An Introduction to Conformal Ricci Flow

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Abstract. We introduce a variation of the classical Ricci flow equation that modifies the unit volume constraint of that equation to a scalar curvature constraint. The resulting equations are named the *conformal Ricci flow equations* because of the role that conformal geometry plays in constraining the scalar curvature and because these equations are the vector field sum of a conformal flow equation and a Ricci flow equation. These new equations are given by

$$\frac{\partial g}{\partial t} + 2\big(\mathrm{Ric}(g) + \tfrac{1}{n}g\big) = -pg$$

$$R(g) = -1$$

for a dynamically evolving metric g and a scalar non-dynamical field p. The conformal Ricci flow equations are analogous to the Navier-Stokes equations of fluid mechanics,

$$\frac{\partial v}{\partial t} + \nabla_v v + \nu \Delta v \;=\; -\operatorname{grad} p$$

$$\operatorname{div} v = 0\,.$$

Because of this analogy, the time-dependent scalar field p is called a *conformal pressure* and, as for the real physical pressure in fluid mechanics that serves to maintain the incompressibility of the fluid, the conformal pressure serves as a Lagrange multiplier to conformally deform the metric flow so as to maintain the scalar curvature constraint.

The equilibrium points of the conformal Ricci flow equations are Einstein metrics with Einstein constant $-\frac{1}{n}$. Thus the term $-2(\mathrm{Ric}(g) + \frac{1}{n}g)$ measures the deviation of the flow from an equilibrium point and acts as a nonlinear restoring force. The conformal pressure $p \geq 0$ is zero at an equilibrium point and positive otherwise. The constraint force $-pg$ acts pointwise orthogonally to the nonlinear restoring force $-2(\mathrm{Ric}(g) + \frac{1}{n}g)$ and conformally deforms g so that the scalar curvature is preserved.

A variety of properties of the conformal Ricci flow are discussed, including the reduced conformal Ricci flow, local existence and uniqueness, a variational formulation using a quasi-gradient of the Yamabe functional, strictly monotonically decreasing global and local volume results, and applications to 3-manifold geometry. The geometry of the conformal Ricci flow is discussed as well as the remarkable analytic fact that the constraint force does not lose derivatives and thus analytically the conformal Ricci equation is a bounded perturbation of the classical unnormalized Ricci equation. Lastly, we discuss potential applications to Perelman's proposed implementation of Hamilton's program to prove Thurston's 3-manifold geometrization conjectures.

Dedicated to Professor Vincent Moncrief
on the occasion of his 60th birthday

To Vince: Scholar, educator, colleague, friend

Contents

1. Introduction and background

1.1. The conformal Ricci flow equations

We introduce a variation of the **classical Ricci flow equation** of Hamilton [24] that modifies the volume constraint $\mathrm{vol}(M, g_t) = 1$ of the evolving metric g_t to a scalar curvature constraint $R(g_t) = -1$. The resulting modified Ricci flow equations are named the **conformal Ricci flow equations** because of the role that conformal geometry plays in maintaining the scalar curvature. Moreover, as we shall see in Sections 4 and 5, the conformal Ricci flow equations are literally the vector sum of a smooth conformal evolution equation and a densely defined Ricci evolution equation.

Since the volume of a Riemannian manifold (M, g) is a positive real number and since the scalar curvature is a real-valued function on M, the constraint on the scalar curvature is considerably more drastic than the volume constraint of the classical Ricci flow equation. Thus the configuration space of the conformal Ricci flow equations is considerably smaller than the configuration space of the classical Ricci flow equation. We discuss why working on a smaller configuration space may have advantages over working on a larger configuration space (see Section 1.6).

We take a unified point of view and show the similarities and differences between the classical and conformal Ricci flow equations. From this point of view, both systems have the structure of a constrained dynamical system with a Lagrange multiplier that enters as a time-dependent parameter field and whose purpose is to preserve the constraints. We also discuss various analogies between the conformal Ricci flow equations and the Euler and Navier-Stokes equations for incompressible fluid flow. In particular, a time-dependent parameter (i.e., non-dynamical) scalar field $p \geq 0$ naturally arises which we call the **conformal pressure** and which has the property that it is zero at an equilibrium point and positive otherwise.

A variety of properties of the conformal Ricci flow equations are discussed, including the reduced conformal Ricci flow equation, local existence and uniqueness, a variational formulation using a quasi-gradient of the Yamabe functional, strictly monotonically decreasing global and local volume results, and possible applications to 3-manifold geometry. The geometry of the conformal Ricci flow is discussed, leading to the remarkable analytic fact that the constraint force does not lose derivatives and thus analytically the conformal Ricci equation is a bounded perturbation of the classical unnormalized Ricci equation. Lastly, we discuss potential applications to Perelman's proposed implementation of Hamilton's program to prove Thurston's 3-manifold geometrization conjectures.

Throughout this paper, M will denote a smooth (C^∞) closed (compact without boundary) connected oriented n-manifold, $n \geq 3$. Let $\mathcal{D} = \mathcal{D}(M) = \mathrm{Diff}(M)$ denote the infinite-dimensional group of smooth diffeomorphisms of M, $S_2 = S_2(M) = C^\infty(T^*M \otimes_{\mathrm{sym}} T^*M)$ the space of smooth symmetric 2-covariant tensor fields on M, $\mathcal{M} = \mathrm{Riem}(M) \subset S_2$ the space of smooth Riemannian (positive-definite) metrics on M, and $\mathcal{F} = C^\infty(M, \boldsymbol{R})$ the space of smooth real-valued functions on M, where $\boldsymbol{R}$ denotes the real numbers. The group $\mathcal{D}$ is an infinite-dimensional ILH (inverse limit Hilbert) Lie group, S_2 and $\mathcal{F}$ are infinite-dimensional ILH linear spaces, and $\mathcal{M}$ is an open ILH submanifold of S_2 (i.e., $\mathcal{M}$ is open in S_2) with tangent space at $g \in \mathcal{M}$ given by

$$T_g\mathcal{M} = \{g\} \times S_2 \approx S_2 \,, \tag{1.1}$$

which we identify with S_2 (for more information about ILH spaces, see Omori [31], Ebin [9], and Ebin-Marsden [10]).

For each $g \in \mathcal{M}$ there exists a unique **volume element** $d\mu_g$ on M determined by g and the orientation of M. In local coordinates $(x^1, \ldots, x^n)$, $d\mu_g = \sqrt{\det g_{ij}} \, dx^1 \wedge \ldots \wedge dx^n$. Let $\mathrm{vol}(M, g) = \int_M d\mu_g$ denote the **volume** of the Riemannian manifold (M, g), let $\boldsymbol{R}^+ = (0, \infty)$, let

$$\mathrm{vol} : \mathcal{M} \longrightarrow \boldsymbol{R}^+ \,, \quad g \longmapsto \int_M d\mu_g = \mathrm{vol}(M, g)$$

denote the **volume functional** on $\mathcal{M}$, and let

$$\mathcal{M}^1 = \{\, g \in \mathcal{M} \mid \mathrm{vol}(M, g) = 1 \,\} = \mathrm{vol}^{-1}(1)$$

denote the **space of Riemannian metrics with unit volume**.

For a Riemannian metric g in $\mathcal{M}$, let $K(g)$ denote the **sectional curvature** and let $\mathrm{Riem}(g)$ denote the **Riemann-Christoffel curvature tensor** defined on vector fields X, Y and Z on M by

$$\mathrm{Riem}(g)(X, Y, Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \,. \tag{1.2}$$

Let $\mathrm{Ric}(g) \in S_2$ denote the **Ricci curvature tensor**, $R(g) = \mathrm{tr}_g(\mathrm{Ric}(g)) = g^{-1}(\mathrm{Ric}(g)) \in \mathcal{F}$ the **scalar curvature**, where tr_g is the metric trace and g^{-1} is the inverse of g, $\mathrm{Weyl}(g)$ the **Weyl conformal curvature tensor**, $\mathrm{Ric}^T(g) = \mathrm{Ric}(g) - \frac{1}{n} R(g) g \in S_2$ the **traceless part of the Ricci tensor**, and $\mathrm{Ein}(g) = \mathrm{Ric}(g) - \frac{1}{2} R(g) g \in S_2$ the **Einstein tensor** of g. In local coordinates, $(\mathrm{Riem}(g))^i{}_{jkl} = R^i{}_{jkl}$, $(\mathrm{Ric}(g))_{ij} = R_{ij} = R^a{}_{iaj}$, $R(g) = g^{ij} R_{ij}$, $(\mathrm{Weyl}(g))^i{}_{jkl} = W^i{}_{jkl}$, $(\mathrm{Ric}^T(g))_{ij} = R_{ij} - \frac{1}{n} R g_{ij}$, and $(\mathrm{Ein}(g))_{ij} = R_{ij} - \frac{1}{2} R g_{ij}$. (The T designating traceless in $(\cdot)^T$ is unrelated to the T that occurs in the semi-open intervals $[0, T)$ introduced later.)

We define a **hyperbolic metric** on M as a Riemannian metric $g \in \mathcal{M}$ with constant negative sectional curvature, not necessarily -1. This definition is slightly more general than the usual definition in which $K(g) = -1$. By allowing $K(g)$ to float, we can normalize it by other conditions, such as requiring that the scalar curvature $R(g) = -1$ in which case $K(g) = -\frac{1}{n(n-1)}$, in contrast to having $K(g) = -1$ and $R(g) = -n(n-1)$. We remark that with this definition of hyperbolic metric, Mostow rigidity only determines a hyperbolic metric up to isometry and homothety. However, by normalizing the *scalar* curvature of hyperbolic metrics to -1, rigidity of hyperbolic metrics up to isometry is recovered.

For $g \in \mathcal{M}$ and ρ a tensor field on M, let $|\rho|_g = (\rho \cdot \rho)^{1/2} \in \mathcal{F}$ denote the pointwise g-metric norm of ρ where center dot " $\cdot$ " denotes the g-metric contraction to a scalar. For example, $|\mathrm{Ric}(g)|_g^2 = \mathrm{Ric}(g) \cdot \mathrm{Ric}(g) = g^{ik} g^{jl} R_{ij} R_{kl}$ is the pointwise squared norm of the Ricci tensor and $|\mathrm{Riem}(g)|_g^2 = g_{ai} g^{bj} g^{ck} g^{dl} R^a{}_{bcd} R^i{}_{jkl}$ is the pointwise squared norm of the Riemann-Christoffel curvature tensor of g.

On $\mathcal{M}$ there is a natural L_2 weak infinite-dimensional Riemannian metric $\mathcal{G}$, first constructed by Palais [32] on November 10, 1961 and first published in Ebin [9], given by

$$\mathcal{G}_g : T_g \mathcal{M} \times T_g \mathcal{M} \approx S_2 \times S_2 \longrightarrow \boldsymbol{R} \,, \quad (h, h) \longmapsto \int_M h \cdot h \, d\mu_g \,. \tag{1.3}$$

Let $(\mathcal{M}, \mathcal{G})$ denote this weak infinite-dimensional Riemannian manifold.

Let

$$\mathcal{M}_{-1} = \{\, g \in \mathcal{M} \mid R(g) = -1 \,\} \tag{1.4}$$

denote the **subspace of Riemannian metrics with constant scalar curvature** -1. Then $\mathcal{M}_{-1} \subset \mathcal{M}$ with the induced weak Riemannian metric $\mathcal{G}_{-1}$ is a weak Riemannian submanifold of $(\mathcal{M}, \mathcal{G})$ which we denote $(\mathcal{M}_{-1}, \mathcal{G}_{-1}) \subset (\mathcal{M}, \mathcal{G})$. The differential topology (but not the geometry) of this space has been studied in detail in Fischer-Marsden [13] and we summarize the needed information in Section 2.

The conformal Ricci flow equations (with smooth initial conditions) are defined as follows.

Definition 1.1 *Let M be a smooth closed connected oriented n-manifold, $n \geq 3$. The* **conformal Ricci flow equations** *on M are defined by the equations*

$$\frac{\partial g}{\partial t} + 2\left(\mathrm{Ric}(g) + \tfrac{1}{n}g\right) = -pg \tag{1.5}$$

$$R(g) = -1 \tag{1.6}$$

for curves

$$g : [0, T) \longrightarrow \mathcal{M}, \quad t \longmapsto g(t), \quad with \quad g(0) = g_0 \in \mathcal{M}_{-1}, \ and$$

$$p : [0, T) \longrightarrow \mathcal{F}, \quad t \longmapsto p(t),$$

continuous on the semi-open interval $[0, T)$, $0 < T \leq \infty$, and smooth on the open interval $(0, T)$. ∎

Remarks:

(i) Because of the constraint equation $R(g) = -1$ and the initial condition $g_0 \in \mathcal{M}_{-1}$, the curve $g : [0, T) \longrightarrow \mathcal{M}$ actually lies in $\mathcal{M}_{-1}$. However, if g is viewed as such, then to assert that g restricted to $(0, T)$ is a smooth curve requires the result given below (Theorem 2.3) that $\mathcal{M}_{-1}$ is a smooth submanifold of $\mathcal{M}$. Thus for now, for differentiability purposes, we take g to be smooth as a curve in $\mathcal{M}$ and constrained to the subspace $\mathcal{M}_{-1}$ of $\mathcal{M}$.

(ii) As we shall see in Proposition 3.3 (see also the Remark following that Proposition), for non-static flows g, the curve $p : [0, T) \longrightarrow \mathcal{P}$ actually lies in the open subspace of positive real-valued functions $\mathcal{P} = C^{\infty}(M, \mathbf{R}^{+}) \subset C^{\infty}(M, \mathbf{R}) = \mathcal{F}$.

We refer to (1.5) as the **evolution equation** (of the conformal Ricci flow equations), (1.6) as the **constraint equation** (of the conformal Ricci flow equations), the curve $g : [0, T) \to \mathcal{M}$ as the **conformal Ricci flow**, and the curve $p : [0, T) \to \mathcal{F}$ as the **conformal pressure**. Because of the **quasi-parabolic** nature of the equations (see Section 4), we do not expect any backward time of existence and so we refer to a solution with a positive semi-infinite time of existence $[0, \infty)$ as an **all-time solution**.

1.2. A brief survey of the geometry of the conformal Ricci flow equations

As a consequence of the constraint equation, the conformal Ricci flow takes place in $\mathcal{M}_{-1}$ which is a non-empty infinite-dimensional closed submanifold of $\mathcal{M}$ (see Theorem 2.3). As a consequence of this constraint, it is of importance to note that the conformal pressure p is *not* on the same footing as the dynamical metric field g and thus *no* initial value of p is given. Rather, p serves as a time-dependent Lagrange multiplier and the term $-pg$ acts as the constraint force necessary to preserve the scalar curvature constraint (see also the discussion in Section 5). Consequently, p must solve a time-dependent elliptic partial differential equation as the metric evolves.

This situation is completely analogous to the incompressible Euler or Navier-Stokes equations where the real physical pressure serves as a Lagrange multiplier in order to maintain the divergence-free constraint of the dynamical vector field v (see Ebin-Marsden [10]). Because of this analogy, we think of the non-dynamical scalar field p as a **conformal pressure** because it pointwise conformally deforms g so as to maintain the scalar curvature constraint.

The conformal Ricci flow equations share characteristics with both the quasilinear elliptic-hyperbolic incompressible Euler equations and the semilinear elliptic-parabolic incompressible Navier-Stokes equations, sharing the elliptic characteristics of both equations, the quasilinear characteristics of the Euler equations, and the parabolic characteristics of the Navier-Stokes equation (see also Section 4).

In fact, one of the design criteria for the conformal Ricci flow equations was that the structural similarities to the Euler and Navier-Stokes equations for an incompressible fluid were apparent. For example, Euler's equations for a unit density incompressible ideal fluid with time-dependent velocity vector field v on a fixed Riemannian manifold (M, g) are

$$\frac{\partial v}{\partial t} + \nabla_v v = -\operatorname{grad} p \,, \qquad \operatorname{div} v = 0 \,, \qquad v(0) = v_0 \,, \qquad (1.7)$$

where p is the time-dependent pressure of the fluid. These equations can also be written as

$$\frac{\partial v}{\partial t} + P(\nabla_v v) = 0 \,, \qquad \operatorname{div} v_0 = 0 \,, \qquad v(0) = v_0 \,, \qquad (1.8)$$

where P is the projection of a vector field onto its divergence-free part. Similarly, for a viscous fluid, the incompressible Navier-Stokes equations are

$$\frac{\partial v}{\partial t} + \nabla_v v + \nu \Delta v = -\operatorname{grad} p$$
$$\operatorname{div} v = 0 \,,$$

where the Laplacian on (1-forms metrically associated with) vector fields is $\Delta = \delta d + d\delta$ and the kinematic viscosity is $\nu > 0$ (see Ebin-Marsden [10], p. 161, and Taylor [40], p. 493, for the "correct" Navier-Stokes equations on a Riemannian manifold; here the simpler form is sufficient for our purposes). These equations can be thought of as a vector field sum of the heat equation

$$\frac{\partial v}{\partial t} = -\nu \Delta v$$

and the Euler equation. A nonlinear Trotter product formula is then applied to prove existence and uniqueness of solutions (Ebin-Marsden [10]). In Section 5, we use a similar approach to prove existence and uniqueness to the conformal Ricci flow equations.

Equation (1.7) should be compared with (1.5–1.6) and (1.8) should be compared with the **projection form** (5.30) of the conformal Ricci flow equations

$$\frac{\partial g}{\partial t} + 2\widetilde{P}_g(\operatorname{Ric}(g)) = 0 \,, \qquad (1.9)$$

where $\widetilde{P}_g$ is a non-orthogonal projection from $T_g\mathcal{M}$ to $T_g\mathcal{M}_{-1}$ described in Section 5.

As we shall see in Proposition 3.1, the equilibrium points of the conformal Ricci flow equations occur at Einstein metrics with Einstein constant $-\frac{1}{n}$, i.e., when $\operatorname{Ric}(g) + \frac{1}{n}g = 0$. Thus the term $-2(\operatorname{Ric}(g) + \frac{1}{n}g)$ can be viewed as a measure of the

deviation from an equilibrium point or, in a rough mechanical analogy, can be thought of as a nonlinear restoring force (of course, since these equations are first order in time, this analogy is only valid for Aristotelian, rather than Newtonian, mechanics). On the other hand, the conformal pressure term $-pg$ is the constraint force necessary to counter-balance the conformal component pg of the nonlinear restoring force $-2(\mathrm{Ric}(g) + \frac{1}{n}g)$. Thus the constraint force acts by conformally deforming g so that the scalar curvature is preserved. The constraint force $-pg$ acts pointwise orthogonally to $-2(\mathrm{Ric}(g) + \frac{1}{n}g)$ and keeps the flow in the submanifold $\mathcal{M}_{-1}$ (see Figure 1 in Section 5 and the ensuing discussion). Moreover, as we shall see in Proposition 3.3, the conformal pressure p is, as expected (or, as designed), zero at an equilibrium point and strictly positive otherwise, and is calculated in (3.7).

Remarks:

(i) When we refer to the conformal Ricci flow equations, we shall always assume all of the above conditions regarding M, n, g, and p. For the reasons for the restriction dim $M = n \geq 3$, see Remark (i) following Proposition 3.1.

(ii) For the most part g and p will denote time-dependent objects and we sometimes write this explicitly as $g = g(t) = g_t$ or $p = p(t) = p_t$. A subscript t will never mean the partial t-derivative, which we will always write explicitly.

(iii) The 2 that appears in (1.5) as well as in (1.15) below comes about because in local coordinates the second order quasilinear terms of the Ricci tensor $R_{ij} = (\mathrm{Ric}(g))_{ij}$ are given by

$$R_{ij} = -\tfrac{1}{2}g^{kl}\frac{\partial^2 g_{ij}}{\partial x^k \partial x^l} + \tfrac{1}{2}g^{kl}\frac{\partial^2 g_{ki}}{\partial x^l \partial x^j} + \tfrac{1}{2}g^{kl}\frac{\partial^2 g_{lj}}{\partial x^k \partial x^i} - \tfrac{1}{2}g^{kl}\frac{\partial^2 g_{kl}}{\partial x^i \partial x^j} + Q_{ij}\,, \quad (1.10)$$

where $Q_{ij} = Q_{ij}(g_{kl}, \frac{\partial g_{mn}}{\partial x^p})$ represents the lower order terms of R_{ij}, which are quadratic in the first order derivatives of g and rational in g. Locally, the evolution equation (1.5) can be written as

$$\frac{\partial g_{ij}}{\partial t} - g^{kl}\frac{\partial^2 g_{ij}}{\partial x^k \partial x^l} + g^{kl}\frac{\partial^2 g_{ki}}{\partial x^l \partial x^j} + g^{kl}\frac{\partial^2 g_{lj}}{\partial x^k \partial x^i} - g^{kl}\frac{\partial^2 g_{kl}}{\partial x^i \partial x^j} + Q_{ij} + \tfrac{2}{n}g_{ij} = -pg_{ij}\,.$$

Thus if we ignore for the moment the three second order **index mixing terms**, then

$$\frac{\partial g_{ij}}{\partial t} - g^{kl}\frac{\partial^2 g_{ij}}{\partial x^k \partial x^l} + Q_{ij} + \tfrac{2}{n}g_{ij} \approx -pg_{ij}\,, \quad (1.11)$$

so that the left hand side is a quasilinear strictly parabolic (or non-mixing) heat operator for g_{ij}. In fact, in a system of time-dependent coordinates which we call **moving harmonic coordinates**, (1.11) is locally the evolution equation of the conformal Ricci flow equations. Moreover, these time-dependent coordinates can be used as a basis for proving existence and uniqueness of solutions for the classical Ricci flow equation (see Fischer [11]).

(iv) Since the conformal Ricci flow satisfies the constraint $R(g) = -1$, $\mathrm{Ric}^T(g) = \mathrm{Ric}(g) - \frac{1}{n}R(g)g = \mathrm{Ric}(g) + \frac{1}{n}g$ and so (1.5) can also be written in the useful form

$$\frac{\partial g}{\partial t} + 2\mathrm{Ric}^T(g) = -pg\,. \quad (1.12)$$

1.3. Some comparisons with the classical Ricci flow equation

For $g \in \mathcal{M}$, let $\bar{R}(g) = \int_M R(g) d\mu_g / \mathrm{vol}(M, g)$ denote the volume-averaged total scalar curvature of g. To put the conformal Ricci flow equations into perspective, we consider the **classical (normalized) Ricci flow equation** of Hamilton [24],

$$\frac{\partial g}{\partial t} = -2\mathrm{Ric}(g) + \tfrac{2}{n}\bar{R}(g)g\,, \tag{1.13}$$

from a slightly different point of view than that in which it is usually considered. Equation (1.13) is called normalized because the volume is conserved; Hamilton's **unnormalized Ricci flow equation**, where the volume is not preserved, is

$$\frac{\partial g}{\partial t} = -2\mathrm{Ric}(g)\,. \tag{1.14}$$

We augment the classical Ricci flow equation (1.13) and write it as a constrained dynamical system

$$\frac{\partial g}{\partial t} = -2\,\mathrm{Ric}(g) + cg \tag{1.15}$$

$$\mathrm{vol}(M, g) = 1 \tag{1.16}$$

for curves

$$g : [0, T) \longrightarrow \mathcal{M}\,, \quad t \longmapsto g(t) \quad \text{with} \quad g(0) = g_0 \in \mathcal{M}^1\,, \text{ and}$$

$$c : [0, T) \longrightarrow \boldsymbol{R}\,, \quad t \longmapsto c(t)\,,$$

continuous on the semi-closed interval $[0, T)$ and smooth on $(0, T)$, $0 < T \le \infty$.

