l)}_{a_k} + \zeta \wedge dt
\]

for convenient one-forms \(\eta_i, \epsilon_j, \gamma_{kl}\) and \(\zeta\) defined on \(U\), for \(i = 1, \ldots, s\), \(j = 1, \ldots, n_a\), \(k = 1, \ldots, m_a\) and \(l = 1, \ldots, \beta\). This means that the finite dimensional codistribution \(\Gamma \oplus \mathcal{Y}_{\beta}\) is integrable, where \(\mathcal{Y}_{\beta} = \text{span}\{dt, dx_{a_1}, du^{(0)}_{a_1}, \ldots, du^{(\beta)}_{a_1}\}\).

An application of finite dimensional Frobenius theorem constructs a local basis \(\mathcal{W} = \{dw_1, \ldots, dw_r\}\) of \(\Gamma \oplus \mathcal{Y}_{\beta}\), where the \(w_i\) are smooth functions defined on some open neighborhood of \(\xi\). Hence we may locally complete the set \(\{dt, dx_{a_1}, du^{(0)}_{a_1}, \ldots, du^{(\beta)}_{a_1}\}\) with elements \(\{dw_1, \ldots, dw_r\}\) \(\subset \mathcal{W}\), forming a local basis of \(\Gamma \oplus \mathcal{Y}_{\beta}\).

In particular, one may chose \(z = (w_{i_1}, \ldots, w_{i_s})\).

Note that \(\text{span}\{dz\} \oplus \mathcal{Y}_{\beta} = \Gamma \oplus \mathcal{Y}_{\beta}\). In particular one has \(\dim \Gamma = \dim \text{span}\{dz\} = \dim \mathcal{W}\) and \(\text{span}\{dz\} + \mathcal{Y} = \Gamma \oplus \mathcal{Y}\). Since \(\Gamma \subset \mathcal{B}\), \(\theta\) is the identity map, and \(\ker \theta = \mathcal{Y}\), then \(\theta(\Gamma \oplus \mathcal{Y}) = \Gamma\). Note also that \(\theta(\text{span}\{dz\} + \mathcal{Y}) = \theta\text{span}\{dz\}\). Hence \(\Gamma = \theta\text{span}\{dz\}\). Since \(\dim \Gamma = \dim \text{span}\{dz\}\) = \(\dim \mathcal{W}\), it follows that the set \(\{\theta dz_{i_1}, \ldots, \theta dz_s\}\) is independent. To conclude the proof, note that \(\sum_{i=1}^s \alpha_i \theta dz_i + \eta_i = 0\) for some \(\eta_i \in \mathcal{Y}\) implies that \(\sum_{i=1}^s \alpha_i \theta dz_i = 0\). Hence \(\text{span}\{dz\} \cap \mathcal{Y} = \{0\}\). \(\square\)

Lemma 4.8: Let \((x, u) = ((x_a, x_b), (u_a, u_b))\) be a state representation that is defined in some open neighborhood \(U\) of \(\xi\) and assume that it is strongly adapted to the output subsystem \(Y\). Let \(z, v\) be sets of smooth functions defined on \(U\) such that

1. \(\{dz, dv^{(0)}, \ldots, dv^{(\delta)}\}\) is linearly independent modulo \(\mathcal{Y}\) at every point \(\xi\) of \(U\).
2. \(\text{span}\{dz\} \subset \text{span}\{dz, dv\} \oplus \mathcal{Y}\),
3. \(\text{span}\{dz_x\} \subset \text{span}\{dz, dv^{(0)}, \ldots, dv^{(\delta-1)}\} \oplus \mathcal{Y}\) and \(\text{span}\{dv_y\} \subset \text{span}\{dz, dv^{(0)}, \ldots, dv^{(\delta)}\} \oplus \mathcal{Y}\) for some \(\delta \in \mathbb{N}\).
Let \( c \in \mathbb{N} \). Let \( \tilde{x}_a = (x_a, u_a^{(0)}, \ldots, u_a^{(c-1)}) \) and \( \tilde{u}_a = u_a^{(c)} \). There exists a convenient \( c \in \mathbb{N} \) and an open neighborhood \( V \subset U \) of \( \xi \) such that, \( ((\tilde{x}_a, z), (\tilde{u}_a, v)) \) is also a local state representation that is (strongly) adapted to \( V \) with local state equations given by

\[
\begin{align*}
\dot{x}_a &= f_a(t, \tilde{x}_a, \tilde{u}_a) \\
\dot{z} &= f_z(t, z, v, \tilde{x}_a, \tilde{u}_a)
\end{align*}
\]

\((26a)\) and \((29a)\) is a straightforward consequence of the definition of \((23)\), \((24)\), and Proposition 3.1. The proof of necessity can be reached using Lemma 4.6.

**Proof** See appendix C. \( \Box \)

**Proof** (of Theorem 4.3.)

(Necessity). By Definition 4.1, there exists a local proper state representation \( ((t, z_a), (v_a, v)) \) of \( S \) that is strongly adapted to \( V \). In particular \( \{t, z_a, (v_a(k), v(k) : k \in \mathbb{N})\} \) is a local coordinate system of the diffiety \( S \). Let \( \mathcal{Y} \) be given by \((8)\). Then, by part (1) of Lemma 4.4, one has \( \text{span} \{dx_b\} \subset \text{span} \{dz, dv^{(0)}, \ldots, dv^{(\beta-1)}\} \oplus \mathcal{Y} \), and \( \text{span} \{du_b\} \subset \text{span} \{dz, dv^{(0)}, \ldots, dv^{(\beta)}\} \oplus \mathcal{Y} \), for \( \beta \) big enough. In the same way, one may locally write\(^1\) \( \text{span} \{dz\} \subset \text{span} \{dx_b, du_b^{(0)}, \ldots, du_b^{(\gamma-1)}\} \oplus \mathcal{Y} \) and \( \text{span} \{dv\} \subset \text{span} \{dx_b, du_b^{(0)}, \ldots, du_b^{(\gamma)}\} \oplus \mathcal{Y} \) for some convenient \( \gamma \). Now take \( \delta = \beta + \gamma \). Since

\[
\text{span} \{dx_b\} \subset \text{span} \{dx_b, du_b\} \oplus \mathcal{Y},
\]

by derivation, it follows that

\[
\text{span} \{dx_b, du_b^{(0)}, \ldots, du_b^{(\gamma)}\} \oplus \mathcal{Y} \subset \text{span} \{dz, dv^{(0)}, \ldots, dv^{(\delta)}\} \oplus \mathcal{Y}.
\]

\((27)\)

Let \( \mathcal{B} = \text{span} \{dx_b, (du_b^{(k)} : k \in \mathbb{N})\} \). Let \( \theta : \mathcal{B} \oplus \mathcal{Y} \to \mathcal{B} \) be the corresponding adapted projection. Recall that \( \theta|\mathcal{B} \) is the identity map and \( \theta(\mathcal{Y}) = 0 \). Let \( \Gamma_0 \) be defined by \((23)\). Then it is clear that \( \theta(\Gamma_0 \oplus \mathcal{Y}) = \Gamma_0 \).

Now define \( \Omega_0 = \text{span} \{dx_b, du_b, \ldots, du^{(\gamma)}_b, dz, dv^{(0)}, \ldots, dv^{(\delta)}\} \). It is clear that \( \Gamma_0 = \theta \Omega_0 \). By Proposition 3.1 for \( k = 0 \), \( \Gamma_0 \oplus \mathcal{Y} = \Omega_0 + \mathcal{Y} \). By \((28)\), one has \( \Omega_0 + \mathcal{Y} = \text{span} \{dz, dv^{(0)}, \ldots, dv^{(\delta)}\} \oplus \mathcal{Y} \). By Definition 4.1 (see also Definition 2.1), one notes that \( \{t, z_a, (v_a(k), v_a(k) : k \in \mathbb{N})\} \) is a local coordinate system and \( \mathcal{Y} = \text{span} \{t, z_a, (v_a(k) : k \in \mathbb{N})\} \). It follows that \( \text{span} \{dz, dv^{(0)}, \ldots, dv^{(\delta)}\} \cap \mathcal{Y} = 0 \). Then \( \Omega_0 + \mathcal{Y} = \text{span} \{dz, dv^{(0)}, \ldots, dv^{(\delta)}\} \oplus \mathcal{Y} \). In particular, from Proposition 3.1 for \( k = 0 \), one shows that, for \( \gamma \) and \( \delta \) previously constructed, then \( \Gamma_0 \oplus \mathcal{Y} = \text{span} \{dz, dv^{(0)}, \ldots, dv^{(\delta)}\} \oplus \mathcal{Y} \).

The proof of necessity can be reached using Lemma 4.6.

(Sufficiency). It will be shown first that

\[
\theta \left( \text{span} \{dv^{(0)}, \ldots, dv^{(\delta-k)}\} \right) \subset \Gamma_k, k = 0, \ldots, \delta
\]

\((29a)\)

and that

\[
\Gamma_k = \Gamma_{k+1} \oplus \theta \left( \text{span} \{dv^{(\delta-k)}\} \right), k = 0, \ldots, \delta
\]

\((29b)\)

Take \( \Omega_0 = \text{span} \{dx_b, du_b^{(0)}, \ldots, du^{(\gamma)}_b, dv^{(0)}, \ldots, dv^{(\delta)}\} \). Note that \( \Gamma_0 = \theta \Omega_0 \). Then, the condition \((29a)\) is a straightforward consequence of the definition of \((23)\), \((24)\), and Proposition 3.1. The equation \((29b)\) will be shown by induction. Note that \((29b)\) coincides with \((iii)\) for \( k = 0 \).

\(^1\)If \( \gamma = 0 \), one assumes that \( \text{span} \{dz\} \subset \text{span} \{dx_b\} \oplus \mathcal{Y} \).
Assume that it holds for some \( k \), with \( 0 \leq k \leq \delta - 1 \). Let \( V_{\delta - k - 1} = \theta(\text{span} \{ dv_j^{(\delta-k-1)} \}) \). By contradiction, assume that \( \{ \Gamma_{k+2} \cap V_{\delta - k - 1} \} \neq \{ 0 \} \) for some \( \nu \) in the neighborhood of definition of the state representation \( ((x_a, x_b), (u_a, u_b)) \). By (iv), (29a), and the nonsingularity of \( \Gamma_{k+1} \), one may construct a local basis of \( \Gamma_{k+1} \) of the form \( \{ \omega_1, \ldots, \omega_h, \theta(dv_j^{(\delta-k-1)}) \} \). Let \( \omega \in \Gamma_{k+2} \cap V_{\delta - k - 1} \) be a smooth one form locally defined on \( S \) such that \( \omega \nu = \omega(\nu) \neq 0 \) and \( \omega \nu \in \{ \Gamma_{k+2} \cap V_{\delta - k - 1} \} \). Since \( \Gamma_{k+2} \subset \Gamma_{k+1} \), this is equivalent to say that \( \omega = \sum_{i=1}^s \alpha_i \omega_i + \sum_{j=1}^{\dim v} \beta_j \theta(dv_j^{(\delta-k-1)}) \), where \( \alpha_i | \nu = 0 \) for all \( i = 1, \ldots, s \), but some \( \beta_i | \nu \neq 0 \). It follows from (24) that

\[ \hat{\omega} = \sum_{i=1}^s [\hat{\alpha}_i \omega_i + \alpha_i \hat{\omega}_i] + \sum_{i=1}^{\dim v} [\hat{\beta}_j \theta(dv_j^{(\delta-k-1)}) + \beta_j \theta(dv_j^{(\delta-k)})] \in \Gamma_{k+1} + \mathcal{Y}. \]

Note that \( \sum_{i=1}^s \hat{\alpha}_i \omega_i + \sum_{i=1}^{\dim v} \hat{\beta}_j \theta(dv_j^{(\delta-k-1)}) \in \Gamma_{k+1} \). It follows that \( \eta | \nu = \left\{ \sum_{i=1}^{\dim v} \beta_j \theta(dv_j^{(\delta-k-1)}) \right\} | \nu \in \Gamma_{k+1} | \nu \oplus \mathcal{Y} | \nu \). But \( \eta | \nu \) is in the image of \( \theta \) and hence it is easy to show\(^1\) that \( \eta | \nu \in \Gamma_{k+1} | \nu \) and \( \eta | \nu \neq 0 \), and this contradicts the induction hypothesis.

Now it is easy to see that (29b) is a consequence of (29a), of (iv), of the fact that \( \Gamma_{k+1} \cap \text{span} \{ \theta(dv_j^{(\delta-k)}) \} = 0 \), and of the fact that \( \dim \{ \Gamma_k - \text{dim} \Gamma_{k+1} = \text{dim} v, \text{for } k = 0, 1, \ldots, \delta + 1 \) Applying Lemma 4.7 to \( \Gamma_{\delta + 1} \), one may write \( \Gamma_{\delta + 1} = \theta(\text{span} \{ dz \}) \), where the set \( \theta(dz) = \{ \theta(dz_1), \ldots, \theta(dz_\delta) \} \) is linearly independent. From (29b) it follows that the set \( \Lambda_k = \{ \theta(dz), \ldots, \theta(dv_j^{(\delta-k)}) \} \), \( k = 0, \ldots, \delta \) is linearly independent and \( \Gamma_k = L_k \) for \( k = 0, \ldots, \delta \).

By (23), one may write

\[ \text{span} \{ du_k \} \subset \Gamma_0 = L_0. \]  

Now, by (27) and by (24) with \( k = 1 \), it follows that

\[ \text{span} \{ dx_k \} \subset \Gamma_1 = L_1. \]

Let \( H_k = \text{span} \{ dz, dv_0^{(0)}, \ldots, dv_0^{(\delta-1)} \} \). Hence \( L_k = \theta H_k \) and from (13), it follows that

\[ H_k + \mathcal{Y} = L_k \oplus \mathcal{Y}, k = 0, \ldots, \delta + 1. \]

Now note that

1. Since \( L_0 \) is linearly independent\(^2\), then \( \{ dz, dv_0^{(0)}, \ldots, dv_0^{(\delta)} \} \) is linearly independent modulo \( \mathcal{Y} \).

2. \( \text{span} \{ dz \} \subset \text{span} \{ dz, dv_0^{(0)} \} \oplus \mathcal{Y} \) (by the definition of \( \Gamma_k \) in (i), from the fact that \( \Gamma_k = L_k \), and from (32)).

3. \( \text{span} \{ dx_k \} \subset \text{span} \{ dz, dv_0^{(0)} , \ldots, dv_0^{(\delta-1)} \} \oplus \mathcal{Y} \) (by (30), (31) and (32)).

The proof of sufficiency then follows from Lemma 4.8.

\[^1\text{Since } \Gamma_{k+1} \subset \mathcal{B}, \theta \mathcal{B} \text{ is the identity map, ker } \theta = \mathcal{Y}, \theta(\eta | \nu) = \eta | \nu \in \theta(\Gamma_{k+1} | \nu \oplus \mathcal{Y} | \nu) = \Gamma_{k+1} | \nu \).

\[^2\text{If } \omega = \sum \alpha_i dz_i + \sum \beta_j dv_j^{(\delta-1)} + \eta = 0, \text{ where } \eta \in \mathcal{Y}, \text{ for convenient smooth functions } \alpha_i, \beta_j, \text{ then } \theta \omega = 0 \text{ and ker } \theta = \mathcal{Y} \text{ implies that } L_0 \text{ is linearly dependent.} \]
Theorem 4.9: Assume that the conditions of theorem E.2 hold (see Appendix E). Consider the explicit system (6). Let \( \mathcal{Y} \) be the codistribution, defined on \( S \) by (8). There exists \( \gamma \in \mathbb{N} \) such that \( \gamma \) is the least integer for which \( \text{span} \{dv\} \subset \text{span} \{dt, dx, du, \ldots, du^{(\gamma)}\} \oplus \mathcal{Y} \). Then the State Representation Problem for the Implicit System (6) with input \( v \) is solvable if and only if there exist a non-negative integer \( \delta \) such that, for the codistribution \( \Gamma_0 \) defined on \( S \) by

\[
\tilde{\Gamma}_0 = \text{span} \left\{ dx, du^{(0)}, \ldots, du^{(\gamma)}, \frac{dx}{du^{(0)}}, \ldots, \frac{dx}{du^{(\delta)}} \right\} + \mathcal{Y}
\]

and for \( \tilde{\Gamma}_k = \text{span} \left\{ \omega \in \tilde{\Gamma}_{k-1} \mid \dot{\omega} \in \tilde{\Gamma}_{k-1} \right\} \), we have

(i) \( \tilde{\Gamma}_k/\mathcal{Y} \) is finite dimensional and nonsingular for \( k = 0, \ldots, \delta+1 \), and \( \dim \frac{\tilde{\Gamma}_{k-1}}{\mathcal{Y}} = \dim \tilde{\Gamma}_k = \dim v \) for \( k = 1, \ldots, \delta \).