The curve $c : [0, T) \to \boldsymbol{R}$ acts as a time-dependent Lagrange multiplier to preserve the constraint equation $\mathrm{vol}(M, g) = 1$. As with any Lagrange multiplier, it can formally and in this case actually be eliminated. Taking the time derivative of the constraint equation (1.16) and using the evolution equation (1.15) yields

$$0 = \frac{d}{dt}\,\mathrm{vol}(M, g) = \frac{d}{dt}\int_M d\mu_g = \int_M \frac{\partial}{\partial t}d\mu_g = \int_M \tfrac{1}{2}\mathrm{tr}_g\left(\frac{\partial g}{\partial t}\right) d\mu_g$$

$$= \int_M \tfrac{1}{2}\mathrm{tr}_g\left(-2\,\mathrm{Ric}(g) + cg\right) d\mu_g = -\int_M R(g)d\mu_g + \tfrac{n}{2}c\int_M d\mu_g$$

$$= -R_{\mathrm{total}}(g) + \tfrac{n}{2}c\,\mathrm{vol}(M, g)\,, \tag{1.17}$$

where $R_{\mathrm{total}}(g) = \int_M R(g)d\mu_g$ is the **total** (or integrated) scalar curvature. Thus

$$c = \tfrac{2}{n}\frac{R_{\mathrm{total}}(g)}{\mathrm{vol}(M, g)} = \tfrac{2}{n}\bar{R}(g) \tag{1.18}$$

where $\bar{R}(g) = \frac{R_{\mathrm{total}}(g)}{\mathrm{vol}(M,g)}$ is the **volume-averaged total scalar curvature of** g. Since c involves integrals over M, it is a global, or nonlocal, function of g and thus depends on g at every point of M (see also Remark (iii) after Proposition 3.2).

Using (1.18), both c and the constraint equation can be eliminated from the system (1.15–1.16) which can then be written in its usual **reduced form**,

$$\frac{\partial g}{\partial t} = -2\,\mathrm{Ric}(g) + \tfrac{2}{n}\bar{R}(g)g\,, \qquad g_0 \in \mathcal{M}^1\,, \tag{1.19}$$

where now the constraint $\mathrm{vol}(M, g_0) = 1$ applies only to the initial condition g_0 rather than the entire flow g_t, $t \in [0, T)$.

Note that the reduced form (1.19) is achieved by using only the time derivative of the constraint equation, (1.17), and not the constraint equation $\mathrm{vol}(M, g) = 1$ itself (see also Section 1.4 below).

The constraint on the entire flow g_t is recovered from (1.19) since

$$\frac{d}{dt} \mathrm{vol}(M, g) = \tfrac{1}{2} \int_M \mathrm{tr}_g \left(\frac{\partial g}{\partial t} \right) d\mu_g = \tfrac{1}{2} \int_M \mathrm{tr}_g \left(-2 \mathrm{Ric}(g) + \tfrac{2}{n} \bar{R}(g) g \right) d\mu_g$$

$$= -\int_M R(g) d\mu_g + \tfrac{n}{2} \tfrac{2}{n} \bar{R}(g) \int_M d\mu_g = -R_{\mathrm{total}}(g) + R_{\mathrm{total}}(g) = 0 \,, \quad (1.20)$$

so that from the initial constraint $g_0 \in \mathcal{M}^1$, $\mathrm{vol}(M, g_t) = 1$. Thus a solution of the reduced equation (1.19) stays in $\mathcal{M}^1$ if it starts in $\mathcal{M}^1$ and thus the constrained system (1.15–1.16) is equivalent to the reduced equation (1.19) with initial conditions in $\mathcal{M}^1$. This same theme with the appropriate spaces will appear for the conformal Ricci flow equations (see Section 3.3).

1.4. A further simplification of the classical Ricci flow equation

As we have seen in Section 1.3, to get to the usual formulation of the classical Ricci flow, the Lagrange multiplier c is eliminated by using the derivative of the constraint equation, $\frac{d}{dt} \mathrm{vol}(M, g) = 0$. However, the undifferentiated constraint equation itself $\mathrm{vol}(M, g) = 1$ is not explicitly used. If we do apply the constraint, then

$$c = \tfrac{2}{n} \bar{R}(g) = \tfrac{2}{n} \frac{R_{\mathrm{total}}(g)}{\mathrm{vol}(M, g)} = \tfrac{2}{n} R_{\mathrm{total}}(g) \quad (1.21)$$

and the classical Ricci equation (1.19) further reduces to the **fully reduced equation**

$$\frac{\partial g}{\partial t} = -2 \mathrm{Ric}(g) + \tfrac{2}{n} R_{\mathrm{total}}(g) g \,, \qquad g_0 \in \mathcal{M}^1 \,. \quad (1.22)$$

Since the constraint equation itself and not just its time derivative have been used in deriving (1.22), solutions to (1.22) no longer satisfy $\frac{d}{dt} \mathrm{vol}(M, g) = 0$. However, the constraint on the entire flow is still recoverable as follows. Let $\tilde{v}(g) = \mathrm{vol}(M, g) - 1$. Then if g satisfies (1.22),

$$\frac{d\tilde{v}(g)}{dt} = \frac{d}{dt} \mathrm{vol}(M, g) = \tfrac{1}{2} \int_M \mathrm{tr}_g \left(\frac{\partial g}{\partial t} \right) d\mu_g = \tfrac{1}{2} \int_M \mathrm{tr}_g \left(-2 \mathrm{Ric}(g) + \tfrac{2}{n} R_{\mathrm{total}}(g) g \right) d\mu_g$$

$$= -\int_M R(g) d\mu_g + \tfrac{n}{2} \tfrac{2}{n} R_{\mathrm{total}}(g) \int_M d\mu_g = -R_{\mathrm{total}}(g) + R_{\mathrm{total}}(g) \mathrm{vol}(M, g)$$

$$= R_{\mathrm{total}}(g)(\mathrm{vol}(M, g) - 1) = R_{\mathrm{total}}(g) \tilde{v}(g) \,.$$

Thus if we let $\tilde{v}(t) = \tilde{v}(g(t))$ and $R_{\mathrm{total}}(t) = R_{\mathrm{total}}(g(t))$, we see that $\tilde{v}(t)$ solves the linear first-order non-autonomous ordinary differential equation

$$\frac{d\tilde{v}(t)}{dt} = R_{\mathrm{total}}(t) \tilde{v}(t) \,. \quad (1.23)$$

Since the initial metric g_0 has $\mathrm{vol}(M, g_0) = 1$, $\tilde{v}(0) = \tilde{v}_0 = 0$, so by uniqueness of solutions of (1.23) (or by the explicit solution $\tilde{v}(t) = \tilde{v}_0 e^{\int_0^t R_{\mathrm{total}}(t') dt'}$), $\tilde{v}(t) = \mathrm{vol}(M, g(t)) - 1 \equiv 0$. Thus the initial constraint is maintained by the fully reduced equation (1.22) and thus (1.15–1.16) with $g(0) = g_0 \in \mathcal{M}^1$, (1.19), and (1.22) are all equivalent.

As we shall see, the fully reduced equation (1.22) is the one most similar to the fully reduced conformal Ricci equation where a linear (partial differential) equation

similar in structure to (1.23) appears (see (3.21) and Remark (i) following) to insure that the constraint is maintained on the entire flow if it is satisfied by the initial condition.

1.5. The conformal Ricci flow on manifolds of Yamabe type -1

In order to explain why we have chosen the name conformal Ricci flow, we recall the following terminology introduced by Fischer-Moncrief [14].

Definition 1.2 (The Yamabe type of a manifold) *Let M be a closed connected n-manifold, $n \geq 3$. Then*

(1) *M is of* **Yamabe-type** -1 *if M admits no metric with $R(g) = 0$;*

(2) *M is of* **Yamabe-type** 0 *if M admits a metric with $R(g) = 0$ but no metric with $R(g) = 1$;*

(3) *M is of* **Yamabe-type** 1 *if M admits a metric with $R(g) = 1$.*

The definition of Yamabe type partitions the class of closed n-manifolds, $n \geq 3$, into three classes that are mutually exclusive and exhaustive (see [14] for details). We note in particular that the three categories of closed manifolds that admit metrics of constant positive, zero, and constant negative sectional curvature fall into the categories of Yamabe type 1, 0, and -1, respectively.

The pointwise multiplicative Abelian group $\mathcal{P} = C^\infty(M, \boldsymbol{R}^+)$ of smooth positive real valued functions on M acts freely, smoothly, and properly on $\mathcal{M}$. The resulting orbit space $\mathcal{M}/\mathcal{P}$ is the space of **(pointwise) conformal classes** on M. In particular, we note that $\mathcal{M}/\mathcal{P}$ is a contractible manifold (see Fischer-Moncrief [15] for more information).

The key fact is that if M is of Yamabe type -1, then every Riemannian metric on M is uniquely pointwise conformally deformable to a metric with scalar curvature $= -1$. Thus $\mathcal{M}/\mathcal{P}$ and $\mathcal{M}_{-1}$ are in bijective correspondence and in fact are ILH diffeomorphic. Thus $\mathcal{M}_{-1}$ is a representation of the space of pointwise conformal structures $\mathcal{M}/\mathcal{P}$ and results regarding the conformal Ricci flow on $\mathcal{M}_{-1}$ can be interpreted in terms of conformal geometry.

When M is not of Yamabe type -1, then $\mathcal{M}_{-1}$ represents only one component of $\mathcal{M}/\mathcal{P}$ and thus the conformal interpretation needs modification. We emphasize, however, that as we have done in this paper, one can work directly on $\mathcal{M}_{-1}$ as a space in its own right without restricting to manifolds of Yamabe type -1. If, however, we wish to interpret our results on the space of (pointwise) conformal geometries $\mathcal{M}/\mathcal{P}$, then we would need to either restrict to manifolds of Yamabe type -1 or augment our results to include other subspaces of $\mathcal{M}$.

Since the conformal Ricci equation was designed with conformal geometry in mind, applications to conformal geometry will, as expected, be of more interest on manifolds of Yamabe type -1 (see for example Proposition 3.5 and also Sections 9 and 10).

1.6. Summary

Comparing the classical and conformal Ricci flow equations, we have the following:

(i) The constraint equation changes from $\mathrm{vol}(M, g) = 1$ for the classical Ricci flow to $R(g) = -1$ for the conformal Ricci flow with the concomitant change of the configuration space from $\mathcal{M}^1$ to $\mathcal{M}_{-1}$. Since $\mathcal{M}^1$ is a codimension-1 submanifold

of $\mathcal{M}$ whereas $\mathcal{M}_{-1}$ is a codimension-$C^\infty(M, \boldsymbol{R})$ submanifold of $\mathcal{M}$, $\mathcal{M}_{-1}$ is a much smaller configuration space than $\mathcal{M}^1$.

(ii) To compensate for this smaller configuration space, the Lagrange multiplier in the evolution equation adjusts from being a time-dependent constant $c = c(t)$ (independent of x) in the classical Ricci system to a time-dependent real-valued function $p = p(t, x)$ in the conformal Ricci system.

(iii) In the augmented classical Ricci flow equations (1.15–1.16), solving for the Lagrange multiplier and eliminating the constraint equation involves solving a time-dependent linear inhomogeneous algebraic equation (1.17) whereas in the conformal Ricci flow equations solving for the Lagrange multiplier and eliminating the constraint equation involves solving a time-dependent linear inhomogeneous elliptic partial differential equation (3.6). Thus, although both the classical and conformal Ricci equations are quasilinear quasi-parabolic systems (see Section 4), the conformal Ricci equations have an additional linear elliptic equation and thus overall are a linear-elliptic quasilinear quasi-parabolic system.

(iv) For the classical and conformal Ricci flow equations, the volume and scalar curvature behave somewhat oppositely. For the classical Ricci flow equation, the volume is preserved, $\mathrm{vol}(M, g) = 1$, but for non-static flows the scalar curvature is not preserved, whereas for the conformal Ricci equation, the scalar curvature is preserved, $R(g) = -1$, but for non-static flows the volume is not preserved. For more specific information, see Proposition 3.5, Table 1 following that Proposition, and Proposition 7.1.

(v) As we shall see in Example 2.2, $\mathrm{Ric}(g)$ is orthogonal to $\mathcal{M}_{-1}$. Thus the classical unnormalized Ricci flow (1.14), at a metric $g \in \mathcal{M}_{-1}$, is orthogonal to $\mathcal{M}_{-1}$, whereas the conformal Ricci flow is tangential to $\mathcal{M}_{-1}$ (see Figure 1 in Section 5). Thus in some sense the classical unnormalized Ricci flow acts like a Riemannian gradient as it is orthogonal to the level hypersurface $\mathcal{M}_{-1}$, whereas the conformal Ricci flow acts like a symplectic gradient as it is tangent to the level hypersurface $\mathcal{M}_{-1}$.

We now discuss why from the point of view of geometry having a smaller configuration space is potentially better. The classical Ricci flow was designed to search out Einstein metrics. But if one is looking for something, it makes sense to look in as small a space as possible if it is known that the object of interest is in that smaller space. Since an Einstein metric with negative Einstein constant can always be scaled so as to lie in $\mathcal{M}_{-1}$, we know that what we are looking for must lie in $\mathcal{M}_{-1}$. Thus it makes sense at the outset to restrict the problem to the smaller space $\mathcal{M}_{-1}$ and in fact the conformal Ricci flow equations were designed so as to restrict to this smaller configuration space $\mathcal{M}_{-1}$ from the beginning.

On the other hand, since *any* metric (and not just an Einstein metric) can be scaled to lie in $\mathcal{M}^1$, restricting the search for Einstein metrics to $\mathcal{M}^1$ does not significantly reduce the search space.

As a second motivation for restricting to the smaller space $\mathcal{M}_{-1}$, we expect that the conformal Ricci flow equations will be of use in searching for curves of metrics that asymptotically achieve the σ-constant of M (see Section 9). Computing $\sigma(M)$ involves a minimax procedure which attempts to obtain $\sigma(M)$ in two steps, first by minimizing the Yamabe functional in a fixed conformal class and then by maximizing over all conformal classes (Anderson [1]). The first step of this procedure, corresponding to the *min* part, is the Yamabe problem and has been solved. The unsolved second step

then involves a maximization over the conformal classes of M. In the case that M is of Yamabe type -1, the conformal classes are represented by $\mathcal{M}_{-1}$ (see Section 1.5). Thus the conformal Ricci flow, which takes place on the space $\mathcal{M}_{-1}$, exploits the fact that the first part of the problem of realizing the σ-constant has been solved and then moves on to the appropriate smaller space $\mathcal{M}_{-1}$ in which to tackle the second part of the problem of finding curves of Riemannian metrics that asymptotically realize $\sigma(M)$.

We also remark that the conformal Ricci flow equation was designed as a parabolic model for understanding the reduced Hamiltonian formulation of Einstein's equations of general relativity ([16],[18],[19]), similar to the fact that the classical Ricci flow equation is somewhat of a parabolic model for the unreduced Hamiltonian formulation of Einstein equations.

In comparing the conformal Ricci flow with the classical Ricci flow, one should also make some remarks concerning the $n = 3$ case. At first, the classical Ricci flow equation had its most spectacular success in the case of strictly positive Ricci tensors in its attempt to prove the Poincaré conjecture. Indeed, the classical Ricci flow equation "prefers" positive curvature (Hamilton [24], p. 276, 279) since it preserves both positive scalar and positive Ricci curvature (see also Proposition 3.5). The conformal Ricci flow equation, on the other hand, was designed more to deal with the complementary situation of negative Yamabe type manifolds (see Section 1.5) and hyperbolic geometry rather than positive Ricci tensors and spherical space forms. Thus in this sense the classical Ricci flow and the conformal Ricci flow are concerned with complementary problems. However, we hasten to add that the classical Ricci flow equation is now intensely used to study general questions regarding 3-manifold topology (see for example Cao-Chow [6], Hamilton ([25],[26]), Perelman ([35],[36],[37]) and Ye [44]) and thus the classical and conformal Ricci flows are best viewed as complementary tools when examining similar problems.

2. The space $\mathcal{M}_{-1}$

The space
$$\mathcal{M}_{-1} = \{g \in \mathcal{M} \mid R(g) = -1\} \tag{2.1}$$
plays an important role in geometry (Fischer-Marsden [13]), in general relativity (Fischer-Moncrief [16]), in Teichmüller theory (Fischer-Tromba [23]), and in the conformal Ricci flow as developed in this paper. The structure of this space has been studied in detail in [13]. The main viewpoint pioneered there is to consider the **scalar curvature function**
$$R : \mathcal{M} \longrightarrow \mathcal{F} \,, \qquad g \longmapsto R(g) \,, \tag{2.2}$$
as a map between infinite-dimensional manifolds and to study the properties of this map. Since $\mathcal{M}$ is open in S_2 and since $\mathcal{F}$ is a linear space, the derivative $DR(g)$ of R at $g \in \mathcal{M}$ maps S_2 to $\mathcal{F}$ and is given by
$$DR(g) : S_2 \longrightarrow \mathcal{F} \,, \qquad h \longmapsto DR(g)h = \Delta_g \mathrm{tr}_g h + \delta_g \delta_g h - \mathrm{Ric}(g) \cdot h \,, \tag{2.3}$$
where Δ_g is the Laplace operator acting on scalar-valued functions, tr_g is the metric trace, and $\delta_g \delta_g$ is the double covariant divergence acting on 2-covariant symmetric tensors. In local coordinates,
$$\begin{aligned} DR(g)h &= - g^{kl}(g^{ij}h_{ij})_{|k|l} + g^{ik}g^{jl}h_{ij|k|l} - g^{ik}g^{jl}R_{ij}h_{kl} \\ &= - g^{kl}(h^j{}_j)_{|k|l} + h^{kl}{}_{|k|l} - R^{kl}h_{kl} \,, \end{aligned}$$

where the vertical bar denotes covariant derivative with respect to g. We note explicitly that both the Laplacian on scalar functions Δ_g and the covariant divergence δ_g are taken with embedded negative signs, $\Delta_g \phi = -g^{ij}\phi_{|i|j}$, $(\delta_g h)_j = -g^{ik}h_{ij|k} = -h^k{}_{j|k}$, and for a 1-form α, $\delta_g \alpha = -g^{ij}\alpha_{i|j}$, so that $\delta_g \delta_g h = h^{kl}{}_{|k|l}$ and $\Delta_g \phi = -g^{ij}(\phi_{|i})_{|j} = \delta_g d\phi$. Thus in this sign convention, the Laplacian $\Delta_g = \delta_g d$ is a positive operator.

Let

$$DR(g)^* : \mathcal{F} \longrightarrow S_2 \,, \qquad \phi \longmapsto DR(g)^*\phi = \mathrm{Hess}_g\phi + g\Delta_g\phi - \mathrm{Ric}(g)\phi \qquad (2.4)$$

denote the L_2-adjoint of $DR(g)$ where $\mathrm{Hess}_g\phi$ denotes the Hessian of the scalar function ϕ. In local coordinates,

$$(DR(g)^*\phi)_{ij} = \phi_{|i|j} - g_{ij}(g^{kl}\phi_{|k|l}) - R_{ij}\phi \,. \qquad (2.5)$$

The second order operator $DR(g)^*$ has injective symbol since for $\xi \in T_x^*M$ and $1 \in \boldsymbol{R}$, its symbol $\sigma_\xi(DR(g)^*)1 = \xi \otimes \xi - g\|\xi\|^2$ is injective if $\xi \neq 0$ since its trace is $(1-n)\|\xi\|^2$. Thus using the operators $DR(g)$ and $DR(g)^*$ we have the following L_2-orthogonal splitting of S_2 (see Berger-Ebin [5]),

$$S_2 = \ker DR(g) \oplus \mathrm{range}\, DR(g)^* \,, \qquad h = \bar{h} + DR(g)^*\phi \,, \qquad (2.6)$$

where $\phi = \big(DR(g)DR(g)^*\big)^{-1}\big(DR(g)h\big)$ and $\bar{h} = h - DR(g)^*\phi$ so that

$$h = (h - DR(g)^*\phi) + DR(g)^*\phi = \bar{h} + DR(g)^*\phi = \bar{P}_g(h) + DR(g)^*\phi \,, \qquad (2.7)$$

where

$$\bar{P}_g : S_2 \longrightarrow \ker DR(g) \,, \qquad h \longmapsto \bar{h} = \bar{P}_g h \,, \qquad (2.8)$$

denotes the L_2-orthogonal projection onto $\ker DR(g)$.

Note that

$$DR(g)DR(g)^* : \mathcal{F} \longrightarrow \mathcal{F} \qquad (2.9)$$

is an elliptic fourth order operator with $\ker(DR(g)DR(g)^*) = \ker DR(g)^*$ which is generically zero as a function of g, as shown in Fischer-Marsden [13]. Also note that in the splitting (2.6), there are no curvature assumptions on g.

We now consider $DR(g)h$ on two types of deformations $h \in S_2$ that we shall need in Section 3.2. We first recall the following. For any metric $g \in \mathcal{M}$, the **doubly contracted (differential) Bianchi identity** asserts that the divergence of the Einstein tensor $\mathrm{Ein}(g) = \mathrm{Ric}(g) - \frac{1}{2}R(g)g$ vanishes,

$$0 = \delta_g(\mathrm{Ein}(g)) = \delta_g\Big(\mathrm{Ric}(g) - \tfrac{1}{2}R(g)g\Big) = \delta_g(\mathrm{Ric}(g)) + \tfrac{1}{2}dR(g) \,, (2.10)$$

where in local coordinates $(\delta_g(R(g)g))_i = -g^{jk}(R(g)g_{ij})_{|k} = -R(g)_{|i} = -(dR(g))_i$. Thus the doubly contracted Bianchi identity is a third-order differential identity (i.e., an equation that is true for all metrics $g \in \mathcal{M}$) that "simplifies" the divergence of the Ricci tensor by expressing it as the derivative of the scalar curvature. The **double divergence** of $\mathrm{Ein}(g)$ then leads to the fourth order differential identity,

$$0 = \delta_g\delta_g(\mathrm{Ein}(g)) = \delta_g\delta_g(\mathrm{Ric}(g)) + \tfrac{1}{2}\delta_g dR(g) = \delta_g\delta_g(\mathrm{Ric}(g)) + \tfrac{1}{2}\Delta_g R(g) \,. \quad (2.11)$$

We consider examples of the splitting (2.6) for two explicit deformations that we shall need later.

Example 2.1 For $g \in \mathcal{M}$ and $\varphi \in \mathcal{F}$, $h = \varphi g \in S_2$ is an **infinitesimal pointwise conformal deformation of** g. On a deformation φg,

$$
\begin{aligned}
DR(g)(\varphi g) &= \Delta_g \mathrm{tr}_g(\varphi g) + \delta_g \delta_g(\varphi g) - \mathrm{Ric}(g) \cdot (\varphi g) \\
&= n\Delta_g \varphi - \Delta_g \varphi - R(g)\varphi \\
&= (n-1)\Delta_g \varphi - R(g)\varphi \,.
\end{aligned}
\tag{2.12}
$$

Thus we define an operator $L_g : \mathcal{F} \to \mathcal{F}$ by

$$
L_g \varphi = DR(g)(\varphi g) \,,
\tag{2.13}
$$

so that from (2.12),

$$
L_g : \mathcal{F} \longrightarrow \mathcal{F} \,, \quad \varphi \longmapsto L_g \varphi = (n-1)\Delta_g \varphi - R(g)\varphi \,.
\tag{2.14}
$$

We remark that, interestingly,

$$
\mathrm{tr}_g(DR(g)^* \varphi) = DR(g)(\varphi g) = L_g \varphi \,,
\tag{2.15}
$$

since $\mathrm{tr}_g(\mathrm{Hess}_g \varphi + g\Delta_g \varphi - \mathrm{Ric}(g)\varphi) = (n-1)\Delta_g \varphi - R(g)\varphi$.