(ii) \( \tilde{\Gamma}_{\delta+1} \) is integrable.

(iii) \( \tilde{\Gamma}_k/\mathcal{Y} = \tilde{\Gamma}_k/\mathcal{Y}^{(\delta)} \), where \( \mathcal{Y}^{(\delta)} = \text{span} \{dv^{(\delta)}\} \mod \mathcal{Y} \).

(iv) The set \( \{dv^{(k)}\} \mod \mathcal{Y} \) is locally linearly independent, for \( k = 0, \ldots, \delta \).

Proof If the conditions of Theorem E.2 hold, then one may construct the strongly adapted state representation \( ((x_a, x_b), (u_a, u_b)) \) following the steps (A), (B), (C), and (D) at the end of that theorem. By construction, \( \text{span} \{dx\} + \mathcal{Y} = \text{span} \{dx_b\} \oplus \mathcal{Y} \) and \( \text{span} \{dx, du\} + \mathcal{Y} = \text{span} \{dx_b, du_b\} \oplus \mathcal{Y} \). By differentiation, \( \text{span} \{dx, du^{(0)}, \ldots, du^{(k)}\} + \mathcal{Y} = \text{span} \{dx_b, du_b^{(0)}, du_b^{(k)}\} \oplus \mathcal{Y} \). The existence of \( \gamma \) is a simple consequence of the last equality, and the fact that \( \{t, x_a, x_b, (u_a^{(k)}, u_b^{(k)} : k \in \mathbb{N})\} \) is a local coordinate system such that \( \mathcal{Y} = \text{span} \{dt, x_a, (u_a^{(k)} : k \in \mathbb{N})\} \) (see the Proof of Corollary 4.5). The result follows easily from the proof of Theorem 4.3, Proposition 3.1 and the properties of adapted projections (see section 3).

Remark 4.10 The conditions of the Theorems 4.3 and 4.9 are equivalent. However, the conditions of Theorem 4.9 are more intrinsic than the ones of theorem 4.3, since they do not rely on a particular choice of the adapted state representation, but only on some geometric properties of the original state representation \( (x, u) \) of \( S \). Theorem 4.3 is more suitable for the computations of a given example, to verify the solvability of the State Representation Problem, and to construct solution, if one can integrate \( \Gamma_{\delta+1} + \mathcal{Y} \). Note also that part (2) of Lemma 4.4 may furnish a bound for \( \delta \). In fact, in the proof of Theorem 4.3, one takes \( \delta = \beta + \gamma \), and so as \( \beta \) is bounded by Lemma 4.4, then \( \delta \) is also bounded by \( n + (m+1)\gamma \), where \( n = \dim x \) and \( m = \dim u \).

5 Examples

Example 1. Consider the implicit system

\[
\begin{align*}
\dot{x}_1(t) &= u_1 + 2x_3u_2 \\
\dot{x}_2(t) &= tx_3 + 2(x_1 + 1)u_1 + 4x_1x_3u_2 \\
\dot{x}_3(t) &= u_2(t) \\
y(t) &= x_1 - x_3^2 = 0
\end{align*}
\]

Let \( x = (x_1, x_2, x_3) \) and \( u = (u_1, u_2) \) The input candidate for this example is \( v = tx_3 + u_1(x_1 - x_3^2) \). First apply Theorem E.1 of Appendix E. In fact, if one writes differentials in the basis \( \{dt, dx, du\} \), representing them as row vectors, one may compute the \( 6 \times 9 \) matrix \( M_1 = [(dy)^T(dy)^T(dt)^T dx^T]^T \).
It can be shown that the assumptions of theorem E.1 holds for $\alpha = 1$. In fact, (2) is consequence of the fact that the submatrix of $M_1$ formed respectively by 1st, 3rd, 4th, 5th and 6th rows has constant rank, and equal to 4. Note that (3) is consequence of the fact that $M_1$ has constant rank, equal to 7. It is easy to deduce from the rows of $M_1$ that (4) follows. Furthermore note that dim (span \{dx\} \cap \text{span} \{dt, dy\}) - dim (\text{span} \{dx\} \cap \text{span} \{dt, dy, d\}) = dim (\text{span} \{dx\}) + dim (\text{span} \{dt, dy\}) - dim (\text{span} \{dt, dx, dy\}) - dim (\text{span} \{dt, dy, d\}) + dim (\text{span} \{dt, dx, dy, d\}) = 3 + 2 - 4 - 3 - 3 + 5 = 0. Hence, (1) holds. Following the steps (A), (B), (C) and (D) at the end of Theorem E.1, it is possible to choose $a = x_1 - x_3^2$, $u_a = \dot{y} = u_1$, $x_b = (x_2, x_3)$ and $u_b = u_2$. If one replace $u_b$ by $v$, then condition (D) will be not respected any more. Then one may proceed with the algorithm of section F.

In order to perform the next computations, it is useful to write the one forms in the basis $B_2 = \{dt, dx_b, du_b, d\tilde{u}_b, dx_a, du_a, d\tilde{u}_a, d\tilde{u}_a\}$, instead of considering the original basis $B_1 = \{dt, dx, du, d\tilde{u}, d\tilde{u}\}$. This allows to compute the adapted projection $\theta$ concerning the adapted state representation $(x_a, x_b), (u_a, u_b))$.

Writing $dv$ in the basis $B_2$, one obtains $dv = x_3 dt + t dx_b$. From this, one concludes that $\gamma = 0$. Now choose $\delta = 2$, and construct

$$\Gamma_0 = \text{span} \{dt, dx_b, du_b, \theta dv, \theta d\tilde{v}, \theta \tilde{d}v\}$$

For a one-form $\omega$, the adapted projection $\theta \omega$ is obtained by writing $\omega$ in the basis $\{dt, dx_b, (du_b^{(k)} : k \in \mathbb{N}), dx_a, (du_a^{(k)} : k \in \mathbb{N})\}$ and by deleting the components in $dt, dx_a, (du_a^{(k)} : k \in \mathbb{N})$ (see Section 3). Now choose $\delta = 2, \tilde{\delta} = 1$ and $\tilde{v} = \tilde{v} = 0$.

In particular, for $t \neq 0$, one obtains $\Gamma_0 = \text{span} \{dt, dx_b, du_b\}$. From (9), it is clear that $\text{span} \{dx_b\} \mod Y \subset \text{span} \{dx_b\} \mod Y$. It follows from (16) that $\Gamma_1 = \text{span} \{dt, dx_b, du_b\}$, and that $\Gamma_2 = \text{span} \{dt, dx_b\}$. To compute $\Gamma_3$, note that: $\theta dx_b = \phi dx_b + 4z_3 x_1 du_a$ and that $\theta dx_b = du_b$. Then, one may show from (16) that $\Gamma_3 = \text{span} \{dt, dx_b, du_b\}$. From (9), it is integrable, and that $\theta dx_b = du_b$. Then, one may show from (16) that $\Gamma_3 = \text{span} \{dt, dx_b, du_b\}$. Now, as $\Gamma_3 + \text{span} \{dy\}$ is integrable, all the assumptions of Theorem 4.3 holds. To see that $\Gamma_3 + \text{span} \{dy\}$ is integrable, it suffices to notice that $dy = dx_1 - 2x_3 x_1 dx_3$, hence $\Gamma_3 + \text{span} \{dy\} = \text{span} \{dt, dx_2 - 2x_1 x_3, dy\}$. Thus, one may choose $z = x_2 - x_3^2$. Now, after some computations, it follows that $\dot{z} = \dot{y} + v - y\dot{y}$. Remember that, by the proof of Theorem 4.3, $((z, x_a), (v, u_a))$ is a strongly adapted state representation with state equations given by

$$\dot{x}_a = u_a$$
$$\dot{z} = v + x_a - x_a u_a$$

Hence, a state representation of the implicit system is obtaining by taking $y = x_a = 0$ and $\dot{y} = u_a = 0$ (see Remark 4.2). In particular one gets the state representation

$$\dot{z} = v. \quad \text{(14)}$$

**Example 2.** Consider the input-output equations (with output $w = (w_1, w_2)$ and input $v = (v_1, v_2)$) given by:

$$-v_1 \dot{w}_1 + (e^{w_1} +1) \dot{w}_1 - w_1 \dot{w}_1 + (e^{w_1} + \dot{w}_1 - w_1 v_1)^2 + v_2 = 0$$
$$\dot{w}_2 + u_2 (e^{w_1} + \dot{w}_1 - w_1 v_1) v_1 = 0$$