As a special case of (2.14), note that for a constant $c \in \mathbf{R}$,

$$
L_g c = DR(g)(cg) = -cR(g) \,.
\tag{2.16}
$$

Thus if $R(g) = -1$, $L_g c = DR(g)(cg) = c$. Thus we have the following L_2-orthogonal splitting of $h = g$ under the assumption that $R(g) = -1$, in which case $DR(g)g = 1$,

$$
g = \bar{g} + DR(g)^* \big((DR(g)DR(g)^*)^{-1}(1) \big) \,,
\tag{2.17}
$$

where $\bar{g} = \bar{P}_g(g) = g - DR(g)^* \big((DR(g)DR(g)^*)^{-1}(1) \big)$.

Example 2.2 (See also Examples 5.2 and 5.3.) For $g \in \mathcal{M}$, consider the deformation $h = \mathrm{Ric}(g)$. From (2.4),

$$
DR(g)^*(-1) = \mathrm{Hess}_g(-1) + g\Delta_g(-1) - \mathrm{Ric}(g)(-1) = \mathrm{Ric}(g) \,, \quad (2.18)
$$

so that $\mathrm{Ric}(g) \in \mathrm{range}\, DR(g)^*$. Thus $\bar{P}_g(\mathrm{Ric}(g)) = \overline{\mathrm{Ric}(g)} = 0$ since $\mathrm{Ric}(g)$ is L_2-orthogonal to $\ker DR(g)$. Thus $\mathrm{Ric}(g)$ does not split (non-trivially) L_2-orthogonally (see Example 5.2 for a non-trivial non-orthogonal splitting of $\mathrm{Ric}(g)$).

This fact is also "discoverable" by integrating (2.3) over M, to give

$$
\int_M DR(g)h \, d\mu_g = \int_M (\Delta_g \mathrm{tr}_g h + \delta_g \delta_g h - \mathrm{Ric}(g) \cdot h) \, d\mu_g = -\int_M \mathrm{Ric}(g) \cdot h \, d\mu_g \,.
$$

Thus for all $h \in \ker DR(g)$, $\int_M \mathrm{Ric}(g) \cdot h \, d\mu_g = 0$, so that $\mathrm{Ric}(g)$ is L_2-orthogonal to $\ker DR(g)$.

Analytically, if we try to split $\mathrm{Ric}(g)$, we first note that from (2.11),

$$
\begin{aligned}
DR(g)\mathrm{Ric}(g) &= \Delta_g \mathrm{tr}_g \mathrm{Ric}(g) + \delta_g \delta_g \mathrm{Ric}(g) - \mathrm{Ric}(g) \cdot \mathrm{Ric}(g) \\
&= \Delta_g R(g) - \tfrac{1}{2}\Delta_g R(g) - |\mathrm{Ric}(g)|_g^2 \\
&= \tfrac{1}{2}\Delta_g R(g) - |\mathrm{Ric}(g)|_g^2 \,,
\end{aligned}
\tag{2.19}
$$

which is an expression that we shall need frequently. Thus, from (2.7) and (2.19),

$$
\begin{aligned}
\mathrm{Ric}(g) &= \overline{\mathrm{Ric}(g)} + DR(g)^* \Big((DR(g)DR(g)^*)^{-1}(DR(g)\mathrm{Ric}(g)) \Big) \\
&= \overline{\mathrm{Ric}(g)} + DR(g)^* \Big((DR(g)DR(g)^*)^{-1} \big(\tfrac{1}{2}\Delta_g R(g) - |\mathrm{Ric}(g)|^2 \big) \Big) \,.
\end{aligned}
\tag{2.20}
$$

Then, from (2.18) and (2.19),

$$
DR(g)(DR(g)^*(-1)) = DR(g)\mathrm{Ric}(g) = \tfrac{1}{2}\Delta_g R(g) - |\mathrm{Ric}(g)|^2 \,,
$$

so that $(DR(g)DR(g)^*)^{-1}(\tfrac{1}{2}\Delta_g R(g) - |\mathrm{Ric}(g)|^2) = -1$. Thus (2.20) reduces to $\mathrm{Ric}(g) = \overline{\mathrm{Ric}(g)} + DR(g)^*(-1) = \overline{\mathrm{Ric}(g)} + \mathrm{Ric}(g)$ so that $\overline{\mathrm{Ric}(g)} = 0$ (see also the remarks after Theorem 2.3). ∎

The information that we shall need about $\mathcal{M}_{-1}$ is contained in the following theorem. We first recall that Aubin [3] has shown that every closed n-manifold M, $n \geq 3$, has a Riemannian metric with constant negative scalar curvature and thus there are no topological obstructions to the constraint $R(g) = -1$. In particular, $\mathcal{M}_{-1} \neq \emptyset$.

Theorem 2.3 (The structure of the space $\mathcal{M}_{-1}$ (Fischer-Marsden [13])) *Let M be a closed connected orientable n-manifold with either $n \geq 3$ or $n = 2$ and* genus$(M) \geq 2$. *Then $\mathcal{M}_{-1}$ is a non-empty closed (as a subset) smooth infinite-dimensional* ILH*-submanifold of $\mathcal{M}$. Moreover, $\mathcal{M}_{-1}$ is $\mathcal{D}$-invariant and has codimension $C^\infty(M, \boldsymbol{R})$ in $\mathcal{M}$.*

For $g \in \mathcal{M}_{-1}$, the tangent space to $\mathcal{M}_{-1}$ at g is given by

$$T_g \mathcal{M}_{-1} \approx \ker DR(g) = \{\, h \in S_2 \mid \Delta_g \mathrm{tr}_g h + \delta_g \delta_g h - \mathrm{Ric}(g) \cdot h = 0 \,\}, \tag{2.21}$$

which is the first summand in the L_2-orthogonal splitting,

$$T_g \mathcal{M} = T_g \mathcal{M}_{-1} \oplus T_g^\perp \mathcal{M}_{-1}. \tag{2.22}$$

Equivalently,

$$S_2 = \ker DR(g) \oplus \mathrm{range}\, DR(g)^*, \tag{2.23}$$

where $T_g^\perp \mathcal{M}_{-1} \approx \mathrm{range}\, DR(g)^*$ is the L_2-orthogonal complement of $T_g \mathcal{M}_{-1}$ in the natural L_2-Riemannian metric $\mathcal{G}$ (see (1.3)) on $\mathcal{M}$.

If M is of Yamabe type -1, then the space of pointwise conformal classes $\mathcal{M}/\mathcal{P}$ on M and $\mathcal{M}_{-1}$ are ILH *diffeomorphic.*

Proof (sketch): It is shown in [13] that the scalar curvature map $R : \mathcal{M} \to \mathcal{F}$ is a smooth ILH submersion almost everywhere. In particular, -1 is a regular value of R and thus from the inverse function theorem adapted to the ILH topology, the level set

$$\mathcal{M}_{-1} = R^{-1}(-1) \tag{2.24}$$

is a smooth closed ILH submanifold of $\mathcal{M}$ with the indicated tangent space.

If $g \in \mathcal{M}$ and $f \in \mathcal{D}$, let f^*g denote the pullback of g by f, defined by $f^*g(x)(v_1, v_2) = g(f(x))(T_x f(v_1), T_x f(v_2))$ for $x \in M$, $v_1, v_2 \in T_x M$, the tangent space to M at $x \in M$.

By the **covariance of the scalar curvature operator**

$$R(f^*g) = f^*(R(g)) = R(g) \circ f. \tag{2.25}$$

The $\mathcal{D}$-invariance of $\mathcal{M}_{-1}$ then follows easily, for if $g \in \mathcal{M}_{-1}$,

$$R(f^*g) = f^*(R(g)) = f^*(-1) = -1,$$

so that $f^*g \in \mathcal{M}_{-1}$.

If M is of Yamabe type -1, then every Riemannian metric on M is uniquely pointwise conformally deformable to a metric with scalar curvature $= -1$. An analysis of the proof shows that this bijection is an ILH diffeomorphism, so the space of conformal structures $\mathcal{M}/\mathcal{P}$ is ILH diffeomorphic to $\mathcal{M}_{-1}$. ∎

Thus in the case of Yamabe type -1 manifolds, the space $\mathcal{M}_{-1}$ is an important representation of $\mathcal{M}/\mathcal{P}$ and thus plays an important role in conformal geometry.

As discussed in Example (ii) above, for any $g \in \mathcal{M}$, $\mathrm{Ric}(g) \in \mathrm{range}\, DR(g)^* = (\ker DR(g))^\perp$ (in the L_2-metric). Thus we have the following interesting geometrical fact about the Ricci tensor: If $\mathrm{Ric}(g) \neq 0$, then in an L_2-sense, the direction

$\text{Ric}(g) \in (\ker DR(g))^{\perp}$ infinitesimally deforms g in a direction orthogonal to those directions preserving the scalar curvature of g.

This can be formalized as follows. Let

$$\mathcal{M}_{\rho} = R^{-1}(\rho) = \{ g' \in \mathcal{M} \mid R(g') = \rho \}$$

denote the space of Riemannian metrics with scalar curvature ρ. Then, generically, $\mathcal{M}_{\rho}$ is a $C^{\infty}(M, \boldsymbol{R})$-codimensional submanifold of $\mathcal{M}$ (Fischer-Marsden [13]) and $T_g\mathcal{M} = T_g\mathcal{M}_{\rho} \oplus T_g^{\perp}\mathcal{M}_{\rho}$, in which case $\text{Ric}(g) \in \text{range } DR(g)^* \approx T_g^{\perp}\mathcal{M}_{\rho}$ and so is L_2-orthogonal to $\mathcal{M}_{\rho}$ (see also Figure 1 in Section 5).

Somewhat similarly, for $g \in \mathcal{M}$ with $\text{vol}(M, g) = c > 0$, let

$$\mathcal{M}^c = \{ g' \in \mathcal{M} \mid \text{vol}(M, g') = c \}$$

denote the codimension-1 submanifold of metrics with volume c. Then

$$T_g\mathcal{M}^c \approx \{ h \in S_2 \mid \int_M \text{tr}_g h \, d\mu_g = \int_M g \cdot h \, d\mu_g = 0 \},$$

so that

$$T_g^{\perp}\mathcal{M}^c \approx \{ h' \in S_2 \mid \int_M h' \cdot h \, d\mu_g = 0 \text{ for all } h \in T_g\mathcal{M}^c \} = \{ \lambda g \mid \lambda \in \boldsymbol{R} \} = \boldsymbol{R}g.$$

Thus, in an L_2-sense, any metric $g \in \mathcal{M}$, if viewed as a "vector" $g \in \boldsymbol{R}g \approx T_g^{\perp}\mathcal{M}^c$, is the direction which maximally increases the volume $\text{vol}(M, g)$.

An interesting and useful kinematical observation regarding $\mathcal{M}_{-1}$ is the following.

Proposition 2.4 *For $g \in \mathcal{M}$,*

$$|\text{Ric}(g)|_g^2 \quad = |\text{Ric}^T(g)|_g^2 + \tfrac{1}{n}R^2(g) \tag{2.26}$$

$$|\text{Riem}(g)|_g^2 = |\text{Weyl}(g)|_g^2 + \tfrac{4}{n-2}|\text{Ric}(g)|_g^2 - \tfrac{2}{(n-1)(n-2)}R^2(g) \tag{2.27}$$

$$= |\text{Weyl}(g)|_g^2 + \tfrac{4}{n-2}|\text{Ric}^T(g)|_g^2 + \tfrac{2}{n(n-1)}R^2(g) \tag{2.28}$$

(where $R^2(g) = (R(g))^2$). Consequently, for $g \in \mathcal{M}_{-1}$,

$$|\text{Ric}(g)|_g^2 \quad = |\text{Ric}^T(g)|_g^2 + \tfrac{1}{n} \geq \tfrac{1}{n} \tag{2.29}$$

$$|\text{Riem}(g)|_g^2 = |\text{Weyl}(g)|_g^2 + \tfrac{4}{n-2}|\text{Ric}^T(g)|_g^2 + \tfrac{2}{n(n-1)} \geq \tfrac{2}{n(n-1)}. \tag{2.30}$$

Thus, kinematically, any set (or curve) of metrics in $\mathcal{M}_{-1}$ has Ricci and Riemann curvature norm bounded away from zero. Thus the constraint $R(g) = -1$ prevents the norms $|\text{Ric}(g)|_g$ and $|\text{Riem}(g)|_g$ from collapsing to zero.

Proof: For $g \in \mathcal{M}$, $\text{Ric}(g) = \text{Ric}^T(g) + \tfrac{1}{n}R(g)g$. Squaring and using the facts that $\text{Ric}^T(g)$ and g are pointwise orthogonal and that $|g|_g^2 = g^{ik}g^{jl}g_{ij}g_{kl} = g^{kl}g_{kl} = \text{tr}_g g = n$ (not n^2) gives

$$|\text{Ric}(g)|_g^2 = |\text{Ric}^T(g) + \tfrac{1}{n}R(g)g|_g^2 = |\text{Ric}^T(g)|_g^2 + \tfrac{1}{n^2}R^2(g)|g|_g^2 = |\text{Ric}^T(g)|_g^2 + \tfrac{1}{n}R^2(g)$$

In a similar (but longer) manner, the expressions for $|\text{Riem}(g)|_g^2$ follows by squaring the local coordinate expression which defines the Weyl conformal curvature tensor,

$$R_{ijkl} = W_{ijkl} + \tfrac{1}{n-2}(g_{jk}R_{il} + g_{il}R_{jk} - g_{jl}R_{ik} - g_{ik}R_{jl})$$
$$+ \tfrac{1}{(n-1)(n-2)}R(g)(g_{ik}g_{jl} - g_{jk}g_{il}).$$

■

3. The conformal Ricci flow equations

3.1. The equilibrium points of the conformal Ricci flow equations

Proposition 3.1 (The equilibrium points of the conformal Ricci flow equations) *A metric $g \in \mathcal{M}_{-1}$ is an equilibrium point of the conformal Ricci flow equations if and only if g is an Einstein metric with negative Einstein constant $-\frac{1}{n}$,*

$$\mathrm{Ric}(g) = -\tfrac{1}{n}g\,, \tag{3.1}$$

if and only if $\mathrm{Ric}^T(g) = 0$, in which case the conformal pressure

$$p = 0\,. \tag{3.2}$$

If $n = 3$, then $g \in \mathcal{M}_{-1}$ is an equilibrium point if and only if g is a hyperbolic metric with constant negative sectional curvature $K(g) = -\frac{1}{6}$ and $\mathrm{Ric}(g) = -\frac{1}{3}g$, in which case g is unique up to isometry.

Proof: Suppose $g \in \mathcal{M}_{-1}$ is an equilibrium point of the conformal Ricci flow equations with corresponding conformal pressure $p = p(g) \in \mathcal{F}$. Then

$$2\mathrm{Ric}(g) + (\tfrac{2}{n} + p)g = 0 \tag{3.3}$$

and the trace yields $2R(g) + 2 + np = -2 + 2 + np = np = 0$. Substituting $p = 0$ back into (3.3) gives

$$\mathrm{Ric}(g) = -\tfrac{1}{n}\,g\,. \tag{3.4}$$

Conversely, if $\mathrm{Ric}(g) = -\frac{1}{n}\,g$, from (3.7) below, $p = p(g) = 0$ so that $2\mathrm{Ric}(g) + (\frac{2}{n} + 0)g = 0$ so that g is an equilibrium point (see also Remark (ii) following Proposition 3.2). For $g \in \mathcal{M}_{-1}$, $\mathrm{Ric}^T(g) = \mathrm{Ric}(g) - \frac{1}{n}R(g)g = \mathrm{Ric}(g) + \frac{1}{n}g$ so that g is an equilibrium point if and only if $\mathrm{Ric}^T(g) = 0$.

If $n = 3$, then $g \in \mathcal{M}_{-1}$ is an equilibrium point if and only if $\mathrm{Ric}(g) = -\frac{1}{3}\,g$ if and only if g is a hyperbolic metric with constant sectional curvature $K(g) = -\frac{1}{6}$, in which case, by Mostow rigidity, g is unique up to isometry when restricted to $\mathcal{M}_{-1}$ (see definition of hyperbolic metric in Section 1.1). ∎

Remarks:

(i) If $M = \boldsymbol{S}^1$, $\boldsymbol{S}^2$, or $\boldsymbol{T}^2$, $\mathcal{M}_{-1}$ is empty for these manifolds and so the conformal Ricci flow equations (1.5–1.6) have no solutions. If $n = 2$, genus $(M) \geq 2$, every $g \in \mathcal{M}_{-1} \neq \emptyset$ satisfies $\mathrm{Ric}(g) = -\frac{1}{2}\,g$ and thus every $g \in \mathcal{M}_{-1}$ is an equilibrium point of (1.5–1.6) with $p = 0$. Thus in this case the evolution equation (1.5) reduces to the trivial equation $\frac{\partial g}{\partial t} \equiv 0$. Thus we restrict to $n \geq 3$.

(ii) At an equilibrium point $g \in \mathcal{M}_{-1}$, the conformal pressure is zero since in that case trivially no constraint force is necessary to constrain the flow to $\mathcal{M}_{-1}$. In Proposition 3.3 below we will see that, conversely, if at any time t_0 the conformal pressure $p(t_0, x_0) = 0$ for some $x_0 \in M$, then p is identically zero and g must be an equilibrium point, $\mathrm{Ric}(g) = -\frac{1}{n}g$.

(iii) A 3-manifold M has an equilibrium point $g \in \mathcal{M}_{-1}$ if and only if M supports a hyperbolic metric. Contrapositively, if M does not support a hyperbolic metric, then there are no equilibrium points for the conformal Ricci flow. For example, if M is diffeomorphic to any spherical space form, flat manifold, handle $\boldsymbol{S}^1 \times \boldsymbol{S}^2$, or to any non-trivial connected sum $M_1 \# M_2$ of 3-manifolds $(M_1 \neq \boldsymbol{S}^3,\ M_2 \neq \boldsymbol{S}^3)$, then M would not support a hyperbolic metric and thus there would be no equilibrium points for the conformal Ricci flow on M.

(iv) We note the significant differences between the cases $n = 3$ and $n \geq 4$. If $n = 3$, equilibrium points are hyperbolic metrics and thus, since they are normalized by $R(g) = -1$, are by Mostow rigidity unique up to isometry. On the other hand, for $n \geq 4$, there may exist continuous families of *non-isometric* Einstein metrics with $\mathrm{Ric}(g) = -\frac{1}{n}g$ which would be equilibrium points of the conformal Ricci flow. A similar situation also arises for the higher dimensional reduced Einstein equations (see Fischer-Moncrief [21] for more information regarding this situation).

(v) If we let g_e denote an equilibrium point, where the "e" stands for equilibrium (or einstein), then the static (or equilibrium, or trivial) curve $g(t) \equiv g_e$ is a solution of the conformal Ricci system with initial value g_e and $p = 0$. On the other hand, if $g(t)$ is a non-static (or non-equilibrium, or non-trivial) solution, then no equilibrium point can be on the image of $g(t)$. Thus for a non-static solution g, $\mathrm{Ric}^T(g(t)) \neq 0$ for any $t \in [0, T)$.

(vi) We remark that the equilibrium points for both the classical and conformal Ricci flow equations are Einstein metrics.

3.2. Maintaining the constraint

The conformal pressure p can be thought of as a time-dependent Lagrange multiplier necessary to maintain the constraint equation $R(g) = -1$. Here we find the elliptic equation satisfied by p. In Section 3.3 we show how p and the constraint equation itself can be eliminated.

Proposition 3.2 (Maintaining the constraint equation) *Let* $g : [0, T) \to \mathcal{M}_{-1}$ *be a solution of the conformal Ricci flow equations with conformal pressure* $p : [0, T) \to \mathcal{F}$. *Then for each* $t \in [0, T)$,

$$\frac{\partial g}{\partial t} \in \ker DR(g),\tag{3.5}$$

and p *satisfies the linear inhomogeneous elliptic equation*

$$L_g p = (n-1)\Delta_g p + p = 2(|\mathrm{Ric}(g)|_g^2 - \tfrac{1}{n}) = 2(|\mathrm{Ric}^T(g)|_g^2),\tag{3.6}$$

where the operator $L_g = (n-1)\Delta_g + 1 : \mathcal{F} \to \mathcal{F}$ *is an isomorphism and is the operator* L_g *defined in (2.13–2.14) when* $g \in \mathcal{M}_{-1}$. *Thus for each* $t \in [0, T)$, *the conformal pressure* p *is uniquely determined by*

$$p = 2L_g^{-1}\left(|\mathrm{Ric}(g)|_g^2 - \tfrac{1}{n}\right) = 2L_g^{-1}\left(|\mathrm{Ric}^T(g)|_g^2\right) = 2L_g^{-1}\left(|\mathrm{Ric}(g)|_g^2\right) - \tfrac{2}{n}.\tag{3.7}$$

Proof: Differentiating the constraint equation $R(g) = -1$ with respect to time yields

$$\frac{\partial R(g)}{\partial t} = DR(g)\frac{\partial g}{\partial t} = 0,\tag{3.8}$$

so that $\frac{\partial g}{\partial t} \in \ker DR(g)$. Then from (1.5), (2.12), and (2.19),

$$0 = DR(g)\frac{\partial g}{\partial t} = DR(g)\left(-2\mathrm{Ric}(g) - (\tfrac{2}{n} + p)g\right) = -2DR(g)\mathrm{Ric}(g) - DR(g)((\tfrac{2}{n} + p)g)$$

$$= -2\left(\Delta_g \mathrm{tr}_g \mathrm{Ric}(g) + \delta_g\delta_g \mathrm{Ric}(g) - \mathrm{Ric}(g) \cdot \mathrm{Ric}(g)\right)$$

$$\qquad - \left(\Delta_g \mathrm{tr}_g((p + \tfrac{2}{n})g) + \delta_g\delta_g((\tfrac{2}{n} + p)g) - \mathrm{Ric}(g) \cdot ((\tfrac{2}{n} + p)g)\right)$$

$$= -2\left(\tfrac{1}{2}\Delta_g R(g) - |\mathrm{Ric}(g)|_g^2\right) - \left((n-1)\Delta_g(\tfrac{2}{n} + p) - R(g)(\tfrac{2}{n} + p)\right)$$

$$= -\Delta_g R(g) + 2|\mathrm{Ric}(g)|_g^2 - (n-1)\Delta_g p + (p + \tfrac{2}{n})R(g).\tag{3.9}$$

In local coordinates, (3.8) and (3.9) are given by

$$0 = \frac{\partial R}{\partial t} = g^{ij} R_{|i|j} + 2R^{ij} R_{ij} + (n-1)g^{ij} p_{|i|j} + (p + \tfrac{2}{n})R.$$

Applying the constraint $R(g) = -1$, (3.9) reduces to

$$0 = 2|\text{Ric}(g)|_g^2 - (n-1)\Delta_g p - (p + \tfrac{2}{n}) = 2|\text{Ric}(g)|_g^2 - L_g p - \tfrac{2}{n},$$

so that using the equality part of (2.29),

$$L_g p = 2(|\text{Ric}(g)|_g^2 - \tfrac{1}{n}) = 2|\text{Ric}^T(g)|_g^2, \qquad (3.10)$$

where $L_g = (n-1)\Delta_g + 1 = L_g^*$ is a linear self-adjoint elliptic operator. Since $\ker L_g = \ker L_g^* = 0$, $L_g = L_g^*$ is injective and thus by ellipticity of L_g,

$$\mathcal{F} = \text{range}\, L_g \oplus \ker L_g^* = \text{range}\, L_g.$$

Thus L_g is surjective and thus an isomorphism. Let $L_g^{-1} : \mathcal{F} \to \mathcal{F}$ denote its inverse. Then $p = 2L_g^{-1}(|\text{Ric}(g)|_g^2 - \tfrac{1}{n}) = 2L_g^{-1}(|\text{Ric}^T(g)|_g^2)$ uniquely solves (3.10). Lastly, since for any constant c, $L_g c = (n-1)\Delta_g c + c = c$, $L_g^{-1}(c) = c$, so that $p = 2L_g^{-1}(|\text{Ric}(g)|_g^2 - \tfrac{1}{n}) = 2L_g^{-1}(|\text{Ric}(g)|_g^2) - \tfrac{2}{n}$. ∎

Remarks:

(i) Equation (3.7), $p(t) = 2L_{g(t)}^{-1}(|\text{Ric}^T(g(t))|_{g(t)}^2)$, links the conformal pressure $p(t)$ to the flow $g(t)$ after imposing the scalar curvature constraint $R(g(t)) = -1$. This equation should be compared with $c(t) = \tfrac{2}{n}R_{\text{total}}(g(t))$ for the classical Ricci flow which links $c(t)$ to $g(t)$, also *after* imposing the volume constraint $\text{vol}(M, g) = 1$ (see the remark above (1.22)).