1. The reader may verify that the assumption (iii) of Theorem 4.3 fails for $\delta = 0$ and for $\delta = 1$.
2. Note that every point of $S$ such that $t = 0$ is a singular point of $\Gamma_0$.
3. These computations were performed by Matlab/Maple®.
One may convert these equations into a system of the form (6) in the following way. Choose $x = (x_1, \ldots, x_8) = (w_1, \dot{w}_1, \dot{w}_2, \dot{w}_3, \dot{v}_1, \dot{v}_2, v_1, v_2)$ and let $u = (u_1, u_2, u_3, u_4) = (w_1^{(3)}, \dot{w}_2, v_1^{(1)}, \dot{v}_2)$. The input-output equations can be written as:

$$
\dot{x} = f(x, u)
$$

$$
y_1 = -x_6 x_2 + (e^{x_2} + 1) x_3 - x_1 x_7 + (e^{x_2} + x_2 - x_1 x_6)^2 + x_8 = h_1(x, u) = 0
$$

$$
y_2 = x_5 + x_4 x_6 (e^{x_2} + x_2 - x_1 x_6) = h_2(x, u) = 0
$$

with $f(x, u) = (x_2 x_3 u_1 x_5 u_2 x_7 u_3 u_4)^T$ which is in the form (6). Note that $w_1 = x_1$, $w_2 = x_4$, $v_1 = x_6$ and $v_2 = x_8$. Now, after some symbolic computations, one can show that the assumptions of Theorem E.1 holds for $\alpha = 1$. Since $\frac{\partial y_1}{\partial x_2} = 1$ and $\frac{\partial y_2}{\partial x_2} = 1$, it is possible to choose $x_a = y = (y_1, y_2)$, $u_a = \dot{y}$ and $x_b = (x_1, x_2, x_3, x_4, x_6, x_7)$. Since $\frac{\partial y_1}{u_3} = 1$ and $\frac{\partial y_2}{u_3} = 1$, then one can choose $u_b = (u_1, u_3)$. Since $u_b$ cannot be replaced by $v$ (while respecting the construction of the step (D) of Theorem E.1), one must proceed with Algorithm F. As in the previous example, it is easy to show that $\gamma = 0$. One will choose $\delta = 2$. Denote $x_b = (x_{b_1}, x_{b_2}, x_{b_3}, x_{b_4}, x_{b_5}, x_{b_6})$ and $u_b = (u_{b_1}, u_{b_2})$. After some symbolic computations, one obtains

$$
\Gamma_0 = \text{span} \{dx_b, du_b, [(1 - e^{x_2})d\dot{u}_b + x_1 d\dot{u}_b]\}
$$

$$
\Gamma_1 = \text{span} \{dx_b, [(1 - e^{x_2})du_b + x_1 du_b]\}
$$

$$
\Gamma_2 = \text{span} \{dx_b, dx_{b_1}, dx_{b_2}, dx_{b_3}, dx_{b_4}, [(1 - e^{x_2})dx_{b_5} + x_1 dx_{b_5}]\}
$$

$$
\Gamma_3 = \text{span} \{dx_b, dx_{b_1}, [(1 - e^{x_2})dx_{b_2} + x_1 dx_{b_2}]\}
$$

Note that $\Gamma_3 = \text{span} \{dx_1, dx_4, [(1 - e^{x_2})dx_2 + x_1 dx_6]\} = \text{span} \{dx_1, dx_4, d(-x_2 - e^{x_2} + x_1 x_6)\}$. In particular, $\Gamma_3$ is integrable. One may take $z = (z_1, z_2, z_3) = (x_1, -x_2 - e^{x_2} + x_1 x_6, x_4)$. Note that the function $\phi : \mathbb{R} \to \mathbb{R}$ defined by $x_2 \mapsto [x_2 + e^{x_2}]$ is invertible with inverse $\psi : \mathbb{R} \to \mathbb{R}$. In particular, one may write $x_2 = \psi(-z_2 + x_1 x_6)$. After some computations one obtains the strongly adapted state representation: $\dot{x}_a = u_a$, $\dot{z}_1 = \psi(-z_2 + z_1 v_1)$, $\dot{z}_2 = z_3^2 + v_2 - x_1$, $\dot{z}_3 = z_2 z_3 v_1 + x_{a_2}$. Taking $y = x_a \equiv 0$, one obtains the following (classical) realization for the original input-output equations (with input $(v_1, v_2)$ and output $(w_1, w_2)$):

$$
\dot{z}_1 = \psi(-z_2 + z_1 v_1)
$$

$$
\dot{z}_2 = z_3^2 + v_2
$$

$$
\dot{z}_3 = z_2 z_3 v_1
$$

$$
w_1 = z_1
$$

$$
w_2 = z_3
$$

---

1. These symbolic computations were also performed by Matlab/Maple®.
2. The reader may verify that the assumption (iii) of Theorem 4.3 does not hold for $\delta = 0$ and $\delta = 1$.
3. Without the need of summation with $\dot{y}$.
Example 3. Consider the implicit system:

\[
\begin{align*}
\dot{x}_1 &= u_1 + ax_4 + ax_6 + (ax_1 - ax_3)^2 + (ax_2 - ax_6)u_3 \\
\dot{x}_2 &= e^{(ax_3 + ax_5)}u_1 + u_2 + u_4 \\
\dot{x}_3 &= -u_1 + ax_4 + ax_6 + (ax_1 - ax_3)^2 + (ax_2 - ax_6)u_3 \\
\dot{x}_4 &= -e^{(ax_3 + ax_5)}u_1 + u_2 + u_4 \\
\dot{x}_5 &= u_1 + ax_4 + ax_6 - (ax_1 - ax_3)^2 - (ax_2 - ax_6)u_3 \\
\dot{x}_6 &= e^{(ax_3 + ax_5)}u_1 + u_2 - u_4 \\
\dot{x}_7 &= ax_1 - ax_3 + (ax_2 - ax_4)u_2 \\
y_1 &= ax_1 - ax_3 = 0 \\
y_2 &= ax_2 - ax_4 = 0
\end{align*}
\]

where \(a = 1/2\). Let \(v = [u_3, u_4]\). Computations with MATLAB/MAPPLE shows that the assumptions of theorem E.1 holds for \(\alpha = 3\) (but they hold neither for \(\alpha = 1\), nor for \(\alpha = 2\)). From steps (A), (B), (C), and (D) at the end of that theorem, it is possible to show that one may choose \(x_b = (x_5, x_6, x_7)\) with \(u_b = v = [u_3, u_4]\), \(x_a = y\) and \(u_a = \dot{y}\). In particular, the Problem of State Representation with Input \(v\) for this implicit system is solvable, and one may take \(z = x_b = (x_5, x_6, x_7)\) (see Remark 4.2).

Example 4. Consider the same system of Example 3, but include the new component \(y_3 = x_7\) to the output. Now, choose \(\dot{y} = (y_1, y_2)\) and \(\ddot{y} = y_3\). Since \(\dim \text{span}\{dt, dy, d\dot{y}, d\ddot{y}\} = \dim \text{span}\{dt, d\dot{y}, d\ddot{y}, d\dddot{y}\}\), it is easy to show that the assumptions of Theorem E.2 hold. One may show that, in this case, the steps (A), (B), (C), (D) at the end of that theorem are satisfied for \(x_b = (x_5, x_6)\), \(u_b = v\), \(x_a = \dot{y}\), \(u_a = \ddot{y}\). In particular, from Remark 4.2 the Problem of State Representation with Input \(v\) for this implicit system is solvable, and one may take \(z = x_b = (x_5, x_6)\).

Example 5. Recall that the proof of Theorem 4.3 shows that system (7) admits (locally) a strongly adapted state representation (21). From this, one may take \(z_a \equiv 0\) and \(v_a \equiv 0\), obtaining the state representation of the implicit system given by (22).