(ii) If g is an equilibrium point, then from Proposition 3.1, $\text{Ric}^T(g) = 0$ and thus $p = 2L_g^{-1}(0) = 0$ as also known from that Proposition. Conversely, from (3.7), if $\text{Ric}^T(g) = 0$, then $p = 0$ and so g is an equilibrium point.

(iii) The conformal pressure $p = 2L_g^{-1}(|\text{Ric}^T(g)|_g^2)$ is an integral or nonlocal function of g in contrast to a local, or differential function of g, such as $R(g)$. Thus at a point $x \in M$, $p(x)$ depends globally on g whereas $R(g)(x)$ depends on g only locally in a neighborhood $U_x \subset M$ of x. This global behavior of p is completely analogous to the pressure that appears in the incompressible Euler or Navier-Stokes equations and is also analogous to the nonlocal Lagrange multiplier c in the classical Ricci equation (1.15). Similarly, the nonlocal equation (3.7) that determines p is completely analogous to the nonlocal equation (1.21) that determines c. ∎

Using the strong maximum principle, we can find additional important and useful information about the conformal pressure p.

By a **domain** $D \subseteq M$ we mean an open connected set of M.

Proposition 3.3 (The strong maximum principle and the conformal pressure) *For $g \in \mathcal{M}_{-1}$, let $p \in \mathcal{F}$ be a solution of*

$$L_g p = (n-1)\Delta_g p + p = 2|\text{Ric}^T(g)|_g^2. \qquad (3.11)$$

Then $p \geq 0$, and more particularly, either

(a) $p = 0$ and $\text{Ric}(g) = -\tfrac{1}{n}g$, or (b) $p > 0$ and $\text{Ric}(g) \neq -\tfrac{1}{n}g$.

Proof: Equation (3.11) is in a form in which we can apply the **strong maximum principle** (also known as the maximum principle of H. Hopf; see e.g. Protter-Weinberger [38], pp. 61, 64) to $-p$. The corresponding strong minimum principle asserts in our notation that if $\phi \in \mathcal{F}$ satisfies the inequality

$$-\Delta_g \phi - u\phi \leq 0 \,, \qquad u \geq 0 \,, \tag{3.12}$$

on any domain $D \subseteq M$ and if $\phi_{\min}$ denotes the minimum value of ϕ on D, then if $\phi_{\min} \leq 0$, then $\phi \equiv \phi_{\min}$ is a constant function on D. Applied to the negative of (3.11) with $u = 1$,

$$-(n-1)\Delta_g p - p = -2|\mathrm{Ric}^T(g)|_g^2 \leq 0 \,, \tag{3.13}$$

and letting $p_{\min}$ denote the minimum value of p on M, if $p_{\min} \leq 0$, then $p = p_{\min}$ is a constant function. But if p is constant, then from (3.13),

$$p = 2|\mathrm{Ric}^T(g)|_g^2 \geq 0 \,, \tag{3.14}$$

and since $p = p_{\min} \leq 0$, it follows that $p = p_{\min} = 0$. Thus from (3.14),

$$0 = 2|\mathrm{Ric}^T(g)|_g^2 \Longrightarrow \mathrm{Ric}^T(g) = 0 \Longrightarrow \mathrm{Ric}(g) = \tfrac{1}{n}R(g)g \Longrightarrow \mathrm{Ric}(g) = -\tfrac{1}{n}g \,. \tag{3.15}$$

If on the other hand $p_{\min} > 0$, then $p \geq p_{\min} > 0$ and thus from (3.7), $\mathrm{Ric}(g) \neq -\tfrac{1}{n}g$. Thus overall, either (1) $p = 0$ and $\mathrm{Ric}(g) = -\tfrac{1}{n}g$, or (2) $p > 0$ and $\mathrm{Ric}(g) \neq -\tfrac{1}{n}g$. ∎

Remark: As a consequence of this proposition, if $g : [0, T) \to \mathcal{M}_{-1}$ is a non-static conformal Ricci flow, then its conformal pressure $p : [0, T) \to \mathcal{P}$ must lie in the space of positive functions $\mathcal{P} = C^\infty(M, \mathbf{R}^+)$. To put this another way, if $p(t_0, x_0) = 0$ vanishes at a "spacetime" point $(t_0, x_0) \in [0, T) \times M$, then $p(t, x) = 0$ for any $(t, x) \in [0, T) \times M$.

3.3. Elimination of the constraint and the reduced conformal Ricci flow

Since the conformal Ricci flow equations are a constrained dynamical system with the conformal pressure p acting as a Lagrange multiplier, both p and the constraint equation $R(g) = -1$ can be formally eliminated to give the **(fully) reduced conformal Ricci equation**. This is analogous to the elimination of c to give the classical Ricci flow equation as discussed in Section 1.3 and the elimination of the fluid pressure p in the incompressible Euler or Navier-Stokes equations (see Ebin-Marsden [10]).

Proposition 3.4 (Elimination of the constraint and the reduced conformal Ricci flow equation) *A smooth curve $g : [0, T) \to \mathcal{M}$ with initial value $g_0 \in \mathcal{M}_{-1}$ is a solution of the* **(fully) reduced conformal Ricci flow equation**

$$\frac{\partial g}{\partial t} + 2\,\mathrm{Ric}(g) = -2\Big(L_g^{-1}(|\mathrm{Ric}(g)|_g^2)\Big)g \,, \tag{3.16}$$

where $L_g = (n-1)\Delta_g + 1 : \mathcal{F} \to \mathcal{F}$ is a curve of invertible elliptic operators with inverses $L_g^{-1} : \mathcal{F} \to \mathcal{F}$ associated with the curve $g : [0, T) \to \mathcal{M}$, if and only if g is a solution of the conformal Ricci flow equations

$$\frac{\partial g}{\partial t} + 2\big(\mathrm{Ric}(g) + \tfrac{1}{n}g\big) = -pg \tag{3.17}$$

$$R(g) = -1 \tag{3.18}$$

with conformal pressure $p = 2L_g^{-1}(|\mathrm{Ric}(g)|_g^2) - \tfrac{2}{n}$.

Proof: Proposition 3.2 asserts that if g is a conformal Ricci flow with conformal pressure p, then $p = 2L_g^{-1}(|\text{Ric}(g)|_g^2) - \frac{2}{n}$ with $L_g = (n-1)\Delta_g + 1$. Substituting p into (3.17) shows that a conformal Ricci flow g is also a solution to the reduced conformal Ricci equation (3.16).

To show the converse, let g be a solution to the reduced conformal Ricci flow equation (3.16) with initial value $g(0) = g_0 \in \mathcal{M}_{-1}$. Set $p = 2L_g^{-1}(|\text{Ric}(g)|_g^2) - \frac{2}{n}$ so that g and p satisfy (3.17). The task is to show that the constraint $R(g) = -1$ is maintained. From the definition of p and noting that $L_g(\frac{2}{n}) = \frac{2}{n}$, we see that p satisfies

$$L_g p = (n-1)\Delta_g p + p = 2(|\text{Ric}(g)|_g^2 - \frac{1}{n}). \tag{3.19}$$

Let $\tilde{R}(g) = 1 + R(g)$. Then using the expression for $DR(g)\frac{\partial g}{\partial t}$ from (3.9) (which is valid for any solution of (3.17) whether or not the constraint equation $R(g) = -1$ is satisfied) and using (3.19),

$$\begin{aligned}
\frac{\partial \tilde{R}(g)}{\partial t} = \frac{\partial R(g)}{\partial t} &= DR(g)\frac{\partial g}{\partial t} = DR(g)\left(-2\text{Ric}(g) - (\frac{2}{n} + p)g\right) \\
&= -\Delta_g R(g) + 2|\text{Ric}(g)|_g^2 - (n-1)\Delta_g p + (p + \frac{2}{n})R(g) \\
&= -\Delta_g(1 + R(g)) + \left(2|\text{Ric}(g)|_g^2 - (n-1)\Delta_g p\right) + (p + \frac{2}{n})R(g) \\
&= -\Delta_g \tilde{R}(g) + (p + \frac{2}{n}) + (p + \frac{2}{n})R(g) \\
&= -\Delta_g \tilde{R}(g) + (p + \frac{2}{n})\tilde{R}(g). \tag{3.20}
\end{aligned}$$

Since $g = g(t)$ is given as a solution to the reduced conformal Ricci system, $p(t) = 2L_{g(t)}^{-1}(|\text{Ric}(g(t))|_{g(t)}^2) - \frac{2}{n}$ is a known function of t. Thus the function $\tilde{R}(t) = \tilde{R}(g(t))$ satisfies a linear heat equation on M,

$$\frac{\partial \tilde{R}(t)}{\partial t} = -\Delta_{g(t)}\tilde{R}(t) + \left(p(t) + \frac{2}{n}\right)\tilde{R}(t), \quad \tilde{R}(0) = \tilde{R}(g_0) = 1 + R(g_0), \tag{3.21}$$

with time-dependent Laplacian and time-dependent coefficients. Uniqueness of solutions to (3.21) follows by modifying standard arguments to take into account the time-dependent Laplacian. For example, although it is more than is needed, uniqueness of solutions to quasi-linear parabolic systems with time-dependent coefficients is shown in Taylor [40], Section 15.7 (see Proposition 7.3, p. 332).

Since $R(g_0) = -1$, $\tilde{R}(0) = \tilde{R}(g_0) = 1 + R(g_0) = 0$ and thus $\tilde{R}(t) = \tilde{R}(g(t)) \equiv 0$ on $[0, T]$. Thus the constraint $R(g(t)) = -1$ is maintained on $[0, T]$ and thus the reduced unconstrained equation (3.16) with initial values in $\mathcal{M}_{-1}$ is equivalent to the unreduced constrained system (3.17–3.18). ∎

Remarks:

(i) The unreduced conformal Ricci flow equations (3.17–3.18) require the constraint $R(g) = -1$ on the entire flow whereas the reduced conformal Ricci flow equation (3.16) requires the constraint $R(g_0) = -1$ only on the initial metric. The constraint is then automatically maintained on the entire flow as a consequence of the reduced evolution equation. This situation is exactly analogous to the relation of the unreduced classical Ricci flow equations (1.15–1.16) to the fully reduced classical Ricci flow equation (1.22) (see the remarks after (1.20) and (1.23)). In this analogy, the linear non-autonomous *partial* differential equation (3.21) that ensures that the constraint $R(g) = -1$ is preserved if it holds initially is analogous to the linear non-autonomous *ordinary* differential equation (1.23) that insures that $\text{vol}(M, g) = 1$ is preserved if it holds initially.

(ii) The reduced conformal Ricci equation uses the "simpler" operator
$L_g = (n-1)\Delta_g + 1$, rather than $L_g = (n-1)\Delta_g - R(g)$, even though the curve
$g : [0, T) \to \mathcal{M}$ is at first only known to be in $\mathcal{M}$ (with initial value $g_0 \in \mathcal{M}_{-1}$).
This is analogous to using the "simpler" $c = \frac{2}{n}R_{\text{total}}(g)$ in the fully reduced
classical Ricci equation (1.22) (with initial value $g_0 \in \mathcal{M}^1$) rather than $\frac{2}{n}\frac{R_{\text{total}}(g)}{\text{vol}(M,g)}$.
In both cases the task is then to show that the constraint is maintained on the
entire domain of definition of any solution of the reduced equation that initially
satisfies the constraint.

3.4. Increasing scalar curvature for the classical Ricci flow equation

In this section we discuss the somewhat opposite behavior of the classical and
conformal Ricci flow equations with respect to scalar curvature (see Remark (iv) in
Section 1.6). For similar somewhat opposite behavior with respect to volume, see
Proposition 7.1. Both results are summarized in Table 1 below.

For a curve of metrics $t \to g_t$, let $R_{\min}(g_t) = \min_{x \in M}\{R(g_t)(x)\}$ denote the time-
dependent spatial minima of the scalar curvatures $R(g_t)$. Augmenting a result of
Hamilton [26] on increasing scalar curvature to manifolds of Yamabe type -1 or 0
gives the following.

Proposition 3.5 (Increasing scalar curvature) *Let $g : [0, T) \to \mathcal{M}^1$ be a solution
of the classical Ricci flow equation*

$$\frac{\partial g}{\partial t} = -2\operatorname{Ric}(g) + \tfrac{2}{n}\bar{R}(g)g\,, \qquad g_0 \in \mathcal{M}^1\,,$$

such that for each $t \in [0, T)$, $R_{\min}(g_t) \leq 0$. Then

$$\frac{d}{dt}R_{\min}(g_t) \geq 0\,, \tag{3.22}$$

so $R_{\min}(g_t)$ is an increasing function of t.

*If M is of Yamabe type -1 or 0, then (3.22) holds for all classical Ricci flows on
M (without any additional assumptions on $R_{\min}(g_t)$).*

Proof: From (2.16), (2.19), and (2.26), the scalar curvature $R(g)$ satisfies

$$\begin{aligned}
\frac{\partial R(g)}{\partial t} &= DR(g)\frac{\partial g}{\partial t} = DR(g)\left(-2\operatorname{Ric}(g) + \tfrac{2}{n}\bar{R}(g)g\right) \\
&= -\Delta_g R(g) + 2|\operatorname{Ric}(g)|_g^2 - \tfrac{2}{n}\bar{R}(g)R(g) \\
&= -\Delta_g R(g) + 2|\operatorname{Ric}^T(g)|_g^2 + \tfrac{2}{n}R^2(g) - \tfrac{2}{n}\bar{R}(g)R(g)\,. \tag{3.23}
\end{aligned}$$

Note that $\bar{R}(g)$ is a (time-dependent) real number (but is constant in x) whereas $R(g)$
is a (time-dependent) scalar function. Since $|\operatorname{Ric}^T(g)|_g^2 \geq 0$ and since at a minimum
point $x_{\min}(t) \in M$ of $R(g_t)$, $-\Delta_{g_t}R(g_t)(x_{\min}(t)) \geq 0$, (3.23) yields

$$\frac{d}{dt}R_{\min}(g) \geq \tfrac{2}{n}R_{\min}^2(g) - \tfrac{2}{n}\bar{R}(g)R_{\min}(g) = \tfrac{2}{n}R_{\min}(g)\left(R_{\min}(g) - \bar{R}(g)\right) \geq 0\,, \tag{3.24}$$

where the last inequality follows by the assumption that $R_{\min}(g) \leq 0$ and from the
inequality $R_{\min}(g) - \bar{R}(g) \leq 0$.

If M is of Yamabe type -1 or 0, then $R_{\min}(g) \leq 0$ for any g on M. For if
$R_{\min}(g) > 0$, then $R(g) > 0$ and thus by Yamabe's Theorem there would exist a
pointwise conformally equivalent metric $g' = pg$, $p \in \mathcal{P}$, with scalar curvature $R(g') =$
constant > 0 (see Aubin [4] and Schoen [39]), contradicting the assumption that M is

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where the solutions are no longer volume preserving,

$$\frac{\partial g}{\partial t} = -2\mathrm{Ric}(g)\,, \tag{4.2}$$

is **quasi-parabolic** and the problem of the non-ellipticity of the Ricci tensor has been overcome by Hamilton ([24],[25]) and DeTurck [7] who have shown that there exists a unique short-time solution to the initial value problem of (4.2) for any closed n-manifold, $n \geq 2$, with arbitrary initial metric $g_0 \in \mathcal{M}$.

The second issue regarding existence and uniqueness is that the conformal Ricci flow equations as well as being quasi-parabolic are also elliptic because the constraint $R(g) = -1$ leads to a linear inhomogeneous elliptic equation (3.6) for the conformal pressure p. Thus overall the conformal Ricci flow equations are a **linear-elliptic quasilinear quasi-parabolic system**. It is this latter issue of the additional elliptic equation for the conformal pressure that we address here.

We proceed as follows. We first show that the (fully) reduced conformal Ricci flow equation defined on $\mathcal{M}_{-1}$ is a smooth bounded perturbation of the unnormalized classical Ricci equation and thus is a vector field sum of a smooth and a densely defined vector field on $\mathcal{M}$. We then consider an evolution equation that involves only the smooth term. That this auxiliary evolution equation has a smooth flow then follows from the usual Picard iteration method for smooth vector fields on Hilbert (or Banach) manifolds. We then "add" back in the Ricci term to this auxiliary equation to get the reduced conformal Ricci flow equation. That this equation has a flow then follows from a nonlinear Trotter product formula.

This bootstrap method from a smooth flow to the flow of interest is inspired by a similar method developed by Ebin-Marsden [10] that shows existence and uniqueness of solutions to the Navier-Stokes equations. In this analogy, the Navier-Stokes equation is best viewed as the sum of the Euler equation and the heat equation taken on the tangent bundle $T\mathcal{D}_\mu$ of the group $\mathcal{D}_\mu$ of volume-preserving diffeomorphisms, where μ is the volume element of a fixed Riemannian metric (here we denote volume elements by μ, rather than $d\mu$). On $T\mathcal{D}_\mu$ there is no loss of derivatives for the Euler equation and in fact it is a smooth vector field and hence has a smooth flow by the usual methods of ordinary differential equations applied to Hilbert manifolds. To this smooth flow is added the heat equation, or more accurately, the heat equation and a Ricci curvature term (see Ebin-Marsden [10], p. 162). The resulting system then has a flow by a nonlinear Trotter product formula and, pulling back this flow to the space of divergence-free vector fields on M, proves existence and uniqueness for the classical Navier-Stokes equation.

In this analogy, the constraint phase space $T\mathcal{D}_\mu$ corresponds to the constraint space $\mathcal{M}_{-1}$, the smooth Euler equation on $T\mathcal{D}_\mu$ corresponds to the smooth auxiliary equation (4.7) below, the heat equation on $T\mathcal{D}_\mu$ corresponds to the unnormalized Ricci flow equation, and the Navier-Stokes equation on $T\mathcal{D}_\mu$ corresponds to the reduced conformal Ricci flow equation on $\mathcal{M}_{-1}$. However, a difference between these two systems is that whereas the components of the Navier-Stokes equation, namely, the Euler and heat equations, both live on the constraint space $T\mathcal{D}_\mu$, the components of the reduced conformal Ricci flow equation, the auxiliary equation (4.7) below and the unnormalized Ricci flow equation, do not separately flow in the constraint space $\mathcal{M}_{-1}$ (only their sum does; see Figure 1 in Section 5).

The reduced conformal Ricci flow equation can also be viewed as a reaction-diffusion equation. In this view the reaction part of the equation corresponds to the

smooth auxiliary equation (4.7) and the diffusion part corresponds to the unnormalized Ricci equation.

For $s > \frac{n}{2} + 1$, let $\mathcal{M}^s_{-1} = \{\, g \in \mathcal{M}^s \mid R(g) = -1 \,\}$.

Theorem 4.1 (Local existence and uniqueness of solutions for the conformal Ricci flow equations) *Let M be a closed connected oriented n-manifold, $n \geq 3$, $s > \frac{n}{2} + 2$, and $g_0 \in \mathcal{M}^s_{-1}$. Then there exists a $T > 0$, a C^0 curve $g : [0, T) \to \mathcal{M}^s$, C^1 as a curve $g : [0, T) \to \mathcal{M}^{s-2}$, and a C^0 curve $p : [0, T) \to \mathcal{F}^s$, such that g has initial condition $g(0) = g_0$ and such that g and p satisfy the conformal Ricci flow equations,*

$$\frac{\partial g}{\partial t} + 2\big(\mathrm{Ric}(g) + \tfrac{1}{n}g\big) = -\,pg \tag{4.3}$$

$$R(g) = -1. \tag{4.4}$$

If $s > \frac{n}{2} + 3$, then g and p are unique.

Proof: We sketch a proof that views the reduced conformal Ricci flow equation as a bounded perturbation of the unnormalized Ricci equation. Our starting point is to note that Proposition 3.4 asserts that the conformal Ricci initial value problem is equivalent to the reduced conformal Ricci initial value problem,

$$\frac{\partial g}{\partial t} + 2\mathrm{Ric}(g) = -2\big(L_g^{-1}(|\mathrm{Ric}(g)|_g^2)\big)g\,, \qquad g_0 \in \mathcal{M}^s_{-1}\,, \tag{4.5}$$

where associated with the yet-to-be-determined C^0 curve $g : [0, T) \to \mathcal{M}^s$ is the C^0 curve of invertible elliptic operators $L_g = (n-1)\Delta_g + 1 : \mathcal{F}^s \to \mathcal{F}^{s-2}$ with inverses $L_g^{-1} : \mathcal{F}^{s-2} \to \mathcal{F}^s$. Note that if such a curve g exists, then from Proposition 3.4 (adapted to the Sobolev setting), g maps to $\mathcal{M}^s_{-1}$ as a C^0 curve and to $\mathcal{M}^{s-2}_{-1}$ as a C^1 curve.

The reduced conformal Ricci flow equation (4.5) differs from the unnormalized Ricci flow equation (4.2) by the term $-2(L_g^{-1}(|\mathrm{Ric}(g)|_g^2))g$. Thus we define the difference vector field on $\mathcal{M}^s$,

$$Z : \mathcal{M}^s \longrightarrow S_2^s\,, \qquad g \longmapsto Z(g) = -2\big(L_g^{-1}(|\mathrm{Ric}(g)|_g^2)\big)g \tag{4.6}$$

and consider the corresponding "heat" equation

$$\frac{\partial g}{\partial t} = Z(g) = -2\big(L_g^{-1}(|\mathrm{Ric}(g)|_g^2)\big)g = -(\tfrac{2}{n} + p)g\,, \tag{4.7}$$

where $p = 2\,(L_g^{-1}(|\mathrm{Ric}(g)|^2) - \tfrac{1}{n})$. The key fact is that Z does not lose derivatives, i.e., Z actually maps H^s to H^s, and is a smooth vector field on the indicated spaces.

To show that Z maps H^s to H^s, we first make the following remarks. Let $g \in \mathcal{M}^s$, $s > \frac{n}{2} + 2$. Then the components of the inverse g^{-1} are a rational combination of the components of g with non-zero denominator $\det g$. Thus g^{-1} as a function of g is a smooth map. From the Schauder ring property of Sobolev functions, g^{-1} is an H^s contravariant metric and then from the local formula (1.10) for the Ricci tensor and the multiplicative properties of Sobolev functions, $\mathrm{Ric}(g) \in S_2^{s-2}$ and $|\mathrm{Ric}(g)|_g^2 \in \mathcal{F}^{s-2}$. Note that the Schauder ring property states that H^s is a ring if $s > \frac{n}{2}$. Thus to be able to conclude that $|\mathrm{Ric}(g)|_g^2$ is of class H^{s-2} requires $s > \frac{n}{2} + 2$ since $\mathrm{Ric}(g) \in S_2^{s-2}$.