The present example shows that the conditions of Theorem 4.3 are not necessary for the existence of a classical state representation of the implicit system with input \(v\). In fact an implicit system (6) may admit a classical state representation (5) that is not associated to any strongly adapted state representation (21) of (7).

For instance, consider the system \(\dot{x}_1 = x_2, \dot{x}_2 = x_3 + u_2, \dot{x}_3 = u_1, \dot{x}_4 = x_3 + x_1 u_1^2, y = x_1 - \epsilon = 0\), with \(\epsilon \in \mathbb{R}\). Considering the explicit system \(S\) (by disregarding the constraint \(y \equiv 0\)), let \(x_a = (x_1, x_2), v_a = \dot{x}_2 = x_3 + u_2, x_b = (x_3, x_4), u_b = u_1\). Then it is possible to show that Theorem E.2 holds with \(\ddot{y} = y\), \(\dot{y} = \emptyset\) and \(\alpha = 2\). So \(((x_a, x_b), (u_a, u_b))\) is a strongly adapted state representation of \(S\) with state equations given by:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= v_a \\
\dot{x}_3 &= u_1 \\
\dot{x}_4 &= x_3 + x_1 u_1^2
\end{align*}
\]

One concludes that the implicit system admits the state representation:

\[
\begin{align*}
\dot{x}_3 &= u_1 \\
\dot{x}_4 &= x_3 + \epsilon u_1^2
\end{align*}
\]
Considering $\epsilon = 0$, simple computations shows that the implicit system admits a proper state representation with input $v = x_3$, namely, $\dot{x}_4 = v$. However, it is not difficult to show that the assumptions of Theorem 4.3 are not satisfied for any $\delta \in \mathbb{N}$ (for any $\epsilon \in \mathbb{R}$).

6 Conclusions

The main result of this paper may be interpreted in the following way. Recall that the conditions obtained in Pereira da Silva and Batista (2009) rely on an integrability test based on a derived flag obtained from the codistribution $\Gamma_0 = \text{span}\{dt, dx, du(0), \ldots, du(\gamma), dv(0), \ldots, dv(\delta)\}$, where $\gamma$ and $\delta$ are convenient integers. Theorem 4.3 of the present paper can be viewed as a quotient version of (Pereira da Silva and Batista 2009, Theo. 1), where this quotient is taken with respect to the codistribution $\mathcal{Y} = \text{span}\{dt, (dy)^{(k)} : k \in \mathbb{N}\}$, that is generated by the differentials of time and the constraint functions $y$ and their derivatives. When there is no constraints, then $\mathcal{Y} = \text{span}\{dt\}$ and Theorem 4.3 reduces to (Pereira da Silva and Batista 2009, Theo. 1).

The class of systems of the form (6) include input-output equations (see Example 2) and implicit systems of the form

$$
F(t, w(t), \dot{w}(t)) = 0.
$$

(33)

In fact, given a system (33), let $x = w$ and $u = \dot{w}$. Then the system $\dot{x}(t) = u(t), y(t) = F(x(t), u(t)) = 0$ is of the form (6).

Acknowledgements

The first author was partially supported by CNPq – Conselho Nacional de Desenvolvimento Científico e Tecnológico (Brazil), Grant Number 308465/2006-7. The authors are indebted to Felipe M. Pait for many interesting suggestions.

Appendix A: Diffieties and Systems

This appendix is a very brief summary of some facts about the infinite dimensional approach of Fliess et al. (1993), Pomet (1995), Fliess et al. (1999). A survey about this subject can be found in Pereira da Silva (2008).

$\mathbb{R}^A$-Manifolds, Diffieties and Systems. The infinite dimensional approach of Fliess et al. (1999) relies on $\mathbb{R}^A$ manifolds. For an introduction to this kind of manifolds the reader may refer to Zharinov (1992).

An ordinary diffiety is an $\mathbb{R}^A$ manifold for which there exists a field $\frac{d}{dt}$, called Cartan field.

A system $S$ is a pair $(S, t)$, where $S$ is an ordinary diffiety, and $t : S \to \mathbb{R}$ is a function, called time, such that $\frac{d}{dt}(t) = 1$ and such that around any point $\xi \in S$ there exists local coordinates of $S$ of the form $(t, \eta)^2$.

State Space Representation and Outputs. A local state representation of a system $(S, t)$ is a local coordinate system $\psi = \{t, x, U\}$, where $x = \{x_i, i \in [n]\}, U = \{u_j^{(k)} : j \in [m], k \in \mathbb{N}\}$. The set of functions $x = (x_1, \ldots, x_n)$ is called state and the set $u = (u_1, \ldots, u_m)$ is called input.

1 Explicit systems can be also converted to the form (33) (see Lévine (2006)).

2 This is equivalent to saying that the function $t$ is a submersion, and the fact that $\frac{d}{dt}(t) = 1$ is equivalent to saying that that the function $t$ is Lie-Bäcklund, when $\mathbb{R}$ is regarded as a diffiety with trivial Cartan field.
In these coordinates the Cartan field is locally written by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i} + \sum_{k \in \mathbb{N}, j \in [m]} u_j^{(k+1)} \frac{\partial}{\partial u_j^{(k)}} \tag{A1}$$

It follows from (A1) that $L_x u^{(k)} = \frac{d}{dt} u^{(k)} = u^{(k+1)}$. So the notation $u^{(k)}$ is consistent with the fact that, along a solution\(^1\), it represents the differentiation of $u^{(k-1)}$ with respect to time.

A state representation of a system $S$ is completely determined by the choice of the state $x$ and the input $u$ and will be denoted by $(x, u)$. An output $y$ of a system $S$ is a set of functions defined on $S$. A state representation is said to be classical (or proper) if $f_i$ does not depend on $u^{(a)}$ for $a > 1$. A control system $S$ is a system such that there exists a local state representation around every $\xi \in S$.

**System associated to differential equations.** Now assume that a control system is given by a set of equations

$$
\begin{align*}
\dot{t} &= 1 \\
\dot{x}_i &= f_i(t, x, u, \ldots, u^{(a_i)}), \ i \in [n] \\
y_j &= \eta_j(x, u, \ldots, u^{(a_j)}), \ j \in [p]
\end{align*} \tag{A2}
$$

One can always associate to these equations a diffiety $S$ of global coordinates $\psi = \{t, x, U\}$ and Cartan field given by (A1).

**Solutions.** A solution of a system $S$ with Cartan field $\frac{d}{dt}$ is a smooth map $\sigma : (a, b) \rightarrow S$, where $(a, b) \subset \mathbb{R}$, such that $\sigma(t) = \frac{d}{dt}(\sigma(t))$.

**Subsystems.** A (local) subsystem $S_a$ of a system $S$ with time notion $t$ is a pair $(S_a, \pi)$, where $S_a$ is a system with a time notion $\tau_a$ and Cartan field $\partial_a$, and $\pi$ is a Lie-Bäcklund submersion $\pi : U \subset S \rightarrow S_a$ between the system $U \subset S$ and $S_a$ such that $\tau_a \circ z \pi = t$. A local state representation $x = (x_a, x_b)$, $u = (u_a, u_b)$ is said to be adapted to a subsystem $S_a$ if we locally have

$$
\begin{align*}
\dot{x}_a &= f_a(t, x_a, u_b) \tag{A3a} \\
\dot{x}_b &= f_b(t, x_a, x_b, u_a, u_b) \tag{A3b}
\end{align*}
$$

and $(x_a, u_a)$ is a local state representation of $S_a$ with state equations (A3a).