Since $|\mathrm{Ric}(g)|_g^2 \in \mathcal{F}^{s-2}$, from ellipticity of L_g, $L_g^{-1}(|\mathrm{Ric}(g)|_g^2) \in \mathcal{F}^s$ and thus $Z(g) = -(L_g^{-1}(|\mathrm{Ric}(g)|_g^2))g \in S_2^s$. Thus, remarkably, Z does not lose derivatives relative to $g \in \mathcal{M}^s$. Although this does not prove that Z is smooth, it does make it plausible since now the range space does not have a weaker topology than the domain space.

To show that Z is smooth, we proceed as in Fischer-Marsden [13] and consider the Ricci tensor as a map, the **Ricci map** (or **operator**),

$$\text{Ric} : \mathcal{M}^s \longrightarrow S_2^{s-2} , \qquad g \longmapsto \text{Ric}(g) , \qquad (4.8)$$

which can also be interpreted as a vector field on $\mathcal{M}^{s-2}$ defined on the dense domain $\mathcal{M}^s \subset \mathcal{M}^{s-2}$. This vector field loses derivatives and is only an everywhere defined smooth vector field when $s = \infty$, in which case the theorem of ordinary differential equations on Banach manifolds fails.

First we show that for $s > \frac{n}{2}+1$, Ric is a smooth function on the indicated spaces. Because differentiation is a continuous linear map between the spaces indicated, the smoothness of Ric depends on the multiplications that occur in computing $\text{Ric}(g)$. The second-order derivatives appear linearly with components of g^{-1} as coefficients whose terms are various natural (i.e., non-metric) contractions of the homothetically invariant expression $g^{ij} \frac{\partial^2 g_{kl}}{\partial x^m \partial x^n}$ (i.e., invariant under the homothetic transformation $g \mapsto cg$, $c > 0$) which we write symbolically as $g^{-1} \otimes D^2 g$, to yield a symmetric 2-covariant expression. Since g^{-1} is a smooth function of g, by the multiplicative properties for Sobolev spaces, for $s > \frac{n}{2}$, the pointwise bilinear map $g^{-1} \otimes D^2 g$ induces a multiplication $H^s \times H^{s-2} \to H^{s-2}$ which is continuous bilinear and hence smooth. Thus the second order terms are smooth functions of g and we note that $s > \frac{n}{2}$ suffices for smoothness of the second-order terms.

The first order terms are various natural contractions of the homothetically invariant generic form $g_{ij} g^{kl} g^{mn} g^{pq} \frac{\partial g_{ab}}{\partial x^c} \frac{\partial g_{de}}{\partial x^f}$ to yield a symmetric 2-covariant expression. Thus the first order terms are rational combinations of the components of g with non-zero denominator $\det g$ and quadratic functions of Dg. Thus again by the multiplicative properties for Sobolev spaces, for $s > \frac{n}{2} + 1$, H^{s-1} is a ring and the pointwise multilinear map $g \otimes g^{-1} \otimes g^{-1} \otimes g^{-1} \otimes Dg \otimes Dg$ induces a multiplication $H^s \times H^s \times H^s \times H^s \times H^{s-1} \times H^{s-1} \to H^{s-1}$ which is continuous multilinear and hence smooth. Thus the first order terms are smooth functions of g and we note that since Dg appears quadratically, $s > \frac{n}{2} + 1$ is necessary for smoothness of the first-order terms. Thus Ric is C^∞ for the indicated spaces under the condition $s > \frac{n}{2} + 1$.

For $s > \frac{n}{2} + 2$, H^{s-2} is a ring and again by the multiplicative properties of Sobolev spaces, the pointwise squared Ricci norm map

$$\mathcal{M}^s \to S_2^{s-2} \times S_2^{s-2} \to \mathcal{F}^{s-2} , \qquad g \longmapsto (\text{Ric}(g), \text{Ric}(g)) \longmapsto |\text{Ric}(g)|_g^2$$

is a composition of the smooth map $g \mapsto \text{Ric}(g) \times \text{Ric}(g)$ and a multiplication $H^{s-2} \times H^{s-2} \to H^{s-2}$ which is continuous bilinear and hence smooth. Thus $g \mapsto |\text{Ric}(g)|_g^2$ is smooth. We note that since $\text{Ric}(g) \in S_2^{s-2}$ appears quadratically in $|\text{Ric}(g)|_g^2$, $s > \frac{n}{2} + 2$ is necessary for smoothness of $g \mapsto |\text{Ric}(g)|_g^2$.

In a similar manner, one shows that $g \mapsto L_g$ is smooth as a function of g and that the map $g \mapsto L_g^{-1}(|\text{Ric}(g)|_g^2)$ is smooth. Thus the vector field $Z : \mathcal{M}^s \longrightarrow S_2^s$, $g \mapsto -2(L_g^{-1}(|\text{Ric}(g)|_g^2))g$ is smooth. Since $\mathcal{M}^s$ is a smooth Hilbert manifold, the existence and uniqueness of local flows follows by the fundamental theorem of ordinary differential equations on Hilbert manifolds (see, for example, Lang [28], Theorem 2.6).

If we now consider the sum of the two vector fields -2Ric and Z,

$$-2\text{Ric} + Z : \mathcal{M}^s \longrightarrow S_2^{s-2} , \qquad g \longmapsto -2\text{Ric}(g) - 2\left(L_g^{-1}(|\text{Ric}(g)|_g^2)\right) g , \qquad (4.9)$$

the corresponding "heat" equation

$$\frac{\partial g}{\partial t} = -2\text{Ric}(g) - 2\left(L_g^{-1}(|\text{Ric}(g)|_g^2)\right)g \qquad (4.10)$$

is the reduced conformal Ricci flow equation. This equation does lose derivatives but this can be overcome by using a nonlinear Trotter product formula (see Ebin-Marsden [10], pp. 142, 157 and Taylor [40], Section 15.5). To do so, one approximates solutions of (4.10) by successively solving the two component equations

$$\frac{\partial g}{\partial t} = -2\big(L_g^{-1}(|\mathrm{Ric}(g)|_g^2)\big)g \tag{4.11}$$

$$\frac{\partial g}{\partial t} = -2\mathrm{Ric}(g) \tag{4.12}$$

over small time intervals and then composing the resulting solution operators.

As we have seen, the first equation is smooth and the second equation is the unnormalized Ricci flow equation which, as discussed earlier, is known to have unique local semi-flows $F_t : \mathcal{M}^s \to \mathcal{M}^s$, $t \in [0,\tau)$, $\tau > 0$, generated by the densely defined vector field $-2\mathrm{Ric} : \mathcal{M}^s \subset \mathcal{M}^{s-2} \to S_2^{s-2}$ (for existence and uniqueness results of the classical Ricci flow equation in the H^s-setting, see Fischer [11]). Thus the reduced conformal Ricci flow equation is the vector field sum of a smooth vector field and a densely defined vector field, the former of which has unique local flows by classical methods of ordinary differential equations and the latter of which is known to have unique local semi-flows by quasi-parabolic methods. The resulting vector field sum then fits the setting for the nonlinear Trotter product formula as it appears in Ebin-Marsden [10]. After checking the hypotheses of that formulation as they apply to (4.11–4.12), we can conclude that the reduced conformal Ricci equation also has unique local semi-flows.

That uniqueness requires $s > \frac{n}{2} + 3$ (instead of $s > \frac{n}{2} + 2$ as for existence) is discussed in Fischer [11]. ∎

Remarks:

(i) Each term on the left hand side of (4.3) is separately a C^0 curve in S_2^{s-2} whereas the right hand side (and thus the *sum* of the terms on the left) is a C^0 curve in S_2^s (actually in $-\mathcal{M}^s$ for non-static curves since $p > 0$). However, in the rearrangement $\frac{\partial g}{\partial t} = \mathrm{Ric}(g) - (p + \frac{2}{n}g)g$, each side is a C^0 curve in S_2^{s-2}.

(ii) Since $s > \frac{n}{2} + 2$, the solutions are genuine, i.e., the equations are satisfied in the classical sense.

(iii) Although the auxiliary equation (4.6) is smooth and in particular does not lose derivatives, the conformal Ricci flow equations themselves do lose derivatives since if g_t is H^s, $\mathrm{Ric}(g_t)$ is only H^{s-2}. ∎

By adding $\frac{2}{n}g$ to the right hand side of (4.11) and subtracting $\frac{2}{n}g$ from (4.12), (4.11–4.12) can be re-written in the more geometrical (but analytically equivalent) form

$$\frac{\partial g}{\partial t} = Z_1(g) = -2\big(L_g^{-1}(|\mathrm{Ric}(g)|_g^2)\big)g + \tfrac{2}{n}g = -pg \tag{4.13}$$

$$\frac{\partial g}{\partial t} = Z_2(g) = -2(\mathrm{Ric}(g) + \tfrac{1}{n}g), \tag{4.14}$$

where $p = p(g) = 2L_g^{-1}(|\mathrm{Ric}(g)|_g^2) - \frac{2}{n}$ and where the vector field sum of the right hand sides is again the reduced conformal Ricci flow equation. Analytically, the first vector field is smooth (see Proposition 5.5) and thus has a local flow. The second vector field is densely defined on $\mathcal{M}^s$ but modulo the bounded perturbation term $-\frac{2}{n}g$ is the unnormalized Ricci flow equation and thus also has unique local semi-flows.

The vector fields (4.13–4.14) have the geometrical feature that they are pointwise orthogonal on $\mathcal{M}_{-1}$ since

$$Z_1(g) \cdot Z_2(g) = (-pg) \cdot -2(\mathrm{Ric}(g) + \tfrac{1}{n}g) = 2p(R(g) + 1) = 0 \,. \quad (4.15)$$

We shall further explore this feature in Section 5. Here we note that neither vector field Z_1 nor Z_2 preserves the volume nor the scalar curvature of the initial metric $g(0) = g_0$ but that Z_1 preserves the pointwise conformal class of the initial metric. Thus if g is a solution to (4.13), then $\frac{\partial g(t)}{\partial t} = -p(g(t))g(t)$ so that g must satisfy

$$g(t) = g_0 e^{-\int_0^t p(g(t'))dt'} \in Pg_0 \,. \quad (4.16)$$

Thus the flow of (4.13) lies in the $\mathcal{P}$-orbit of the initial metric g_0 and thus preserves the pointwise conformal class of g_0.

We also remark that a split-step algorithm based on the pair of equations (4.13–4.14) can also be used as a basis for a numerical solution of the reduced conformal Ricci flow equation.

Lastly, we note that (4.14) is similar in structure to the historically first Ricci flow equation (see Hamilton [24], p. 256),

$$\frac{\partial g}{\partial t} = -2(\mathrm{Ric}(g) - \tfrac{1}{n}R(g)g) \,, \quad (4.17)$$

which has a backwards heat equation in $R(g)$. Indeed, calculating as in (3.23), the scalar curvature of a solution g to (4.17) must satisfy

$$
\begin{aligned}
\frac{\partial}{\partial t}R(g) = DR(g)\frac{\partial g}{\partial t} &= DR(g)\left(-2\,\mathrm{Ric}(g) + \tfrac{2}{n}R(g)g\right) \\
&= -\Delta_g R(g) + 2|\mathrm{Ric}(g)|_g^2 + \tfrac{2}{n}DR(g)(R(g)g) \\
&= -\Delta_g R(g) + 2|\mathrm{Ric}(g)|_g^2 + \tfrac{2}{n}(n-1)\Delta_g R(g) - \tfrac{2}{n}R^2(g) \\
&= \tfrac{n-2}{n}\Delta_g R(g) + 2|\mathrm{Ric}(g)|_g^2 - \tfrac{2}{n}R^2(g) \,, \quad (4.18)
\end{aligned}
$$

where, because of the change in sign of the Laplacian term from $-\Delta_g$ to $\tfrac{2}{n}(n-1)\Delta_g$, is a backwards heat equation in $R(g)$. Thus (4.17) *in general* cannot have short-time solutions. However, in the locally homogeneous case (i.e., where there exists a local isometry between neighborhoods U_x, U_y of every pair of points x, y in (M, g)), $R(g) = \bar{R}(g)$ and so (4.17) is equal to the classical Ricci flow equation and thus is well-posed (note that in this case $\Delta_g R(g) = 0$; see also Proposition 8.1 and the remark following that Proposition).

Although (4.17) is similar to (4.14) (and the equations are identical when $R(g) = -1$), (4.14) does not suffer from a backwards heat equation in $R(g)$, since

$$
\begin{aligned}
\frac{\partial}{\partial t}R(g) = DR(g)\frac{\partial g}{\partial t} &= DR(g)\left(-2\,\mathrm{Ric}(g) - \tfrac{2}{n}g\right) \\
&= -\Delta_g R(g) + 2|\mathrm{Ric}(g)|_g^2 - \tfrac{2}{n}DR(g)(g) \\
&= -\Delta_g R(g) + 2|\mathrm{Ric}(g)|_g^2 - \tfrac{2}{n} \,, \quad (4.19)
\end{aligned}
$$

which is a proper heat equation for $R(g)$.

Similarly, we remark in passing that for the conformal Ricci flow equations, since $R(g) = -1$, there is no issue of whether or not $R(g)$ solves a backwards or proper heat equation.

5. The geometry of the conformal Ricci flow

To put the fact that the conformal Ricci flow equations do not lose derivatives *relative* to the unnormalized Ricci equation into further geometrical perspective, we consider a non-orthogonal L_2-splitting of S_2^s. We then apply this splitting to $\mathrm{Ric}(g)$ and show why this splitting considerably simplifies when the constraint $R(g) = -1$ is satisfied. Indeed, as we shall see, when $R(g) = -1$, the map $g \mapsto -pg$ does not lose derivatives and is smooth as a function of g. Thus the reduced conformal Ricci flow equation is a bounded perturbation of the unnormalized Ricci equation.

For $n \geq 3$ and $s > \frac{n}{2} + 1$, we continue to let

$$R : \mathcal{M}^s \longrightarrow \mathcal{F}^{s-2} , \quad g \longmapsto R(g) \tag{5.1}$$

denote the smooth scalar curvature map in the Sobolev setting, with derivative at $g \in \mathcal{M}^s$

$$DR(g) : S_2^s \longrightarrow \mathcal{F}^{s-2} , \tag{5.2}$$

and with $\bar{S}_2^s(g) = \ker DR(g) \subset S_2^s$ the kernel of $DR(g)$.

If $g \in \mathcal{M}^s$ and $p \in \mathcal{P}^s$, then the product $pg \in \mathcal{M}^s$. Let

$$\mathcal{P}^s g = \{ pg \mid p \in \mathcal{P}^s \} \subset \mathcal{M}^s \tag{5.3}$$

denote the space of **(pointwise) conformal deformations of** g, or the **orbit under the group action of** $\mathcal{P}^s$ **on** $\mathcal{M}^s$. The orbit $\mathcal{P}^s g$ is a closed submanifold of $\mathcal{M}^s$.

If $g \in \mathcal{M}^s$ and $\varphi \in \mathcal{F}^s$, then the product $\varphi g \in S_2^s$. Then the tangent space

$$T_g(\mathcal{P}^s g) \approx \mathcal{F}^s g = \{ \varphi g \mid \varphi \in \mathcal{F}^s \} \subset S_2^s \tag{5.4}$$

is the space of **infinitesimal (pointwise) conformal deformations of** g. Both $\bar{S}_2^s(g)$ and $\mathcal{F}^s g$ are closed subspaces of S_2^s.

Let

$$S_2^T(g)^s = \{ h \in S_2^s \mid \mathrm{tr}_g h = 0 \}$$

denote the space of traceless (with respect to $g \in \mathcal{M}^s$) symmetric 2-covariant tensor fields on M. Thus we have the pointwise orthogonal (and hence L_2-orthogonal) splitting

$$S_2^s = \mathcal{F}^s g \oplus S_2^T(g)^s , \qquad h = \tfrac{1}{n}(\mathrm{tr}_g h)g + h^T \tag{5.5}$$

where $h^T = h - \frac{1}{n}(\mathrm{tr}_g h)g$ is the traceless part of h. This splitting can be written more geometrically as

$$T_g \mathcal{M}^s = T_g(\mathcal{P}^s g) \oplus T_g^\perp(\mathcal{P}^s g) , \tag{5.6}$$

where $T_g^\perp(\mathcal{P}^s g) \approx S_2^T(g)^s$ represents the directions orthogonal to the infinitesimal pointwise conformal deformations of g. We also remark that at $g \in \mathcal{M}^s$, the orthogonal space $T_g^\perp(\mathcal{P}^s g)$ can be expressed as the tangent space of the **manifold of metrics** $\mathcal{M}_{d\mu_g}$ **with fixed volume element** $d\mu_g$, the volume element of g (see Figure 1). Thus, if

$$\mathcal{M}_{d\mu_g} = \{ g' \in \mathcal{M}^s \mid d\mu_{g'} = d\mu_g \} , \tag{5.7}$$

then $T_g \mathcal{M}_{d\mu_g}^s \approx S_2^T(g) \approx T_g^\perp(\mathcal{P}^s g)$ since $D(d\mu_g)h = \frac{1}{2}(\mathrm{tr}_g h)d\mu_g = 0$.

Now we combine the two orthogonal splittings (2.23) and (5.5) to give a third non-orthogonal splitting.

Proposition 5.1 (A non-orthogonal L_2-splitting of S_2^s) *For $n \geq 3$, $s > \frac{n}{2} + 1$, and $g \in \mathcal{M}^s$, assume that the scalar curvature $R(g)$ is such that the operator*

$$L_g : \mathcal{F}^s \longrightarrow \mathcal{F}^{s-2} , \quad \varphi \longmapsto L_g\varphi = (n-1)\Delta_g\varphi - R(g)\varphi \qquad (5.8)$$

is an isomorphism. Let $L_g^{-1} : \mathcal{F}^{s-2} \longrightarrow \mathcal{F}^s$ denote its inverse. Then S_2^s splits into a non-orthogonal direct sum

$$S_2^s = \ker DR(g) \oplus \mathcal{F}^s g = \bar{S}_2^s(g) \oplus \mathcal{F}^s g , \qquad h = \tilde{h} + \varphi g \qquad (5.9)$$

where $\varphi = L_g^{-1}(DR(g)h) \in \mathcal{F}^s$, $\varphi g \in \mathcal{F}^s g$, and

$$\tilde{h} = h - \varphi g = h - (L_g^{-1}(DR(g)h))g \in \bar{S}_2^s(g) . \qquad (5.10)$$

Equivalently, but more geometrically, if $\rho = R(g) \in \mathcal{F}^{s-2}$ is such that $\mathcal{M}_\rho = R^{-1}(\rho) = \{ g \in \mathcal{M}^s \mid R(g) = \rho \}$ is a submanifold of $\mathcal{M}^s$, the tangent space $T_g\mathcal{M}^s$ of $\mathcal{M}^s$ at g splits

$$T_g\mathcal{M}^s = T_g\mathcal{M}_\rho^s \oplus T_g(\mathcal{P}^s g) \qquad (5.11)$$

into two non-orthogonal tangent spaces corresponding to the two closed submanifolds $\mathcal{M}_\rho^s$ and $\mathcal{P}^s g$ of $\mathcal{M}^s$ intersecting transversally at g (see Figure 1).

Notational remark: When we let $\bar{S}_2^s(g) = \ker DR(g)$, we are considering the kernel of the map $DR(g) : S_2^s \to \mathcal{F}^{s-2}$. Also, for $h \in S_2^s$, we use $\tilde{h} \in \bar{S}_2^s$ to denote the component of h in $\bar{S}_2^s$ with respect to this non-orthogonal splitting, in contrast to $\bar{h} \in \bar{S}_2^s$ which denotes the component of h in $\bar{S}_2^s$ using the L_2-orthogonal splitting given by 2.23.

Proof: For $g \in \mathcal{M}^s$, we first note that

$$\bar{S}_2^s(g) \cap \mathcal{F}^s g = \{0\} , \qquad (5.12)$$

since if $\tilde{h} = \varphi g \in \bar{S}_2^s(g) \cap \mathcal{F}^s g$, then since $s > \frac{n}{2} + 1$, $R : \mathcal{M}^s \to \mathcal{F}^{s-2}$ is a smooth mapping and from (2.12),

$$DR(g)(\varphi g) = (n-1)\Delta_g\varphi - R(g)\varphi = L_g\varphi = 0 . \qquad (5.13)$$

Since the curvature assumption on $R(g)$ is that L_g is an isomorphism, (5.13) has unique solution $\varphi = 0$.

To show that the direct sum $\bar{S}_2^s \oplus \mathcal{F}^s g$ of the indicated closed subspaces exhausts S_2^s, for $h \in S_2^s$ let $\varphi = L_g^{-1}(DR(g)h)$. Since $DR(g)h \in \mathcal{F}^{s-2}$, by the ellipticity of L_g, $\varphi = L_g^{-1}(DR(g)h) \in \mathcal{F}^s$, $\varphi g = L_g^{-1}(DR(g)h)g \in \mathcal{F}^s g$, and thus

$$\tilde{h} = h - \varphi g \in S_2^s \qquad (5.14)$$

We need to show that $\tilde{h} \in \ker DR(g) = \bar{S}_2^s$, which follows from

$$DR(g)\tilde{h} = DR(g)h - DR(g)(\varphi g) = DR(g)h - L_g\varphi$$
$$= DR(g)h - L_g(L_g^{-1}(DR(g)h)) = 0$$

Thus $h = \tilde{h} + \varphi g$ splits h according to (5.9). ∎

Note that although neither the L_2-orthogonal splitting (2.23) nor the pointwise orthogonal splitting (5.5) requires a curvature condition on g, the L_2-non-orthogonal splitting (5.9) does since it is required that g is such that $L_g = (n-1)\Delta_g - R(g)$ is an isomorphism.

Note also that the summands in the L_2-non-orthogonal splitting (5.11) of S_2^s can be interpreted geometrically as a splitting of the tangent space $T_g\mathcal{M}^s$ into the tangent

spaces of two closed transversally intersecting submanifolds of $\mathcal{M}^s$, namely, $\mathcal{M}^s_\rho$ and $\mathcal{P}^s g$.

Associated with the splitting (5.9) $h = \tilde{h} + \varphi g$, we let

$$\widetilde{P}_g : S_2^s \longrightarrow \bar{S}_2^s, \quad h \longmapsto \widetilde{P}_g(h) = \tilde{h} = h - \varphi g = h - \left(L_g^{-1}(DR(g)h) \right) g \qquad (5.15)$$

denote the projection onto $\bar{S}_2^s(g) = \ker DR(g)$ and refer to $\tilde{h}$ as the **tangential component** (or **part**) of h and φg as the **(infinitesimal) conformal component** (or **part**) of h. We use this terminology since, generically, for $\rho \in \mathcal{F}^{s-2}$, $\mathcal{M}^s_\rho = \{\, g \in \mathcal{M}^s \mid R(g) = \rho \,\}$ is a closed submanifold of $\mathcal{M}^s$ with tangent space at $g \in \mathcal{M}^s_\rho$ given by $T_g \mathcal{M}_\rho \approx \ker DR(g)$. Thus these directions infinitesimally preserve the scalar curvature. Similarly, the space of (pointwise) conformal deformations $\mathcal{P}^s g$ of g is a closed submanifold of $\mathcal{M}^s$ with tangent space $T_g(\mathcal{P}^s g) \approx \mathcal{F}^s g$ so that the (infinitesimal) conformal directions preserve the pointwise conformal class of g. Although the conformal directions are also tangential directions (to $\mathcal{P}^s g$), by the tangential part of $h \in S_2^s$ we shall mean the tangential directions to $\mathcal{M}^s_{-1}$ since this is the constraint space for the conformal Ricci flow.