**Appendix B: Proof of Lemma 4.4**

To prove part (1), let $a \in \mathbb{N}$ and let $H_a$ stand for span $\{dx_b, du_b^{(0)}, \ldots, du^{(a)}_b\}$. If span $\{dz\} \subset \text{span} \{dx_b\} \oplus \mathcal{Y}$, one may take $\alpha = 0$. So assume that span $\{dz\} \not\subset \text{span} \{dx_b\} \oplus \mathcal{Y}$. Let $\beta \in \mathbb{N}$ be such that span $\{dz\} \subset H_{\beta-1} \oplus \mathcal{Y}$, but span $\{dz\} \not\subset H_{\beta-1} \oplus \mathcal{Y}$, where $\mathcal{Y}$ is defined by (8). Since $\{t, x_a, x_b, (u_a^{(k)}, u_b^{(k)} : k \in \mathbb{N})\}$ is a local coordinate system, and $\mathcal{Y} = \text{span} \{dt, dx_a, (du_a^{(k)} : k \in \mathbb{N})\}$, as one assumes that and span $\{dz\} \not\subset \text{span} \{dx_b\} \oplus \mathcal{Y}$, it is clear that there exists such $\beta$ (locally). As $((z_a, z), (v_a, v))$ is proper, then span $\{dz\} \subset \text{span} \{dz, dv\} \oplus$

\(^1\)See the definition of solution given in this section.
\[ \mathcal{Y}. \text{ Note that} \]

\[ dz = \sum_{i=1}^{n_u} \gamma_i dx_{b_i} + \sum_{j=0}^{\beta} \sum_{k=1}^{m_k} \epsilon_{jk} du_{b_{kj}} + \eta\]

where \( \eta \in \mathcal{Y} \) and \( \gamma_i, \epsilon_{jk} \) are smooth functions defined on \( U \subset S \). Since span \{dz\} \not\subset H_{\beta-1} \oplus \mathcal{Y} \), then some function \( \epsilon_{\beta j} \) is not identically zero for some \( j \in \{1, \ldots m_k\} \). So,

\[ d \dot{z} = \sum_{i=1}^{n_u} (\dot{\gamma}_i dx_{b_i} + \gamma_i \dot{dx}_{b_i}) + \sum_{j=0}^{\beta} \sum_{k=1}^{m_k} (\dot{\epsilon}_{jk} du_{b_{kj}} + \epsilon_{jk} \dot{du}_{b_{kj}}^{(j+1)}) + \dot{\eta} \]

The properness of the state representation implies that span \{d\dot{x}_b\} \subset span \{dx_b, du_b\} \oplus \mathcal{Y}. It follows that span \{d\dot{z}\} \subset H_{\beta+1} \oplus \mathcal{Y}, but span \{d\dot{z}\} \not\subset H_{\beta} \oplus \mathcal{Y}. Since span \{d\dot{z}\} \subset span \{dz, dv\} \oplus \mathcal{Y}, it follows that span \{dv\} \not\subset H_{\beta} \oplus \mathcal{Y}.

Now assume, that for some \( \alpha \in \mathbb{N} \), one has span \{dv\} \subset H_{\alpha} \oplus \mathcal{Y}. Assume by contradiction that span \{dz\} \not\subset H_{\alpha-1} \oplus \mathcal{Y}. Then \( \alpha \leq \beta \), with \( \beta \) defined above. Then, from the reasoning above, span \{dv\} \not\subset H_{\beta}, and so span \{dv\} \not\subset H_{\alpha}(\subset H_{\beta}), which is an absurd. The part (2) is a direct consequence of (Pereira da Silva 2008, Lemma 1, part 2).

**Appendix C: Proof of Lemma 4.8**

The following two results are instrumental for the proof of Lemma 4.8.

**Lemma C.1:** (Lemma 2 of Pereira da Silva (2008)) Let \((x, u)\) be a local proper state representation of a system \( S \) around some \( \xi \in S \), and let \( \bar{x} = (\bar{x}_1, \ldots, \bar{x}_s) \) and \( \bar{v} = (\bar{v}_1, \ldots, \bar{v}_m) \) be sets of functions defined on the diffiety \( S \). Suppose that span \{d\bar{x}, \bar{v}\} \subset span \{dt, dx, du\}. Then \((\bar{x}, \bar{v})\) is a local state representation of \( S \) around \( \xi \) if and only if there exist \( \alpha \in \mathbb{N} \) such that

- The set \( S = \{dt, d\bar{x}, d\bar{v}, \ldots, d\bar{v}^{(\alpha)}\} \) is linearly independent pointwise in an open neighborhood of \( \xi \).
- One has span \{dx\} \subset span \{dt, d\bar{x}, d\bar{v}^{(0)}, \ldots, d\bar{v}^{(\alpha-1)}\}, in an open neighborhood of \( \xi \).
- One has span \{d\bar{x}, du\} \subset span \{dt, d\bar{x}, d\bar{v}^{(0)}, \ldots, d\bar{v}^{(\alpha)}\} in an open neighborhood of \( \xi \).

**Lemma C.2:** Let \( S \) be a \( \mathbb{R}^A \) manifold and let \( \xi \in S \). Let \( \psi = \{ (i, \bar{x}, \bar{v}) \} \) be a local coordinate system defined in an open neighborhood \( V_\xi \) of \( \xi \). Let \( \gamma \) be a set of 1-forms defined on \( V_\xi \). Let \( \omega \in \gamma \) be a 1-form such that \( \omega \in \gamma \subset \omega_{1}, \ldots, \omega_{s} \) \( \supset \mathcal{Y} \). Then there exists a finite subset \( F \subset B \) such that \( \omega \in \gamma \subset \omega_{1}, \ldots, \omega_{s} \) \( \subset \mathcal{Y} \) as \( C^\infty(V_\xi)\)-module. Hence, for convenient smooth

\[ \omega_l = \sum_{j=1}^{\gamma_l} \alpha_j dy_j, \quad (C1) \]

\[ \omega_l = \sum_{i=1}^{\epsilon_l} \beta_i dx_i \]

In particular, \( \omega \in \gamma \subset \omega_{1}, \ldots, \omega_{s} \) \( \subset \mathcal{Y} \) as \( C^\infty(V_\xi)\)-module. Hence, for convenient smooth

---

1. Since \( \{t, x_a, u_a(w_a^{(k)}, u_b^{(k)}; k \in \mathbb{N})\} \) is a local coordinate system, such \( \alpha \) always exists (locally).
functions $a_l, l = 1, \ldots, s$, and $b_j, j \in B$, one may write
\[ \omega = \sum_{l=1}^{s} a_l \omega_l + \sum_{j \in B} b_j dy_j = \sum_{l=1}^{s} \frac{\partial \gamma_l}{\partial x} dx_i + \sum_{j \in B} b_j dy_j \] (C2)

One may locally write, perhaps after restricting the open neighborhood of $\xi$ to some $W_\xi \subset V_\xi$:
\[ \omega = \sum_{i \in F_1} e_i dx_i + \sum_{j \in F_2} b_j dy_j \] (C3)

for convenient finite subsets $F_1 \subset A$ and $F_2 \subset B$. On $W_\xi$, as $\psi$ is a local coordinate system, the expression (C3) is unique. In particular, subtracting (C3) from (C2) on $W_\xi$ gives zero. From the independence of the differentials of a local coordinate system, one concludes that $\omega = \sum_{l=1}^{s} a_l \omega_l + \sum_{j \in F_2} b_j dy_j$. The proof is concluded from the last equation and (C1).

Proof (of Lemma 4.8) From Assumptions 1, 2, and 3, and Lemma C.2 it is clear that there exists $c \in \mathbb{N}$ big enough such that

(A) $\text{span} \{d\xi \} \subset \text{span} \{dt, dz, dv^{(0)}, dx_a, du_a^{(0)}, \ldots, du_a^{(c)} \}$.

(B) $\text{span} \{d\bar{x}_b \} \subset \text{span} \{dt, dz, dv^{(0)}, \ldots, dv^{(\delta-1)}, dx_a, du_a^{(0)}, \ldots, du_a^{(c-1)} \}$.

(C) $\text{span} \{d\bar{u}_b \} \subset \text{span} \{dt, dz, dv^{(0)}, \ldots, dv^{(\delta)}, dx_a, du_a^{(0)}, \ldots, du_a^{(c)} \}$.

By Assumption 1 and from the fact that $\gamma = \text{span} \{dt, dz, (du_a^{(k)} : k \in \mathbb{N}) \}$, it follows also that

(D) The set $\{dt, dz, dv^{(0)}, \ldots, dv^{(\delta)}, dx_a, du_a^{(0)}, \ldots, du_a^{(c)} \}$ is locally linearly independent.

Now let $\bar{x} = \{z, v^{(0)}, \ldots, v^{(\delta-1)}, x_a, u_a^{(0)}, \ldots, u_a^{(c-1)} \}$ and $\bar{v} = (v^{(\delta)}, u_a^{(c)})$. By (A), (B), (C), (D) it is clear that

- $\text{span} \{d\bar{x} \} \subset \text{span} \{dt, d\bar{x}, d\bar{v} \}$
- $\text{span} \{d\bar{x} \} \subset \text{span} \{dt, d\bar{x} \}$ and $\text{span} \{du \} \subset \text{span} \{dt, d\bar{x}, d\bar{u} \}$.
- $\{dt, d\bar{x}, d\bar{v} \}$ is linearly independent.

Hence, by Lemma C.1, it follows that $(\bar{x}, \bar{v})$ is also a local state representation around $\xi$. Since $\{(\bar{x}_a, z), (\bar{u}_a, v), (k, k) \in \mathbb{N} \}$ is a local coordinate system around $\xi$, it is then clear that $((\bar{x}_a, z), (\bar{u}_a, v))$ is also a local state representation. By (A) it follows that the state equations are of the form (26). As the original state representation $((x_a, x_b), (u_a, u_b))$ is (strongly) adapted to the subsystem $Y$, it follows easily from definition 2.1 that $((\bar{x}_a, z), (\bar{u}_a, v))$ is also (strongly) adapted to the subsystem $Y$.

Appendix D: Proof of Proposition 3.2

Proof Write the 1-forms $\omega_l$ and $\dot{\omega}_l$ in local coordinates $x = \{x_i, i \in A\}$ defined around $\xi$. One obtains $\omega_l = \sum_{i \in F} \alpha_l(x) dx_i, \dot{\omega}_l = \sum_{i \in F} \beta_l(x) dx_i, l = 1, \ldots, s$. The subset $F$ can be chosen finite for a convenient open neighborhood $V_\xi$ of $\xi$. Without loss of generality, assume that $F = \{1, \ldots, k\}$. Hence one may identify $\omega_l(\nu)$ and $\dot{\omega}_l(\nu)$ respectively with the row vectors $\omega_l(\nu) =$...
(α₁(ν), ..., αₖ(ν)) and \( \dot{ω}_i(ν) = (β₁(ν), ..., βₖ(ν)) \). Define the \( 2s \times k \) matrix

\[
N_ν = \begin{bmatrix}
    ω₁(ν) \\
    \vdots \\
    ω_s(ν) \\
    \dot{ω}_1(ν) \\
    \vdots \\
    \dot{ω}_s(ν)
\end{bmatrix}
\]

and let \( M_ν = N_ν^T \), the transpose of \( M_ν \). Let \( π_1 : \mathbb{R}^s \times \mathbb{R}^s \to \mathbb{R}^s \) the projection \((z₁, z₂) \mapsto z₁\). To say that \( ω_ν ∈ Γ_1|_ν \) is equivalent to say that \( ω_ν = \sum_{i∈F} α_iω_i(ν) \), where \((α₁, ..., α_s)^T ∈ π_1(\ker M_ν)\).

Hence, \( Γ_1 \) is nonsingular around \( ξ \in \xi \) if and only if \( π_1(\ker M_ν) \) is locally constant dimensional around \( ξ \). As the family \( \{ω₁, ..., ω_s\} \) is pointwise independent, then \( \ker M_ν \cap \ker π₁ = \{0\} \).

Hence, the map \( π_1 \) restricted to \( \ker M_ν \) is an isomorphism into its image, and so \( \dim \ker M_ν = \dim π₁(\ker M_ν) \). Hence \( ξ \) is also a regular point of \( \ker M_ν \). As \( \dim \ker M_ν + \dim \ker \ker M_ν = 2s \), \( ξ \) is also a regular point of \( \im M_ν \).

Note now that \( \dim Γ_1|_ν = \dim \ker \ker M_ν = 2s - s - k = s - k \). In particular, it follows from a dimensional argument, that the set \( \{η₁, ..., η_{s-k}\} \) is a local basis of \( Γ_1 \), where

\[
η_j = ω_{k+j} - \sum_{l=1}^{k} β_l^k \dot{ω}_l
\]

This shows the smoothness of \( Γ_1 \) on some open neighborhood \( W_ξ \) of \( ξ \).

Now, it is clear from smoothness of \( η_j \) that \( η_j ∈ Ω₁ \) (when one restricts the open neighborhood of definition to \( W_ξ \)), and so \( Γ_1 ⊂ Ω₁ \). By (17) and (18) it follows that \( Ω₁ ⊂ Γ_1 \).

\[\square\]

**Appendix E: Existence of strongly adapted state equations**

The following result gives sufficient conditions for the existence of output subsystems and strongly adapted state equations in the invertible case. It generalizes previous results whose proofs were based on the properties of the dynamic extension algorithm (see Pereira da Silva and Corrêa Filho (2001)). Theorem E.2 is a generalization of this result for the non-invertible case. These two results allows also the application of Theorem 4.3 from the algorithmic point of view (see Appendix F).

**Theorem E.1** (Existence of strongly adapted state equations – invertible case) :  Let \( S \) be a system with state representation \((x, u)\) and output \( y \) defined² by (7). Assume that there exists

¹ Complete locally the independent one-forms of \( B \), regarded as column vectors, to a local basis of \( T^*_ξ U \) forming a locally invertible matrix \( T \) (one may need to restrict the open set \( V_ξ \)). The coefficients \( a_i^j \) and \( b_i^j \) may be computed from the first \( s + k \) components of \( T^{-1} \dot{ω}_{i+k} \), locally around \( ξ \).

² From (7), it is clear that the state representation \((x, u)\) is classic (that is, span \( \{dx\} \subset \{dt, du, dx\} \)) and the output is also classic (that is span \( \{dy\} \subset \{dt, du, dx\} \)).
some $\alpha \in \mathbb{N}$ such that, locally around some $\nu \in S$, one has

\begin{enumerate}
\item span \{dx\} \cap \text{span} \left\{ dt, dy^{(0)}, \ldots, dy^{(\alpha-1)} \right\} = \text{span} \{dx\} \cap \text{span} \left\{ dt, dy^{(0)}, \ldots, dy^{(\alpha)} \right\}.
\item span \{dt, dx, dy^{(0)}, \ldots, dy^{(\alpha-1)}\} is locally nonsingular around $\xi$.
\item span \{dt, dx, du, dy^{(0)}, \ldots, dy^{(\alpha)}\} is locally nonsingular around $\xi$.
\item The set \{dt, dy^{(0)}, \ldots, dy^{(\alpha)}\} is pointwise independent in an open neighborhood of $\xi$.
\end{enumerate}

Then there exists a local output subsystem $Y$ defined around $\nu$ that admits a strongly adapted state representation $(\tilde{x}, \tilde{u})$, where $\tilde{x} = (x_a, x_b)$ and $\tilde{u} = (u_a, u_b)$. Moreover:

\begin{enumerate}
\item One may choose $u_a = y^{(\alpha)}$.
\item One may choose $x_a \subset \{y^{(0)}, \ldots, y^{(\alpha-1)}\}$ such that \{dt, dx_a\} is a local basis of \text{span} \{dy^{(0)}, \ldots, dy^{(\alpha-1)}\}.
\item One may choose $x_b$ in a way that $dx_b$ completes \{dt, dx_a\} to a local basis of \text{span} \{dt, dx, dy, \ldots, dy^{(\alpha-1)}\}.
\item One may choose $u_b$ in order to complete \{dt, dx_a, dx_b, du_a\} to a basis \{dt, dx_a, dx_b, du_a, du_b\} of \text{span} \{dt, dx, du, dy, \ldots, dy^{(\alpha)}\}.
\end{enumerate}

In particular, if $\tilde{\Delta}$ is nonempty and these assumptions hold around all $\xi \in \tilde{\Delta}$, then the corresponding implicit system (6) is regular. Furthermore, span \{dx\} + $\mathcal{Y}$ = span \{dx_b\} $\oplus$ $\mathcal{Y}$ and span \{dx, du\} + $\mathcal{Y}$ = span \{dx_b, du_b\} $\oplus$ $\mathcal{Y}$.

\textbf{Proof} The proof of the theorem is an easy consequence of the proof of Theorem 3 of Pereira da Silva (2008).

The next theorem is a generalization of Theorem E.1. One consider that the output $y$ is partitioned into two parts $(\hat{y}, \tilde{y})$ in a way that the given system $S$ with output $\hat{y}$ obeys the assumptions of Theorem E.1, that is, $\hat{y}$ is such that the system with input $u$ and output $\hat{y}$ is right-invertible. The subset $\hat{y}$ of the output $y$ represents the “dependent” part of the output, that is, it does not contribute to the output rank.