Example 5.2 (Splitting of $\mathrm{Ric}(g)$**)** (Compare Example 2.2.) As an important example, for $n \geq 3$, $s > \frac{n}{2} + 3$, $g \in \mathcal{M}^s$, we split the Ricci tensor $h = \mathrm{Ric}(g) \in S_2^{s-2}$ of g according to (5.9) under the assumption that $L_g = (n-1)\Delta_g - R(g)$ is an isomorphism. Note that $s - 2 > \frac{n}{2} + 1$ so the hypothesis on s of Proposition 5.1 is satisfied. Thus $\mathrm{Ric}(g)$ splits as

$$\mathrm{Ric}(g) = \widetilde{\mathrm{Ric}(g)} + \varphi g, \qquad (5.16)$$

where from (2.19), $DR(g)\mathrm{Ric}(g) = \frac{1}{2}\Delta_g R(g) - |\mathrm{Ric}(g)|^2 \in \mathcal{F}^{s-4}$ since $|\mathrm{Ric}(g)|^2$ and $R(g)$ are of class H^{s-2} so that $\Delta_g R(g) \in \mathcal{F}^{s-4}$. Then, by the ellipticity of L_g,

$$\varphi = L_g^{-1}(DR(g)\mathrm{Ric}(g)) = L_g^{-1}\left(\tfrac{1}{2}\Delta_g R(g) - |\mathrm{Ric}(g)|^2 \right) \in \mathcal{F}^{s-2}, \ (5.17)$$

so that $\varphi g \in S_2^{s-2}$. Thus $\mathrm{Ric}(g)$ splits as

$$\mathrm{Ric}(g) = \widetilde{\mathrm{Ric}(g)} + \left(L_g^{-1}\left(\tfrac{1}{2}\Delta_g R(g) - |\mathrm{Ric}(g)|^2 \right) \right) g, \qquad (5.18)$$

where

$$\widetilde{P}_g(\mathrm{Ric}(g)) = \widetilde{\mathrm{Ric}(g)} = \mathrm{Ric}(g) - \varphi g \in \bar{S}_2^{s-2}. \qquad (5.19)$$

Note that each summand of $\mathrm{Ric}(g) \in S_2^{s-2}$ is also of class H^{s-2} as required by the splitting. ∎

We remark that $\mathrm{Ric}(g)$ splits non-orthogonally into two non-trivial parts (see also Figure 1 below). This is in contrast with the L_2-orthogonal splitting (2.23) where $\mathrm{Ric}(g) \in \mathrm{range}\, DR^*(g)$ and thus does not split non-trivially (see Example 2.2).

In the splitting (5.16), since $\mathrm{Ric}(g)$ is H^{s-2}, each term in the splitting is, as expected, also H^{s-2}. However, relative to $g \in \mathcal{M}^s$, this splitting loses two derivatives, which is also expected since $\mathrm{Ric}(\cdot)$ is a second-order operator. What is completely unexpected is the following.

Example 5.3 (Splitting of $\mathrm{Ric}(g)$ **when** $R(g) = -1$**)** Suppose $g \in \mathcal{M}^s_{-1}$, $s > \frac{n}{2} + 3$. Then $L_g = (n-1)\Delta_g + 1$ is an isomorphism and thus the necessary curvature condition on g for the splitting is satisfied. Thus we can split $\mathrm{Ric}(g) \in S_2^{s-2}$ as above.

When the constraint is not satisfied, the argument $DR(g)\mathrm{Ric}(g) = \frac{1}{2}\Delta_g R(g) - |\mathrm{Ric}(g)|^2$ of L_g^{-1} in (5.18) consists of two terms, the first of which involves fourth

order derivatives of g and the second of which involves only second order derivatives of g. Thus the loss of derivatives comes from the first term involving the fourth order derivatives of g.

Now suppose that g satisfies the constraint equation $R(g) = -1$. Then the term $\Delta_g R(g) = 0$ involving the fourth order derivatives of g drops out and thus the remaining argument $-|\text{Ric}(g)|^2$ of L_g^{-1} is H^{s-2} rather than H^{s-4} as before. Thus from the ellipticity of L_g, $\varphi = L_g^{-1}(DR(g)\text{Ric}(g)) = -L_g^{-1}(|\text{Ric}(g)|^2) \in \mathcal{F}^s$ rather than $\mathcal{F}^{s-2}$ as before and $\varphi g \in \mathcal{F}^s g$ rather than $\mathcal{F}^{s-2} g$ as before. Thus (5.18) reduces to

$$\text{Ric}(g) = \widetilde{\text{Ric}}(g) - (L_g^{-1}(|\text{Ric}(g)|^2))g \,. \tag{5.20}$$

Thus, remarkably, if the constraint $R(g) = -1$ is satisfied, $\varphi g = -(L_g^{-1}(|\text{Ric}(g)|^2))g = \text{Ric}(g) - \widetilde{\text{Ric}}(g) = (I - \widetilde{P}_g)\text{Ric}(g) \in \mathcal{F}^s g$ does not lose derivatives relative to $g \in \mathcal{M}^s$.

Note also that since $g \in \mathcal{M}^s$, we do not expect that φg can in general be any smoother than H^s as it is in this case. ∎

It is important to remark that this improvement in differentiability of $\text{Ric}(g) - \widetilde{\text{Ric}}(g)$ only takes place in the special case when $\Delta_g R(g)$ in (5.17) is of class H^{s-2} (instead of H^{s-4}), as occurs for example when the constraint equation $R(g) = -1$ is satisfied. We also note that the individual terms $\text{Ric}(g)$ and $\widetilde{\text{Ric}}(g) = \text{Ric}(g) - \varphi g$ are still H^{s-2} as before. It is only the difference $\varphi g = \text{Ric}(g) - \widetilde{\text{Ric}}(g)$ that is H^s rather than H^{s-2}.

Thus, summarizing, if we compare the map $g \mapsto (I - P_g)\text{Ric}(g) = \text{Ric}(g) - \widetilde{\text{Ric}}(g)$ when the constraint equation is not satisfied with when it is satisfied, we have

$$\mathcal{M}^s \longrightarrow S_2^{s-2}\,, \quad g \longmapsto \text{Ric}(g) - \widetilde{\text{Ric}}(g) = \left(L_g^{-1}\left(\tfrac{1}{2}\Delta_g R(g) - |\text{Ric}(g)|^2\right)\right)g = \varphi g \tag{5.21}$$

$$\mathcal{M}_{-1}^s \longrightarrow S_2^s\,, \quad g \longmapsto \text{Ric}(g) - \widetilde{\text{Ric}}(g) = -\left(L_g^{-1}(|\text{Ric}(g)|^2)\right)g = \varphi g\,. \tag{5.22}$$

Thus, in general, if the constraint equation is not satisfied, (5.21) maps H^s Riemannian metrics to H^{s-2} tensors with loss of derivatives whereas if the constraint $R(g) = -1$ is satisfied, (5.22) maps H^s Riemannian metrics to H^s tensors $\varphi g \in \mathcal{F}^s g \subset S_2^s$ without loss of derivatives.

This improvement when $R(g) = -1$ is the basis for the fact that the conformal Ricci flow does not lose derivatives relative to the unnormalized Ricci flow. This discussion can be rephrased in terms of the conformal pressure as follows.

Example 5.4 (Splitting of $-2(\text{Ric}(g) + \tfrac{1}{n}g)$ when $R(g) = -1$) Continuing with the assumptions of Example 5.3, we split $-2(\text{Ric}(g) + \tfrac{1}{n}g)$ when $R(g) = -1$. From (3.7) and (5.20),

$$-2(\text{Ric}(g) + \tfrac{1}{n}g) = -2\widetilde{\text{Ric}}(g) + \left(2L_g^{-1}(|\text{Ric}(g)|^2) - \tfrac{2}{n}\right)g$$

$$= -2\widetilde{\text{Ric}}(g) + pg\,, \tag{5.23}$$

where $-2\widetilde{\text{Ric}}(g)$ is the tangential component, the pressure term pg is the (infinitesimal) conformal component, and $p = 2(L_g^{-1}(|\text{Ric}(g)|^2) - \tfrac{1}{n})$. ∎

By rearranging (5.23), we can write the splitting of $-2\text{Ric}(g)$ as

$$-2\text{Ric}(g) = -2\widetilde{\text{Ric}}(g) + (p + \tfrac{2}{n})g\,, \tag{5.24}$$

with tangential component $-2\widetilde{\text{Ric}}(g)$ and conformal component $(p + \tfrac{2}{n})g$. Also, for the purpose of Figure 1, we re-write (5.23) as

$$-2\widetilde{\text{Ric}}(g) = -2(\text{Ric}(g) + \tfrac{1}{n}g) - pg \tag{5.25}$$

Thus, from (5.25), the conformal Ricci equation can be written in terms of the projection map $\widetilde{P}_g$ as follows,

$$
\begin{aligned}
0 &= \frac{\partial g}{\partial t} + 2(\mathrm{Ric}(g) + \tfrac{1}{n}g) + pg \\
&= \frac{\partial g}{\partial t} + 2\widetilde{\mathrm{Ric}}(g) \\
&= \frac{\partial g}{\partial t} + 2\widetilde{P}_g(\mathrm{Ric}(g)) \,.
\end{aligned}
\tag{5.26}
$$

Since $\widetilde{P}_g(g) = 0$, (5.26) can also be written as $\frac{\partial g}{\partial t} + 2\widetilde{P}_g(\mathrm{Ric}(g) + \tfrac{1}{n}g) = 0$.

The "vector" relationships (5.23), (5.24), and (5.25) are depicted in Figure 1 (see page 51). Note in particular that $-2\mathrm{Ric}(g)$ is orthogonal to $T_g\mathcal{M}_{-1}$ and that on the right hand side of (5.25), the two summands $-2(\mathrm{Ric}(g) + \tfrac{1}{n}g) \in S_2^T(g)$ and $-pg \in \mathcal{F}g \subset S_2$ are (pointwise) orthogonal to each other since $R(g) = -1$ (see (4.15)).

Thus we have the final geometrical picture. The non-orthogonal splitting (5.9) decomposes the nonlinear restoring force $-2(\mathrm{Ric}(g) + \tfrac{1}{n}g)$ into a conformal component pg and a tangential component $-2\widetilde{\mathrm{Ric}}(g)$ (see (5.23)). Equivalently, (5.25) resolves the inertial "vector" $\frac{\partial g}{\partial t} = -2\widetilde{\mathrm{Ric}}(g) = -2(\mathrm{Ric}(g) + \tfrac{1}{n}g) - pg \in \ker DR(g)$ into orthogonal components where the constraint force $-pg \in \mathcal{F}g \approx T_g(\mathcal{P}g)$ acts (pointwise) orthogonally to the nonlinear restoring force $-2(\mathrm{Ric}(g) + \tfrac{1}{n}g) \in S_2^T(g) \approx T_g^{\perp}(\mathcal{P}g)$ and counter-balances the conformal component pg of this force (in the non-orthogonal splitting). The scalar curvature of the resulting conformal Ricci flow is therefore preserved. The resolution of $\dot{g} = \frac{\partial g}{\partial t}$ into these two components is shown in Figure 1 (see page 51).

We also remark that in Figure 1 we have represented the "thick" spaces $\mathcal{M}_{-1}$ and $\mathcal{M}_{d\mu_g}$, which are codimension-$C^{\infty}(M, \boldsymbol{R})$ in $\mathcal{M}$, as 2-dimensional spaces, whereas we have represented the "thin spaces" $\mathcal{P}g$ and $T_g^{\perp}\mathcal{M}_{-1} \approx \mathrm{range}\, DR(g)^* = DR(g)^*(\mathcal{F})$, which have the dimensionality of $C^{\infty}(M, \boldsymbol{R}) = \mathcal{F}$, as 1-dimensional spaces. These thin spaces are represented by thin solid lines whereas the dotted lines are construction lines.

The conformal pressure term $-pg$ measures the "force" or pressure that the metric experiences by being constrained to lie in $\mathcal{M}_{-1}$. If there were no pressure term, the metric would follow the flow lines of the equation $\frac{\partial g}{\partial t} + 2(\mathrm{Ric}(g) + \tfrac{1}{n}g) = 0$ and the resulting motion would be neither volume nor scalar curvature preserving.

From the point of view of the non-orthogonal splitting (5.23), the nonlinear restoring term $-2(\mathrm{Ric}(g) + \tfrac{1}{n}g)$ has a component $-2\widetilde{\mathrm{Ric}}(g)$ in the tangential direction of $\mathcal{M}_{-1}$ and a transverse component pg in the infinitesimal conformal direction $T_g(\mathcal{P}g) \approx \mathcal{F}g$. The conformal pressure term $-pg$ then acts oppositely to provide the necessary counter-force to balance the transversal component pg of $-2(\mathrm{Ric}(g) + \tfrac{1}{n}g)$, thereby keeping the flow in $\mathcal{M}_{-1}$.

Note that $-2(\mathrm{Ric}(g) + \tfrac{1}{n}g) \in S_2^T(g) \approx T_g^{\perp}(\mathcal{P}g)$ and thus has no $\mathcal{P}g$ component when split orthogonally by (5.6) but when split non-orthogonally its $\mathcal{P}g$ component is pg. This is analogous to the fact that $-2\mathrm{Ric}(g) \in \mathrm{range}\, DR(g)^* \approx T_g\mathcal{M}_{-1}^{\perp}$ and thus has no $T_g\mathcal{M}_{-1}$ component when split orthogonally by (2.23) but when split non-orthogonally its $T_g\mathcal{M}_{-1}$ component is $-2\widetilde{\mathrm{Ric}}(g)$.

Note that if we write (5.26) as $\frac{\partial g}{\partial t} = -2\widetilde{P}_g(\mathrm{Ric}(g)) = -2\mathrm{Ric}(g) - \tfrac{2}{n}g - pg$, then the right hand side has zero net conformal component, which can be viewed in two different ways. Either pg is the conformal part of $-2(\mathrm{Ric}(g) + \tfrac{1}{n}g)$ or $(p + \tfrac{2}{n})g$ is the

conformal part of $-2\mathrm{Ric}(g)$, as shown in Figure 1. Since we view the system without the constraint as being $\frac{\partial g}{\partial t} + 2(\mathrm{Ric}(g) + \frac{1}{n}g) = 0$ (and not the unnormalized Ricci equation $\frac{\partial g}{\partial t} + 2\mathrm{Ric}(g) = 0$), we adhere to the first interpretation.

If we consider the conformal pressure as a map, also denoted p,

$$p : \mathcal{M}^s_{-1} \longrightarrow \mathcal{F}^s , \qquad g \longmapsto p(g) = 2L_g^{-1}\big(|\mathrm{Ric}(g)|^2_g\big) - \tfrac{2}{n} , \qquad (5.27)$$

then p is a smooth map which is everywhere defined and does not lose derivatives.

Summarizing,

Proposition 5.5 (The constraint force for the conformal Ricci flow does not lose derivatives) *If $g \in \mathcal{M}^s_{-1}$, $s > \frac{n}{2} + 3$, then $2(\mathrm{Ric}(g) + \frac{1}{n}g)$ and $2\widetilde{P}_g(\mathrm{Ric}(g) + \frac{1}{n}g)$ are H^{s-2} but their difference*

$$2(I - \widetilde{P}_g)(\mathrm{Ric}(g) + \tfrac{1}{n}g) = 2(\mathrm{Ric}(g) + \tfrac{1}{n}g - \widetilde{P}_g(\mathrm{Ric}(g))) = -pg$$

is H^s and not merely H^{s-2}, the same differentiability class of g. The conformal pressure

$$p : \mathcal{M}^s_{-1} \longrightarrow \mathcal{F}^s , \quad g \longmapsto 2\,L_g^{-1}(|\mathrm{Ric}(g)|^2) - \tfrac{2}{n} \qquad (5.28)$$

is a smooth map on the indicated spaces and the associated conformal pressure constraint force

$$\mathcal{M}^s_{-1} \longrightarrow S^s_2 , \quad g \longmapsto -pg \qquad (5.29)$$

is a smooth vector field on $\mathcal{M}^s_{-1}$ in the conformal direction.

The reduced conformal Ricci equation can be written as

$$\frac{\partial g}{\partial t} + 2\widetilde{P}_g(\mathrm{Ric}(g)) = 0 , \qquad (5.30)$$

where $\widetilde{P}_g : S^s_2 \to \bar{S}^s_2(g) = \ker DR(g)$ is the projection onto the $\bar{S}^s_2$ component of the non-orthogonal splitting $S^s_2 = \bar{S}^s_2 \oplus \mathcal{F}^s g \approx T_g\mathcal{M}^s_{-1} \oplus T_g(\mathcal{P}^s g)$. ∎

Lastly, we remark on the general structure of the conformal Ricci flow equations. One might try to use the L_2-orthogonal projection (2.23) instead of the non-orthogonal projection of (5.30) in order to constrain the Ricci flow to $\mathcal{M}_{-1}$. Thus, for example, one might consider the system

$$\frac{\partial g}{\partial t} + 2\,\mathrm{Ric}(g) = -2DR(g)^*\phi \qquad (5.31)$$

$$R(g) = -1 \qquad (5.32)$$

where the function ϕ is to be solved so as to maintain the flow in $\mathcal{M}_{-1}$ (in other words, one might consider the equation $\frac{\partial g}{\partial t} + 2\bar{P}_g(\mathrm{Ric}(g)) = 0$ where $\bar{P}_g$ is L_2-orthogonal projection onto $\ker DR(g) \approx T_g\mathcal{M}_{-1}$). However, the Ricci tensor is orthogonal to $\mathcal{M}_{-1}$, $\mathrm{Ric}(g) \in \mathrm{range}\,DR(g)^* \approx T^\perp_g\mathcal{M}_{-1}$, so that $\bar{P}_g(\mathrm{Ric}(g)) = 0$. Thus one cannot non-trivially constrain the Ricci flow to $\mathcal{M}_{-1}$ by orthogonal projection of $\mathrm{Ric}(g)$. To put this another way, for every $g \in \mathcal{M}_{-1}$, if $\phi = 1$, then $-2\mathrm{Ric}(g) - 2DR(g)^*(1) = -2\mathrm{Ric}(g) - (-2\mathrm{Ric}(g)) = 0$ so that every point of $\mathcal{M}_{-1}$ is a critical point for (5.31–5.32). Thus to get a non-trivial *conformal* Ricci flow in $\mathcal{M}_{-1}$, we have had to use a non-orthogonal projection onto $T_g\mathcal{M}_{-1}$.

6. A variational approach to the conformal Ricci flow equations

In this section we discuss a variational approach to the conformal Ricci flow equation.

Let $v_g = \mathrm{vol}(M, g)$ denote the volume of the Riemannian manifold (M, g). The *Yamabe functional* is defined by

$$Y : \mathcal{M} \longrightarrow \mathbf{R}, \quad g \longmapsto \frac{\int_M R(g)d\mu_g}{(\int_M d\mu_g)^{(n-2)/n}} = v_g^{2/n}\frac{R_{\text{total}}(g)}{v_g} = v_g^{2/n}\bar{R}(g)$$

where $\bar{R}(g) = v_g^{-1}R_{\text{total}}(g)$ is the **volume-averaged total scalar curvature**. Thus the Yamabe functional Y is a **volume-normalized total scalar curvature functional** weighted to be invariant under homothetic transformations of g, i.e., if $c \in \mathbf{R}^+$, then

$$Y(cg) = \frac{\int_M R(cg)d\mu_{cg}}{v_{cg}^{(n-2)/n}} = \frac{c^{-1}c^{n/2}}{(c^{n/2})^{(n-2)/n}}\frac{\int_M R(g)d\mu_g}{v_g^{(n-2)/n}} = Y(g)$$

Let $\mathcal{G}$ denote the natural weak L_2-Riemannian metric (see (1.3)) on $\mathcal{M}$ and let $\mathcal{G}_{-1} \equiv \mathcal{G}_{\mathcal{M}_{-1}}$ denote the intrinsic Riemannian metric naturally induced on the submanifold $\mathcal{M}_{-1}$ from $\mathcal{G}$. Thus for $g \in \mathcal{M}_{-1}$,

$$\mathcal{G}_{-1}(g) : T_g\mathcal{M}_{-1} \times T_g\mathcal{M}_{-1} \longrightarrow \mathbf{R},$$

where if $\bar{h}, \bar{k} \in \ker DR(g) \approx T_g\mathcal{M}_{-1}$, then $\mathcal{G}_{-1}(g)(\bar{h}, \bar{k}) = \mathcal{G}(g)(\bar{h}, \bar{k})$. In the next proposition, we first compute the intrinsic gradient of the Yamabe functional restricted to $\mathcal{M}_{-1}$ and see that this does not give the conformal Ricci flow equation. However, by using the non-orthogonal splitting (5.9) and the non-orthogonal projection (5.15), we can formulate the conformal Ricci flow equations in a gradient-like manner, which we refer to as a **quasi-gradient**.

Proposition 6.1 (A quasi-gradient form of the conformal Ricci flow equations)
The gradient of the Yamabe functional

$$Y : \mathcal{M} \longrightarrow \mathbf{R}, \quad g \longmapsto v_g^{(2-n)/n}R_{\text{total}}(g) = v_g^{2/n}\bar{R}(g)$$

in the natural L_2-Riemannian metric $\mathcal{G}$ on $\mathcal{M}$ is given by

$$\mathrm{grad}\, Y : \mathcal{M} \longrightarrow S_2, \quad g \longmapsto (\mathrm{grad}\, Y)(g) = -v_g^{(2-n)/n}\big(\mathrm{Ein}(g) + \tfrac{n-2}{2n}\bar{R}(g)g\big)$$
$$= -v_g^{(2-n)/n}\Big(\big(\mathrm{Ric}(g) - \tfrac{1}{n}\bar{R}(g)g\big) + \tfrac{1}{2}(\bar{R}(g) - R(g))g\Big).$$

When restricted to $\mathcal{M}_{-1}$, $\mathrm{grad}\, Y$ simplifies to

$$(\mathrm{grad}\, Y)_{|\mathcal{M}_{-1}} : \mathcal{M}_{-1} \longrightarrow S_2, \quad g \longmapsto (\mathrm{grad}\, Y)_{|\mathcal{M}_{-1}}(g)$$
$$= -v_g^{(2-n)/n}\big(\mathrm{Ric}(g) + \tfrac{1}{n}g\big) = -v_g^{(2-n)/n}\mathrm{Ric}^T(g).$$

The Yamabe functional Y when restricted to $\mathcal{M}_{-1}$ simplifies to

$$Y_{|\mathcal{M}_{-1}} : \mathcal{M}_{-1} \longrightarrow \mathbf{R}, \quad g \longmapsto Y_{|\mathcal{M}_{-1}}(g) = Y(g) = -v_g^{2/n}.$$

Let

$$\mathrm{grad}_{-1}Y_{|\mathcal{M}_{-1}} : \mathcal{M}_{-1} \to T\mathcal{M}_{-1}$$

denote the gradient of the restricted functional $Y_{|\mathcal{M}_{-1}}$ in the Riemannian manifold $(\mathcal{M}_{-1}, \mathcal{G}_{-1})$. Then with $T_g\mathcal{M}_{-1}$ identified with $\ker DR(g)$, for $g \in \mathcal{M}_{-1}$,

$$(\mathrm{grad}_{-1}Y_{|\mathcal{M}_{-1}})(g) = \bar{P}_g\big((\mathrm{grad}\, Y)_{|\mathcal{M}_{-1}}(g)\big) = -\tfrac{1}{n}v_g^{(2-n)/n}\bar{P}_g(g) \in \ker DR(g),$$

where $\bar{P}_g : S_2 \to \ker DR(g)$ is L_2-orthogonal projection (see (2.8)) and where $\bar{P}_g(g) = \bar{g} = g - DR(g)^*\big((DR(g)DR(g)^*)^{-1}(1)\big)$ (see (2.17)).

Using the non-orthogonal projection $\widetilde{P}_g : S_2 \to \ker DR(g)$ (see (5.15)), the non-orthogonal projection of $(\operatorname{grad} Y)_{|\mathcal{M}_{-1}}$ to $T_g\mathcal{M}_{-1}$ is given by

$$\widetilde{P}_g\big((\operatorname{grad} Y)_{|\mathcal{M}_{-1}}(g)\big) = -v_g^{(2-n)/n}\big(\operatorname{Ric}(g) + \tfrac{1}{n}g + \tfrac{1}{2}pg\big)$$
$$= -v_g^{(2-n)/n}\big(\operatorname{Ric}(g) + L_g^{-1}(|\operatorname{Ric}(g)|_g^2)\big),$$

where $p = 2L_g^{-1}(|\operatorname{Ric}(g)|_g^2) - \tfrac{2}{n}$ is the conformal pressure. Thus the reduced conformal Ricci flow equation can be written in the quasi-gradient form,

$$\frac{\partial g}{\partial t} = 2v_g^{(n-2)/n}\widetilde{P}_g\big((\operatorname{grad} Y)_{|\mathcal{M}_{-1}}(g)\big). \tag{6.1}$$

Proof: First we recall the facts that $(\operatorname{grad} \operatorname{vol})(g) = \tfrac{1}{2}g$ and $(\operatorname{grad} R_{\text{total}})(g) = -\operatorname{Ein}(g)$. That $(\operatorname{grad} \operatorname{vol})(g) = \tfrac{1}{2}g$ follows from the fact that $(\operatorname{grad} \operatorname{vol})(g)$ is defined as the "vector" in S_2 such that for all $h \in S_2$,

$$(d\operatorname{vol})(g)h = \int_M (\operatorname{grad} \operatorname{vol})(g) \cdot h\, d\mu_g.$$

Thus, since

$$(d\operatorname{vol})(g)h = \int_M D(d\mu_g)h = \tfrac{1}{2}\int_M \operatorname{tr}_g h\, d\mu_g = \tfrac{1}{2}\int_M g \cdot h\, d\mu_g = \int_M (\operatorname{grad} \operatorname{vol})(g) \cdot h\, d\mu_g$$

(where "$\cdot$" is the metric contraction) for all $h \in S_2$, it follows that $(\operatorname{grad} \operatorname{vol})(g) = \tfrac{1}{2}g$.

Similarly, since for all $h \in S_2$,

$$(dR_{\text{total}})(g)h = \int_M D(R(g)d\mu_g)h = \int_M DR(g)h\, d\mu_g + \int_M R(g)\, D(d\mu_g)h$$
$$= \int_M (\Delta_g \operatorname{tr}_g h + \delta_g\delta_g h - \operatorname{Ric}(g) \cdot h) + \tfrac{1}{2}\int_M R(g)\operatorname{tr}_g h\, d\mu_g$$
$$= \int_M (-\operatorname{Ric}(g) + \tfrac{1}{2}R(g)g) \cdot h\, d\mu_g$$
$$= -\int_M \operatorname{Ein}(g) \cdot h\, d\mu_g = \int_M (\operatorname{grad} R_{\text{total}})(g) \cdot h\, d\mu_g,$$

we have $(\operatorname{grad} R_{\text{total}})(g) = -\operatorname{Ein}(g)$. With these two basic gradients in hand, we find

$$(\operatorname{grad} Y)(g) = (\operatorname{grad} \operatorname{vol}^{(2-n)/n} R_{\text{total}})(g)$$
$$= (\operatorname{grad} \operatorname{vol}^{(2-n)/n})(g)R_{\text{total}}(g) + v_g^{(2-n)/n}(\operatorname{grad} R_{\text{total}})(g)$$
$$= \tfrac{2-n}{n}(v_g^{(2-n)/n})v_g^{-1}(\tfrac{1}{2}\,g)R_{\text{total}}(g) - v_g^{(2-n)/n}\operatorname{Ein}(g)$$
$$= -v_g^{(2-n)/n}\big(\operatorname{Ein}(g) + \tfrac{n-2}{2n}\bar{R}(g)g\big)$$
$$= -v_g^{(2-n)/n}\big((\operatorname{Ric}(g) - \tfrac{1}{2}R(g)g) + (\tfrac{1}{2} - \tfrac{1}{n})\bar{R}(g)g\big)$$
$$= -v_g^{(2-n)/n}\big((\operatorname{Ric}(g) - \tfrac{1}{n}\bar{R}(g)g) + \tfrac{1}{2}(\bar{R}(g) - R(g))g\big).$$

If $g \in \mathcal{M}_{-1}$, then $\bar{R}(g) = v_g^{-1}R_{\text{total}}(g) = -1 = R(g)$ and thus $\operatorname{grad} Y$ restricted to $\mathcal{M}_{-1}$ simplifies to

$$(\operatorname{grad} Y)_{|\mathcal{M}_{-1}}(g) = -v_g^{(2-n)/n}(\operatorname{Ric}(g) + \tfrac{1}{n}g) = -v_g^{(2-n)/n}\operatorname{Ric}^T(g). \tag{6.2}$$

The Yamabe functional restricted to $\mathcal{M}_{-1}$ is given by $Y_{|\mathcal{M}_{-1}}(g) =$ $v_g^{(2-n)/n} R^T(g) = v_g^{2/n} \bar{R}^T(g) = -v_g^{2/n}$. Denoting its gradient in the weak Riemannian manifold $(\mathcal{M}_{-1}, \mathcal{G}_{-1})$ as grad_{-1}, we have, using (6.2) and $\bar{P}_g(\text{Ric}(g)) = 0$,

$$(\text{grad}_{-1} Y_{|\mathcal{M}_{-1}})(g) = \bar{P}_g\Big((\text{grad}\,Y)_{|\mathcal{M}_{-1}}(g)\Big) = -\tfrac{1}{n} v_g^{(2-n)/n} \bar{P}_g(g) \,,$$

where $\bar{P}_g(g) = \bar{g} = g - DR(g)^* \big((DR(g)DR(g)^*)^{-1}(1)\big)$ as given in (2.20).

Using $\widetilde{P}_g(g) = 0$ and (5.23), the non-orthogonal projection of $(\text{grad}\,Y)_{|\mathcal{M}_{-1}}$ to $T_g \mathcal{M}_{-1}$ is

$$\begin{aligned}
\widetilde{P}_g\big((\text{grad}\,Y)_{|\mathcal{M}_{-1}}(g)\big) &= -v_g^{(2-n)/n} \widetilde{P}_g\big(\text{Ric}(g) + \tfrac{1}{n}g\big) = -v_g^{(2-n)/n} \widetilde{\text{Ric}(g)} \\
&= -v_g^{(2-n)/n}\big(\text{Ric}(g) + \tfrac{1}{n}g + \tfrac{1}{2}pg\big) \\
&= -v_g^{(2-n)/n}\big(\text{Ric}(g) + L_g^{-1}(|\text{Ric}(g)|_g^2)\big)\,,
\end{aligned}$$

where the conformal pressure $p = 2L_g^{-1}(|\text{Ric}(g)|_g^2) - \tfrac{2}{n}$. Thus

$$2\big(\text{Ric}(g) + \tfrac{1}{n}g\big) + pg = -2v_g^{(n-2)/n} \widetilde{P}_g((\text{grad}\,Y)_{|\mathcal{M}_{-1}}(g))$$

and so the reduced conformal Ricci equation can be written in the quasi-gradient form

$$\frac{\partial g}{\partial t} = -2(\text{Ric}(g) + \tfrac{1}{n}g) - pg = 2v_g^{(n-2)/n} \widetilde{P}_g((\text{grad}\,Y)_{|\mathcal{M}_{-1}}(g))\,.$$

∎

Comparing the two projections of $(\text{grad}\,Y)_{|\mathcal{M}_{-1}}$, the L_2-orthogonal projection

$$\begin{aligned}
\bar{P}_g\Big((\text{grad}\,Y)_{|\mathcal{M}_{-1}}(g)\Big) &= (\text{grad}_{-1} Y_{|\mathcal{M}_{-1}})(g) \\
&= -\tfrac{1}{n} v_g^{(2-n)/n}\big(g - DR(g)^*\big((DR(g)DR(g)^*)^{-1}(1)\big)\big)
\end{aligned}$$

and the non-orthogonal projection,

$$\widetilde{P}_g\big((\text{grad}\,Y)_{|\mathcal{M}_{-1}}(g)\big) = -v_g^{(2-n)/n}\big(\text{Ric}(g) + L_g^{-1}(|\text{Ric}(g)|_g^2)\big)\,,$$

we see that the reduced conformal Ricci flow equation is not the intrinsic gradient of the restricted functional $Y_{|\mathcal{M}_{-1}}$ but is closely related to the non-orthogonal projection of the ambient gradient $(\text{grad}\,Y)_{|\mathcal{M}_{-1}}$ of Y restricted to $\mathcal{M}_{-1}$.

Lastly, we remark that as we shall see in Proposition 7.1, for non-static flows the volume of the conformal Ricci flow is a strictly monotonically decreasing real-valued function of time. Since $n \geq 3$, the coefficient $2v_g^{(n-2)/n}$ of equation (6.1) is also strictly monotonically decreasing and thus can be used to renormalize time so that in the new time variable, (6.1) can be written in the quasi-gradient form $\dot{g} = \widetilde{P}_g((\text{grad}\,Y)_{|\mathcal{M}_{-1}}(g))$.

7. Decaying global and local volume for the conformal Ricci flow

In this section we show that for non-static flows, the global and local volumes are strictly monotonically decreasing under the conformal Ricci flow. We then consider some applications of these results.

Proposition 7.1 (Global volume decay for the conformal Ricci flow) *Let $g : [0, T) \to \mathcal{M}_{-1}$ be a conformal Ricci flow with conformal pressure $p : [0, T) \to \mathcal{F}$. Then on $[0, T)$, the volume element $d\mu_g$ satisfies*

$$\frac{\partial}{\partial t} d\mu_g = -\tfrac{n}{2} p \, d\mu_g \tag{7.1}$$

and the volume $\mathrm{vol}(M, g) = \int_M d\mu_g$ satisfies

$$\frac{d}{dt} \mathrm{vol}(M, g) = -\tfrac{n}{2} \int_M p \, d\mu_g = -n \int_M |\mathrm{Ric}^T(g)|_g^2 \, d\mu_g \, . \tag{7.2}$$

Thus if the flow g is non-static, $\mathrm{Ric}^T(g) \neq 0$, $p > 0$, and

$$\frac{d}{dt} \mathrm{vol}(M, g) < 0 \tag{7.3}$$

and so the curve of volumes $\mathrm{vol}(M, g)$ is a strictly monotonically decreasing function along the non-static flow lines of the conformal Ricci flow.

Proof: Computing the time derivative of the volume element $d\mu_g$ yields

$$\frac{\partial}{\partial t} d\mu_g = \tfrac{1}{2} \mathrm{tr}_g \left(\frac{\partial g}{\partial t} \right) d\mu_g = \tfrac{1}{2} \mathrm{tr}_g \left(-2\mathrm{Ric}(g) - (\tfrac{2}{n} + p)g \right) d\mu_g$$
$$= \left(-R(g) - \tfrac{n}{2}(\tfrac{2}{n} + p) \right) d\mu_g = (1 - 1 - \tfrac{n}{2}p) d\mu_g = -\tfrac{n}{2} p \, d\mu_g \, .$$

Integrating (3.6), $(n-1)\Delta_g p + p = 2(|\mathrm{Ric}^T(g)|_g^2)$, over M, using $\int_M \Delta_g p \, d\mu_g = 0$, we find the following relationship between the integrated conformal pressure and the integrated curvature,

$$\int_M p \, d\mu_g = 2 \int_M (|\mathrm{Ric}^T(g)|_g^2) d\mu_g \, . \tag{7.4}$$

Integrating (7.1) over M then yields

$$\frac{d}{dt} \mathrm{vol}(M, g) = \frac{d}{dt} \int_M d\mu_g = \int_M \frac{\partial}{\partial t} d\mu_g = -\tfrac{n}{2} \int_M p \, d\mu_g = -n \int_M |\mathrm{Ric}^T(g)|_g^2 \, d\mu_g \, .$$

Thus for non-static flows, $\mathrm{Ric}^T(g) \neq 0$, $p > 0$ (see Propositions 3.1 and 3.3), and $\frac{d}{dt} \mathrm{vol}(M, g) < 0$. ∎

As an application of Proposition 7.1 we have the following non-existence result.

Proposition 7.2 *There exist no non-static periodic conformal Ricci flows.*

Proof: If $g : [0, \infty) \to \mathcal{M}_{-1}$ is a non-static periodic solution of the conformal Ricci flow, then $g(t_0) = g(t_1)$ for some $t_0, t_1 \in [0, \infty)$, $t_0 < t_1$, and so $\mathrm{vol}(M, g(t_0)) = \mathrm{vol}(M, g(t_1))$, contradicting $\mathrm{vol}(M, g(t_0)) > \mathrm{vol}(M, g(t_1))$ for non-static flows. ∎

Adopting terminology from Perelman [35] for the classical Ricci flow, we define the following.

Definition 7.3 (Conformal Ricci breathers and solitons) *A conformal Ricci flow $g : [0, T) \to \mathcal{M}_{-1}$ with conformal pressure $p : [0, T) \to \mathcal{F}$ and initial condition $g(0) = g_0$ is a **conformal Ricci breather** if for some pair $t_0, t_1 \in [0, T)$, $t_0 < t_1$, there exists a $c_1 > 0$ and a diffeomorphism $f_1 \in \mathcal{D}$ such that*

$$g(t_1) = c_1 f_1^*(g(t_0)) \, . \tag{7.5}$$

For $c_1 < 1$, $c_1 = 1$, and $c_1 > 1$, the breather is called **shrinking**, **steady**, *and* **expanding**, *respectively.*

If, moreover, there exists a one-parameter family of diffeomorphisms, $f : [0, T) \rightarrow \mathcal{D}$, $t \mapsto f(t) = f_t$, and a time-dependent homothetic map $c : [0, T) \rightarrow \boldsymbol{R}^+$, $t \mapsto c(t) = c_t$, such that

$$g(t) = c_t f_t^* g_0 \,, \tag{7.6}$$

then g is a **conformal Ricci soliton**. ∎

Thus, modulo isometry and homothety, a conformal Ricci breather is a periodic solution and a conformal Ricci soliton is a static solution.

Now we have the following additional non-existence result.

Proposition 7.4 *There exist no non-static conformal Ricci breathers or conformal Ricci solitons.*

Proof: Let $g : [0, T) \rightarrow \mathcal{M}_{-1}$ be a conformal Ricci breather and let $t_0, t_1 \in [0, T)$, $t_0 < t_1$, $c_1 > 0$, and $f_1 \in \mathcal{D}$ be such that

$$g(t_1) = c_1 f_1^*(g(t_0)) \tag{7.7}$$

as given in Definition 7.3. Since $g(t_0), g(t_1) \in \mathcal{M}_{-1}$, from (2.25),

$$-1 = R(g(t_1)) = R(c_1 f_1^*(g(t_0))) = c_1^{-1} R(f_1^*(g(t_0)))$$
$$= c_1^{-1}(R(g(t_0)) \circ f_1) = c_1^{-1}((-1) \circ f_1) = -c_1^{-1} \,,$$

so that $c_1 = 1$ and so $g(t_1) = f_1^*(g(t_0))$. Thus from the constraint $R(g) = -1$, any conformal Ricci breather must be steady.

From the invariance of the volume functional with respect to diffeomorphism, $\text{vol}(M, g(t_1)) = \text{vol}(M, f_1^*(g(t_0))) = \text{vol}(M, g(t_0))$. Since $t_1 > t_0$ and the volumes are not strictly monotonically decreasing, from Proposition 7.1, $g(t) = g_0$ must be a static flow.

Lastly, since a conformal Ricci soliton $g : [0, T) \rightarrow \mathcal{M}_{-1}$ is a special case of a conformal Ricci breather (by letting $t_0 = 0$, $t_1 \in (0, T)$, $c_1 = c(t_1)$, and $f_1 = f(t_1)$), there can exist no non-static conformal Ricci solitons. ∎

Thus in essence the constraint $R(g) = -1$ prohibits non-trivial homothetic transformations and the strictly monotonically decreasing volume result for non-static conformal Ricci flows prohibits isometric transformations.

We remark that in Perelman's [35] consideration of classical Ricci flow, the proof that there are no non-static Ricci breathers or Ricci solitons is considerably more involved, necessitating as it does the introduction of a non-scale and a scale invariant functional.

Using the strong maximum principle as contained in Proposition 3.3, we can strengthen the global volume decay of conformal Ricci flow to a local volume decay result. The local result is motivated by a similar result that occurs for the reduced Hamiltonian of general relativity (see Fischer-Moncrief [21], p. 5585, for details).

Theorem 7.5 (Local volume decay for the conformal Ricci flow) *Let $g : [0, T) \rightarrow \mathcal{M}_{-1}$ be a non-static conformal Ricci flow with conformal pressure $p : [0, T) \rightarrow \mathcal{P}$. Let $D \subseteq M$ be a domain in M, possibly all of M, and let $\text{vol}(D, g) = \int_D d\mu_g$ denote the volume of the domain using the time-dependent metric g. Then on $[0, T)$,*

$$\frac{d}{dt} \text{vol}(D, g) = -\frac{n}{2} \int_D p \, d\mu_g < 0 \,. \tag{7.8}$$

Thus $\mathrm{vol}(D, g)$ *is a strictly monotonically decreasing function along the non-static flow lines of the conformal Ricci flow. In particular, if $D = M$, we recover the global volume result for non-static conformal Ricci flows,*

$$\frac{d}{dt}\,\mathrm{vol}(M, g) = -\tfrac{n}{2}\int_M p\,d\mu_g < 0\,. \tag{7.9}$$

Proof: From Proposition 3.3, if the flow g is non-static, then $p > 0$ on M so that the curve of conformal pressures lies in the space of positive functions $\mathcal{P}$ on M. Integrating (7.1) over $D \subseteq M$ yields

$$\frac{d}{dt}\,\mathrm{vol}(D, g) = \frac{d}{dt}\int_D d\mu_g = \int_D \frac{\partial}{\partial t}\,d\mu_g = -\tfrac{n}{2}\int_D p\,d\mu_g < 0\,. \qquad \blacksquare$$

Note that unlike the case for $D = M$, the relationship (7.4) between the integrated conformal pressure and the integrated curvature is not maintained in general since when integrating (3.6) over $D \neq M$, from Gauss' theorem, the integrated Laplacian term $\int_D \Delta_g p\,d\mu_g = -\int_D \mathrm{div}_g(\mathrm{grad}_g p)d\mu_g = -\int_{\partial D}(\mathrm{grad}_g p)^{\perp}d\mu_{i^*g} = -\int_{\partial D} i^*(*dp) \neq 0$ need not drop out. Here $(\mathrm{grad}_g p)^{\perp} = g(\mathrm{grad}_g p, n) = dp{\cdot}n$ denotes the perpendicular projection of $\mathrm{grad}_g p$ at the boundary ∂D, where n denotes the unit outward normal vector field to the boundary, $i : \partial D \to D$ is the inclusion map, i^*g is the induced metric on the boundary, $d\mu_{i^*g}$ is the volume element of the boundary induced by i^*g and the orientation of the boundary, and $*$ is the usual Riemannian star operation from k-forms to $(n - k)$-forms.

A rather surprising consequence of this analysis is that the local volume of an arbitrary domain D cannot remain constant, even for a short time interval, unless the global flow is static. This local result is of potential importance for applications of the conformal Ricci flow to geometrization of 3-manifolds since even if a singularity is forming within a domain D of a manifold, the volume must still be decreasing on that domain (see Section 10).

If $g : [0, \infty) \to \mathcal{M}_{-1}$ is an all-time conformal Ricci flow, by Proposition 7.1 the volume functional $\mathrm{vol}(M, g(t))$ along the flow must converge,

$$\mathrm{vol}(M, g(t)) \longrightarrow v_{\infty} \geq 0 \qquad \text{as} \qquad t \longrightarrow \infty\,. \tag{7.10}$$

If $v_{\infty} = 0$, then we say that the conformal Ricci flow g **volume collapses**.

As an application of the relationship between volume and curvature as reflected in the results of this section, we have the following curvature condition that ensures that the volume does not collapse.

Theorem 7.6 (Curvature condition for no volume collapse) *Let $g : [0, \infty) \to \mathcal{M}_{-1}$ be an all-time non-static conformal Ricci flow with conformal pressure $p : [0, \infty) \to \mathcal{P}$. For $t \in [0, \infty)$, let $A(t) = \max_{x \in M} |\mathrm{Ric}^T(g(t, x))|^2_{g(t,x)} > 0$. Assume that $\int_0^t A(t')dt'$ converges as $t \to \infty$. Let $B = \lim_{t \to \infty}\int_0^t A(t')dt' > 0$. Then for $t_0 \in [0, \infty)$,*

$$\mathrm{vol}(M, g(t)) \geq \mathrm{vol}(M, g(t_0))e^{-n\int_{t_0}^t A(t')dt'} > \mathrm{vol}(M, g(t_0))e^{-nB}\,. \tag{7.11}$$

In particular, the conformal Ricci flow g does not volume collapse, i.e., $\mathrm{vol}(M, g(t)) \to v_{\infty} > 0$ as $t \to \infty$.

Proof: From (7.2) and the definition of $A(t)$,

$$-\frac{d}{dt}\operatorname{vol}(M,g(t)) = n\int_M |\operatorname{Ric}^T(g(t))|^2_{g(t)} d\mu_{g(t)} \leq nA(t)\int_M d\mu_g = nA(t)\operatorname{vol}(M,g(t))\,.$$

Thus

$$\frac{d}{dt}\ln\operatorname{vol}(M,g(t)) \geq -nA(t)\,.$$

Integrating this differential inequality from t_0 to t and exponentiating yields

$$\operatorname{vol}(M,g(t)) \geq \operatorname{vol}(M,g(t_0))e^{-n\int_{t_0}^t A(t')dt'} > \operatorname{vol}(M,g(t_0))e^{-nB}\,.$$

■

This theorem gives a sufficient curvature condition to prevent volume collapse, namely, if the curvature invariant $A(t) = \max_{x\in M}|\operatorname{Ric}^T(g(t,x))|^2_{g(t,x)}$ falls off sufficiently fast as $t \to \infty$, then volume collapse cannot occur. As an example, if $A(t)$ is asymptotic to $1/t^{1+\epsilon}$ for some $\epsilon > 0$, then the hypothesis of the theorem is met and volume collapse cannot occur.

However, the theorem gives only a sufficient condition to prevent volume collapse. Thus, for example, a conformal Ricci flow $g \to g_e$ that converges at any rate as $t \to \infty$ to an Einstein metric $g_e \in \mathcal{M}_{-1}$, $\operatorname{Ric}(g_e) = -\frac{1}{n}g_e$, has $\operatorname{vol}(M,g) \to \operatorname{vol}(M,g_e) > 0$ as $t \to \infty$ and so the volume does not collapse. Similarly, if $A(t) = \max_{x\in M}|\operatorname{Ric}^T(g(t,x))|^2_{g(t,x)} = A = \text{constant}$, or more generally, if $A(t) \geq \epsilon > 0$ is is bounded away from zero, then the hypothesis of the theorem is not met and so, as far as the theorem goes, the volume may or may not collapse.

8. Locally homogeneous conformal Ricci flows

In [27], Isenberg and Jackson consider locally homogenous solutions to the classical Ricci flow equation. Local homogeneity implies that all scalar geometric quantities such as the scalar curvature and the Riemann and Ricci curvature norm are spatially constant. We take advantage of this fact to use the scalar curvature to rescale in space and reparameterize in time locally homogenous classical Ricci flows to give locally homogeneous conformal Ricci flows. Thus the locally homogenous case provides somewhat of an intersection between the classical and conformal Ricci flows. Moreover, by rescaling and reparameterizing locally homogenous classical Ricci flows, one can find explicit locally homogeneous conformal Ricci flows.