\textbf{Theorem E.2} (Existence of strongly adapted state equations – non-invertible case):

Let $S$ be the system with state representation $(x, u)$ and output $y$, defined\footnote{As in Theorem E.1, the state representation $(x, u)$ is classical, and span \{dy\} $\subset$ span \{dt, dx, du\}.} by (7). Let $\nu \in \tilde{\Delta}$, where $\tilde{\Delta} \subset S$ is defined by (10). Assume that there exists a partition $y = (\hat{y}, \tilde{y})$, where $\hat{y}$ is called the independent part, and $\tilde{y}$ is called dependent part of the output. Assume also that there exists some $\alpha \in \mathbb{N}$ such that, locally around $\nu$, one has

\begin{enumerate}
\item span \{dx\} \cap \text{span} \left\{ dt, d\hat{y}^{(0)}, \ldots, d\hat{y}^{(\alpha-1)} \right\} = \text{span} \{dx\} \cap \text{span} \left\{ dt, d\hat{y}^{(0)}, \ldots, d\hat{y}^{(\alpha)} \right\}.
\item span \{dt, dx, d\hat{y}^{(0)}, \ldots, d\hat{y}^{(\alpha-1)}\} is locally nonsingular around $\xi$.
\item span \{dt, dx, du, d\hat{y}^{(0)}, \ldots, d\hat{y}^{(\alpha)}\} is locally nonsingular around $\xi$.
\item The set \{dt, d\hat{y}^{(0)}, \ldots, d\hat{y}^{(\alpha)}\} is pointwise independent in an open neighborhood of $\xi$.
\item span \{d\hat{y}^{(0)}, \ldots, d\hat{y}^{(\alpha-1)}\} $\subset$ span \{dt, dx, d\hat{y}^{(0)}, \ldots, d\hat{y}^{(\alpha-1)}\}.
\item span \{dt, d\hat{y}^{(0)}, \ldots, d\hat{y}^{(k)}\} is nonsingular for $k = \alpha$ and $k = \alpha - 1$.
\item span \{d\hat{y}^{(\alpha)}\} $\subset$ span \{dt, d\hat{y}^{(0)}, d\hat{y}^{(1)}, \ldots, d\hat{y}^{(\alpha-1)}, d\tilde{y}^{(\alpha)}\}.
\item span \{d\hat{y}^{(0)}, d\hat{y}^{(1)}, \ldots, d\hat{y}^{(k)}\} is nonsingular around $\nu$ for $k = \alpha - 1$ and $k = \alpha$.
\end{enumerate}
Then there exists a local output subsystem $Y$ defined around $\nu$ that admits a strongly adapted state representation $(\tilde{x}, \tilde{u})$, where $\tilde{x} = (x_a, x_b)$ and $\tilde{u} = (u_a, u_b)$. Moreover:

(A) One may choose $u_a = \hat{y}(\alpha)$.
(B) One may choose $x_a \subset \{y^{(0)}, \ldots, y^{(\alpha-1)}\}$ such that $\{dt, dx_a\}$ is a local basis of $\text{span}\{dy^{(0)}, \ldots, dy^{(\alpha-1)}\}$.
(C) One may choose $x_b$ in a way that $dx_b$ completes $\{dt, dx_a\}$ to a local basis of $\text{span}\{dt, dx, dy, \ldots, dy^{(\alpha-1)}\}$.
(D) One may choose $u_b$ in order to complete $\{dt, dx_a, dx_b, du_a\}$ to a basis $\{dt, dx_a, dx_b, du_a, du_b\}$ of $\text{span}\{dt, dx, du, dy, \ldots, dy^{(\alpha)}\}$.

In particular, if $\tilde{\Delta}$ is nonempty and these assumptions hold around all $\xi \in \tilde{\Delta}$, then the corresponding implicit system (6) is regular. Furthermore, $\text{span}\{dx\} + \mathcal{Y} = \text{span}\{dx_b\} \oplus \mathcal{Y}$ and $\text{span}\{dx, du\} + \mathcal{Y} = \text{span}\{dx_b, du_b\} \oplus \mathcal{Y}$.

Proof The proof of the theorem is an easy consequence of the proof of Theorem 5 of Pereira da Silva (2008).

Appendix F: Algorithmic Issues

This section is devoted to the algorithmic aspects of the main result, namely, Theorem 4.3. It is important to stress that Theorem 4.9 is an “intrinsec” interpretation of Theorem 4.3, and it is not suitable for computations. One may summarize the theoretical results presented in this paper in the following algorithm, which verifies the solvability conditions of Theorem 4.3.

Algorithm.

Preparation Process I. Let $\xi$ be a point of $S$ defined by (7). Verify for $\alpha \in \{0,1,\ldots,n\}$ if the assumptions of Theorem E.1 hold. Construct $x_a, x_b, u_a, u_b$ as described in the steps (A), (B), (C), (D) in end of the Theorem E.1. If one may choose $u_b = v$, then the Problem is solvable, then stop. (see Remark 4.2). If it is not the case, then continue.

Preparation Process II. If the assumptions of Theorem E.1 do not hold for $\alpha \in \{0,1,\ldots,n\}$ then choose a partition $(\tilde{y}, \hat{y})$ of $y$, and verify if the assumptions of Theorem E.2 hold for some $\alpha \in \{0,1,\ldots,n\}$. If it is the case, construct $x_a, x_b, u_a, u_b$ as described in the steps (A), (B), (C), (D) in end of the Theorem E.2. If one may choose $u_b = v$, so the Problem is solvable with $z_a = x_a$, $v_a = u_a$, and $z = x_b$. Stop. If it is not the case, then continue.

Step 0. Let $\delta = 0$. Let $\omega$ be a one-form given by (11), and let $\theta$ be the map defined by (12).

Step 1. Compute $\gamma$ in the following way. Let

$$\theta(dv) = \sum_{i=1}^{m_a} \beta_idx_{b_i} + \sum_{j=0}^{m_b} \sum_{k=0}^{\nu_j} \epsilon_{jk}du^{(k)}_{b_j}.$$ 

Then $\gamma = \max_{j \in \{0,1,\ldots,m_b\}} \{\kappa_j\}$.

Step 3. Compute $\Gamma_0$ given by (23).

Step 4. Compute the relative derived flag $\Gamma_k$ for $k = 1, \ldots, \delta + 1$, using (16) and the idea of the proof of Proposition 3.3.

Step 5. Verify the assumptions (iii), and (iv) of Theorem 4.3 by direct computation. If these conditions hold, go to step 6. If at least one of these condition fails and $\delta \leq n + \gamma(m + 1)$, then

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1. It shows that the result of Theorem 4.3 is independent of the adapted state representation (9) that is chosen.

2. In this algorithm, one will call the Problem of State Representation for the Implicit System $\Delta$ with input $v$ simply by Problem.

3. As Theorem E.1 is a particular case of Theorem E.2, and the assumptions of Theorem E.2 are generically implied by the properties of the dynamic extension algorithm, if those theorems do not work for $\alpha \leq n$, then $\xi$ is not a generical point (see Pereira da Silva (2008)). In this case it is not known if it useful to try to choose $\alpha > n$.

4. See Remark 4.2.
References


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increment δ and go to step 1. If at least one of these conditions fails and δ = n + γ(m + 1), then the assumptions of Theorem 4.3 fail for all possible values of δ. **Stop.**

**Step 6.** Verify the assumption (i) of Theorem 4.3 by computing the dimensions of the (finite dimensional) codistributions Γ_k. If this condition holds, go to step 7. If this condition fails and δ < n + γ(m + 1), then increment δ and go to step 1. If this condition fails and δ = n + γ(m + 1), then the assumptions of Theorem 4.3 fail for all possible values of δ. **Stop.**

**Step 7.** Verify the assumption (ii) by applying the idea of the proof of Lemma 4.7. If the codistribution Γ_{δ+1} + Y is integrable, it may be possible to compute z by the idea of the proof of Lemma 4.7. If this condition holds, go to step 8. If this condition fails and δ < n + γ(m + 1), then increment δ and go to step 1. If this condition fails and δ = n + γ(m + 1), then the assumptions of Theorem 4.3 fail for all possible values of δ. **Stop.**

**Step 8.** The Problem is solvable by choosing z_a = (x_a, u_a^{(0)}, \ldots, u_a^{(c−1)}) and v_a = u_a^{(c)}, where c ∈ N can be computed as in the proof of Lemma 4.8. **Stop.**

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5See Remark 4.10.
REFERENCES