The following result relates locally homogeneous classical Ricci flows to locally homogeneous conformal Ricci flows.

Proposition 8.1 (Locally homogeneous classical and conformal Ricci flows)
Let

$$g : [0,T) \longrightarrow \mathcal{M}^1\,, \qquad t \longmapsto g(t)$$

be a locally homogeneous solution of the classical Ricci equation (1.19)

$$\frac{\partial g}{\partial t} = -2\operatorname{Ric}(g) + \frac{2}{n}\bar{R}(g)g\,, \qquad g_0 \in \mathcal{M}^1\,.$$

Then g is also a solution of the historical first Ricci flow equation (4.17),

$$\frac{\partial g}{\partial t} = -2\operatorname{Ric}(g) + \frac{2}{n}R(g)g\,, \qquad g_0 \in \mathcal{M}^1\,,$$

and both the volume and the volume element of the flow are constant,

$$d\mu_g = d\mu_{g_0}\,. \tag{8.1}$$

Assume that the spatially constant scalar curvatures satisfy $R(g_t) = c_t < 0$. (If M is of Yamabe type -1, this condition is automatically satisfied, as in Proposition 3.5.) Define a new time parameter

$$s(t) = \int_0^t |R(g_{t'})|dt' \tag{8.2}$$

*for $t \in [0, T)$ and let $\bar{T} = \lim_{t\to T} s(t)$ so that $0 < \bar{T} \le \infty$. Then $\frac{ds(t)}{dt} = |R(g(t))| > 0$ so that $s(t)$ is strictly monotonically increasing. Let $t = t(s)$ denote the inverse of $s(t)$ so that $t : [0, \bar{T}) \to [0, T)$. Define a **rescaled reparameterized flow***

$$\bar{g} : [0, \bar{T}) \longrightarrow \mathcal{M}_{-1}\,, \qquad s \longmapsto \bar{g}(s) = |R(g(t(s)))|g(t(s))\,. \tag{8.3}$$

Then $\bar{g}(s)$ is a locally homogeneous solution to the conformal Ricci equations,

$$\frac{\partial \bar{g}(s)}{\partial s} + 2\left(\mathrm{Ric}(\bar{g}(s)) + \tfrac{1}{n}g\right) = -p(s)\bar{g}(s)$$

$$R(\bar{g}(s)) = -1\,,$$

with initial value $\bar{g}_0 = |R(g_0)|g_0 \in \mathcal{M}_{-1}$, with spatially constant conformal pressure

$$p(s) = 2\,|\mathrm{Ric}^T(\bar{g}(s))|^2_{\bar{g}(s)}\,,$$

and if g is non-static, with strictly monotonically decreasing volume

$$\mathrm{vol}(M, \bar{g}(s)) = |R(g(t(s)))|^{n/2}\,. \tag{8.4}$$

Proof: Since the flow $t \mapsto g_t$ is locally homogeneous, the scalar curvatures are constant, $R(g_t) = c_t = \mathrm{constant}$. Thus $\bar{R}(g_t) = \frac{\int_M R(g_t)d\mu_{g_t}}{\mathrm{vol}(M,g_t)} = R(g_t)$ and thus g_t satisfies (4.17),

$$\frac{\partial g}{\partial t} = -2\mathrm{Ric}(g) + \tfrac{2}{n}R(g)g\,, \tag{8.5}$$

with $R(g)$ spatially constant. Thus from (4.19) (with $R(g)$ spatially constant),

$$\frac{\partial R(g)}{\partial t} = \tfrac{n-2}{n}\Delta_g R(g) + 2|\mathrm{Ric}(g)|^2_g - \tfrac{2}{n}R^2(g) = 2|\mathrm{Ric}(g)|^2_g - \tfrac{2}{n}R^2(g)\,, \tag{8.6}$$

which is no longer a backwards heat equation. Thus in this case (4.17) is equal to the classical Ricci flow equation and is well-posed (see also the remark right after this proof).

Also,

$$\frac{d}{dt}d\mu_g = \tfrac{1}{2}\mathrm{tr}_g\left(\frac{\partial g}{\partial t}\right)d\mu_g = \tfrac{1}{2}\mathrm{tr}_g\left(-2\,\mathrm{Ric}(g) + \tfrac{2}{n}R(g)g\right)d\mu_g = (-R(g) + R(g))d\mu_g = 0\,,$$

so that the volume element (and not just the volume) of the flow is preserved.

Now assume that the spatially constant scalar curvatures satisfy $R(g_t) = c_t < 0$. First we rescale and then we reparameterize the classical Ricci solution $g : [0, T) \to \mathcal{M}^1$. Since for each $t \in [0, T)$, $R(g_t) = c_t < 0$ is constant, we can rescale g_t by a time-dependent homothetic transformation to get $\tilde{g}_t = |R(g_t)|g_t = -R(g_t)g_t$. Then

$$R(\tilde{g}_t) = R(|R(g_t)|g_t) = \frac{1}{|R(g_t)|}R(g_t) = -1\,, \tag{8.7}$$

so that $\tilde{g}_t \in \mathcal{M}_{-1}$ satisfies the constraint equation of the conformal Ricci system.

Noting that for the rescaled curve $\tilde{g}_t = |R(g_t)|g_t = -R(g_t)g_t$, $\text{Ric}(\tilde{g}) = \text{Ric}(g)$ and $|\text{Ric}(\tilde{g})|^2_{\tilde{g}} = |R(g)|^{-2}|\text{Ric}(g)|^2_g$, it follows from (8.6) that $\tilde{g}$ satisfies the evolution equation

$$
\begin{aligned}
\frac{\partial \tilde{g}}{\partial t} &= -\frac{\partial R(g)}{\partial t}g - R(g)\frac{\partial g}{\partial t} \\
&= -\left(2|\text{Ric}(g)|^2_g - \tfrac{2}{n}R^2(g)\right)g - R(g)\left(-2\text{Ric}(g) + \tfrac{2}{n}R(g)g\right) \\
&= -2|\text{Ric}(g)|^2_g\, g + 2R(g)\,\text{Ric}(g) \\
&= -2|R(g)|^2|\text{Ric}(\tilde{g})|^2_{\tilde{g}}\, g - 2|R(g)|\,\text{Ric}(\tilde{g}) \\
&= -2|R(g)||\text{Ric}(\tilde{g})|^2_{\tilde{g}}\, \tilde{g} - 2|R(g)|\,\text{Ric}(\tilde{g})\,.
\end{aligned}
$$

Thus $\tilde{g}_t = |R(g_t)|g_t$ solves the system

$$
\frac{1}{|R(g)|}\frac{\partial \tilde{g}}{\partial t} = -2\,\text{Ric}(\tilde{g}) - 2\,|\text{Ric}(\tilde{g})|^2_{\tilde{g}}\,\tilde{g} \tag{8.8}
$$

$$
R(\tilde{g}_t) = -1\,. \tag{8.9}
$$

Now define a new time parameter $s(t) = \int_0^t |R(g_{t'})|dt'$ so that $s(0) = 0$ and $\frac{ds}{dt}(t) = |R(g_t)| > 0$ so that $s(t)$ is a strictly monotonically increasing function of t. Let $t = t(s)$ denote the inverse of $s(t)$ so that $\frac{dt}{ds}(s) = |R(g(t(s)))|^{-1}$. Let

$$
\bar{g}(s) = \tilde{g}(t(s)) = |R(g(t(s)))|g(t(s)) \tag{8.10}
$$

denote the reparameterized rescaled flow. Then $\bar{g}(s)$ has initial value $\bar{g}_0 = \bar{g}(0) = \tilde{g}(t(0)) = \tilde{g}(0) = |R(g(0))|g(0) = |R(g_0)|g_0$.

From (8.9), the reparameterized $\bar{g}(s)$ also satisfies the constraint $R(\bar{g}(s)) = R(\tilde{g}(t(s))) = -1$. Moreover, from (8.8), $\bar{g}(s)$ satisfies

$$
\begin{aligned}
\frac{\partial \bar{g}(s)}{\partial s} &= \frac{\partial \tilde{g}}{\partial t}(t(s))\frac{\partial t}{\partial s}(s) = \frac{\partial \tilde{g}}{\partial t}(t(s))\frac{1}{|R(g(t(s)))|} \\
&= -2\,\text{Ric}(\tilde{g}(t(s))) - 2\,|\text{Ric}(\tilde{g}(t(s)))|^2_{\tilde{g}(t(s))}\,\tilde{g}(t(s)) \\
&= -2\,\text{Ric}(\bar{g}) - 2\,|\text{Ric}(\bar{g})|^2_{\bar{g}}\,\bar{g}(s)\,.
\end{aligned}
$$

Let $p(s) = 2(|\text{Ric}(\bar{g}(s))|^2_{\bar{g}(s)} - \tfrac{1}{n}) = 2\,|\text{Ric}^T(\bar{g}(s))|^2_{\bar{g}(s)}$. Then $p(s)$ is spatially constant since $g(t)$ and thus $\bar{g}(s)$ are locally homogeneous. Note that since $p(s)$ is spatially constant, it is an explicit solution to (3.6),

$$
L_{\bar{g}(s)}p(s) = (n-1)\Delta_{\bar{g}(s)}p(s) + p(s) = p(s) = 2(|\text{Ric}(\bar{g}(s))|^2_{\bar{g}(s)} - \tfrac{1}{n})\,.
$$

Moreover, $\bar{g}(s)$ satisfies the conformal Ricci system

$$
\frac{\partial \bar{g}(s)}{\partial s} = -2\,\text{Ric}(\bar{g}(s)) - \tfrac{2}{n}\bar{g}(s) - p(s)\bar{g}(s)
$$

$$
R(\bar{g}(s)) = -1
$$

with initial value $\bar{g}_0 = |R(g_0)|g_0$. Lastly, since $\text{vol}(M, g(t)) = 1$,

$$
\begin{aligned}
\text{vol}(M, \bar{g}(s)) &= \text{vol}\Big(M, |R(g(t(s))|g(t(s))\Big) \\
&= |R(g(t(s)))|^{n/2}\text{vol}\Big(M, g(t(s))\Big) = |R(g(t(s)))|^{n/2}\,.
\end{aligned}
$$

Thus if g is non-static, that $\text{vol}(M, \bar{g}(s))$ is strictly monotonically decreasing follows from Proposition 7.1. ∎

We remark that in the locally homogeneous case, $\bar{R}(g_t) = R(g_t)$ and thus (4.17) and Hamilton's Ricci flow equation are equal. In this case $\Delta_g R(g) = 0$ and thus the differential equation for the scalar curvature reduces to the ordinary differential equation given by (8.6) and thus is not a backwards heat equation and is well-posed.

As a corollary of Proposition 8.1, we can deduce the following information about locally homogeneous solutions of the classical Ricci flow equation by using information about conformal Ricci flow.

Corollary 8.2 *Let*

$$g : [0, T) \longrightarrow \mathcal{M} , \qquad t \longmapsto g(t) , \tag{8.11}$$

with initial value $g(0) = g_0$ be a non-static locally homogeneous solution of the classical Ricci flow equation

$$\frac{\partial g}{\partial t} = -2\operatorname{Ric}(g) + \tfrac{2}{n}\bar{R}(g)g , \qquad g_0 \in \mathcal{M}^1 ,$$

with constant negative scalar curvatures $R(g_t) = c_t < 0$. (If M is of Yamabe type -1, then this condition is automatically satisfied.) Then

$$\frac{d}{dt}R(g_t) > 0 ,$$

so that the curve of spatially constant negative scalar curvatures $R(g_t)$ is strictly monotonically increasing in t.

Proof: Let $\bar{g}(s) = \tilde{g}(t(s)) = |R(g(t(s))|g(t(s))$ denote the rescaled reparameterized solution of the conformal Ricci system so that $\bar{g}(s(t)) = |R(g(t)|g(t)$. From (8.4), $\operatorname{vol}(M, \bar{g}(s)) = |R(g(t(s))|^{n/2}$ is strictly monotonically decreasing (Proposition 8.1). From the assumption $R(g(t)) = c_t < 0$,

$$R(g(t)) = -\operatorname{vol}(M, \bar{g}(s(t)))^{2/n} \tag{8.12}$$

and is thus strictly monotonically increasing.

If M is of Yamabe type -1, then from the definition of Yamabe type (Definition 1.2), the scalar curvature of any constant scalar curvature metric must be negative. ∎

Thus a non-static locally homogeneous classical Ricci flow $g(t) \in \mathcal{M}^1$ with $R(g_t) = c_t < 0$ has constant volume ($= 1$) but strictly monotonically increasing scalar curvature, $\frac{d}{dt}R(g_t) > 0$, whereas the transformed locally homogeneous conformal Ricci solution $\bar{g}(s) \in \mathcal{M}_{-1}$ has constant scalar curvature ($= -1$) but strictly monotonically decreasing volume, $\frac{d}{dt}\operatorname{vol}(M, g_t) < 0$.

Since in the locally homogeneous case $R(g_t)$ is spatially constant, under the stronger condition $R(g_t) < 0$ (rather than $R(g_t) \leq 0$ as occurs in Proposition 3.5), we get the stronger conclusion $\frac{d}{dt}R(g_t) > 0$ (rather than $\frac{d}{dt}R(g_t) \geq 0$); see also the remark following Proposition 3.5 and also Table 1 in Section 1.6. The current situation is summarized in Table 2.

9. The conformal Ricci flow and the σ-constant of M

An important question regarding the conformal Ricci flow equation is under what conditions does a non-equilibrium initial condition $g_0 \in \mathcal{M}_{-1}$, $\operatorname{Ric}(g_0) \neq -\frac{1}{n}g_0$, have an all-time solution

$$g : [0, \infty) \longrightarrow \mathcal{M}_{-1} . \tag{9.1}$$

Table 2. The volume and scalar curvature for non-static locally homogeneous solutions to the classical and conformal Ricci flow equation under either the assumption that $R(g_t) < 0$ or that M is of Yamabe type -1.

	Volume	Scalar curvature
Classical Ricci Flow	$\mathrm{vol}(M, g_t) = 1$	$\frac{d}{dt} R(g_t) > 0$
Conformal Ricci Flow	$\frac{d}{dt}\mathrm{vol}(M, g_t) < 0$	$R(g_t) = -1$

Since the conformal Ricci flow $g : [0, T) \to \mathcal{M}_{-1}$ takes place in $\mathcal{M}_{-1}$, the curvature norm $|\mathrm{Riem}(g)|_g$ is always bounded away from zero (see Proposition 2.4). A conformal Ricci flow with curvature norm $|\mathrm{Riem}(g)|_g$ which blows up in either finite or infinite time is called **singular**.

A conformal Ricci flow $g : [0, \infty) \to \mathcal{M}_{-1}$ is **non-singular** if it is an all-time flow and if $|\mathrm{Riem}(g)|_g$ is bounded above, i.e., if there exists a real constant B such that for all $t \in [0, \infty)$,

$$\tfrac{2}{n(n-1)} \leq |\mathrm{Riem}(g)|_g \leq B < \infty \,. \tag{9.2}$$

If $g : [0, \infty) \to \mathcal{M}_{-1}$ is a non-static all-time conformal Ricci flow, then $\mathrm{vol}(M, g)$ is a strictly monotonically decreasing function of t (see Proposition 7.1) and from (7.10), $\mathrm{vol}(M, g(t)) \to v_\infty \geq 0$ as $t \to \infty$. Thus of interest and importance is, firstly, the value of

$$\inf \mathrm{vol}_{-1} \equiv \inf_{g \in \mathcal{M}_{-1}} \mathrm{vol}(M, g) \,, \tag{9.3}$$

and, secondly, whether $\mathrm{vol}(M, g)$ asymptotically approaches this infimum,

$$\mathrm{vol}(M, g) \longrightarrow \inf \mathrm{vol}_{-1} \qquad \text{as} \qquad t \longrightarrow \infty \,. \tag{9.4}$$

If M is of Yamabe type -1, then the value of $\inf \mathrm{vol}_{-1}$ is shown in Fischer-Moncrief [18] to be

$$\inf \mathrm{vol}_{-1} = (-\sigma(M))^{n/2} \,, \tag{9.5}$$

where $\sigma(M)$ denotes the σ-constant of M, an important topological invariant of M defined by a minimax process involving the Yamabe functional (see Anderson [1] and Fischer-Moncrief [20], Section 7). For manifolds of Yamabe type -1, a metric g of constant scalar curvature must have $R(g) = \text{constant} < 0$ and thus from the definition of the σ-constant, $\sigma(M) \leq 0$.

If we assume that M is of Yamabe type -1, that $\sigma(M) = 0$, that there exists a non-singular conformal Ricci flow $g : [0, \infty) \to \mathcal{M}_{-1}$, and that for this flow $\mathrm{vol}(M, g) \to \inf \mathrm{vol}_{-1} = (-\sigma(M))^{n/2} = 0$ as $t \to \infty$, then in this case the conformal Ricci flow volume-collapses M with bounded curvature norm $|\mathrm{Riem}(g)|_g$. Under these circumstances, g cannot converge to a limit metric $g_\infty \in \mathcal{M}_{-1}$ for if it did, by the continuity of the volume functional, $\mathrm{vol}(M, g) \to \mathrm{vol}(M, g_\infty) > 0$, contradicting $\mathrm{vol}(M, g) \to \inf \mathrm{vol}_{-1} = 0$. Thus the flow must degenerate and the nature of the degeneration is of importance.

A closed orientable 3-manifold M is a **graph manifold** if there is a finite collection $\mathcal{T} = \{\boldsymbol{T}_i^2\}$ of disjoint embedded tori $\boldsymbol{T}_i^2 \subset M$ such that each component M_j

of $M\setminus \cup \boldsymbol{T}_i^2$ is a Seifert fibered space, i.e., a space that admits a foliation by circles with the property that a foliated tubular neighborhood $D^2 \times \boldsymbol{S}^1$ of each leaf is either the trivial foliation of a solid torus $D^2 \times \boldsymbol{S}^1$ or its quotient by a standard action of a cyclic group (see Anderson [1] for more information about graph manifolds).

Thus a graph manifold is a union of Seifert fibered spaces glued together by toral automorphisms along toral boundary components. In particular, a Seifert fibered manifold is a graph manifold.

The σ-constant of a graph manifold M satisfies $\sigma(M) \geq 0$ (Anderson [1]). On the other hand, if M is of Yamabe type -1, a metric g of constant scalar curvature must have $R(g) = \text{constant} < 0$ and thus from the definition of the σ-constant, $\sigma(M) \leq 0$. Thus a 3-manifold M that is both a graph manifold and of Yamabe type -1 satisfies $\sigma(M) = 0$. Thus if there exists a non-singular conformal Ricci flow on such a manifold whose volume converges to $\inf \text{vol}_{-1} = (-\sigma(M))^{3/2} = 0$, one expects the volume collapse of M to occur along either $\boldsymbol{S}^1$ or $\boldsymbol{T}^2$-fibers with the embedded graph manifold structure of M describing how the degeneration occurs. Indeed, for the reduced Einstein field equations, cosmological models with closed spatial hypersurfaces of this type are known whose conformal volumes collapse to zero while the conformal metric collapses with bounded Ricci (or equivalently, Riemann) curvature norm along either $\boldsymbol{S}^1$ or $\boldsymbol{T}^2$-fibers (see Fischer-Moncrief [18], [21], Section 9, and [22]) and it is expected that the conformal Ricci flow behaves similarly under similar assumptions.

If $\sigma(M) < 0$, then Anderson [1] has conjectured that either M admits a hyperbolic metric or is the union along incompressible tori of finite volume hyperbolic manifolds and graph manifolds. In this case, if there exists a non-singular conformal Ricci flow whose volume converges to $\inf \text{vol}_{-1}$, then $\text{vol}(M, g) \to \inf \text{vol}_{-1} = (-\sigma(M))^{3/2} > 0$ as $t \to \infty$ so no volume collapse can occur. Nevertheless, if M does not admit a hyperbolic metric, then the conformal Ricci flow cannot converge to a fixed point, which must be hyperbolic by Proposition 3.1 (see also Remark (iii) following that Proposition) but is nevertheless seeking to attain the σ-constant asymptotically insofar as the volume is strictly monotonically decaying to its infimum. Thus again the conformal Ricci flow must degenerate and we expect the degeneration to occur by volume collapsing the graph manifold pieces but not the hyperbolic pieces.

If, on the other hand, M does admit a hyperbolic metric $g_h \in \mathcal{M}_{-1}$, then there exists a static equilibrium solution $g = g_h$ which by Mostow rigidity is unique up to isometry (Mostow rigidity applies because the scalar curvature is normalized to -1; see remark about hyperbolic metrics in Section 1.1). We conjecture that this static solution, as it is for the classical Ricci flow equation (Ye [44]), the Einstein equations (Andersson-Moncrief [2]), and the reduced Einstein equations (Fischer-Moncrief [20]) is **asymptotically stable** under conformal Ricci flow; i.e., given any neighborhood $\mathcal{U}_{g_h} \subset \mathcal{M}_{-1}$ of g_h there exists a *capture neighborhood* $\mathcal{U}'_{g_h} \subseteq \mathcal{U}_{g_h}$ such that if $g_0 \in \mathcal{U}'_{g_h}$ and g is the solution of the conformal Ricci system with initial value g_0, then (i) $g : [0, \infty) \to \mathcal{M}_{-1}$ is non-singular, (ii) for $t \in [0, \infty)$, $g_t \in \mathcal{U}_{g_h}$, and (iii) $g_t \to g_h$ as $t \to \infty$.

10. The conformal Ricci flow and geometrization of 3-manifolds

Using the classical Ricci flow, Hamilton ([25],[26]) has proposed a program to prove Thurston's geometrization conjectures for closed 3-manifolds [41]. The main idea is to try to show that the Ricci flow evolves any given initial Riemannian manifold (M, g_0) to a geometric structure after performing a finite number of surgeries as curvature

singularities arise in finite time during the Ricci flow. The hope is that if the surgeries are done before the singularities arise, the Ricci flow can be continued for all time and the initial manifold (M, g_0) will naturally decompose into geometrical pieces. In this picture the classical Ricci flow causes the initial Riemannian manifold (M, g_0) to self-geometrize to a limit space (M_∞, g_∞) which is the natural geometrical decomposition of (M, g_0).

Recently, Perelman ([35],[36],[37]) has attempted to further implement Hamilton's program. In Perelman's work, much like in the conformal Ricci flow, an important role is played by manifolds of Yamabe type -1, although he doesn't call them that; see [36], p. 17, where Perelman assumes that the initial manifold does not admit a Riemannian metric with non-negative scalar curvature. Since manifolds of Yamabe type 0 and 1 admit metrics with scalar curvature 0 and 1, respectively, these manifold types are excluded and so Perelman's manifolds must be of Yamabe type -1 (see Definition 1.2).

In Perelman's work, after each surgery, the manifold may become disconnected in which case each component is dealt with separately. If the initial manifold develops a component with nonnegative scalar curvature, then the Ricci flow on that component becomes extinct so that that component is removed from further consideration and the Ricci flow then continues on the new manifold ([36], pp. 6, 17). Note that the extinct components which are left behind are no longer of Yamabe type -1, reflecting again the importance of Yamabe type -1 manifolds in Perelman's work.

Some of Perelman's techniques have a natural counterpart for the conformal Ricci flow. For example, for the unnormalized Ricci flow, the volume $\mathrm{vol}(M, g)$ satisfies $\frac{d}{dt}\mathrm{vol}(M, g) = -\int_M R(g)d\mu_g$ and Perelman shows that the rescaled volume functional $\mathrm{vol}(M, g(t))(t + \frac{1}{4})^{-3/2}$ is a non-increasing function in t ([36], p. 17). However, under the conformal Ricci flow, the volume functional itself without rescaling is automatically non-increasing and is in fact strictly monotonically decreasing if the flow is non-static (Proposition 7.1).

As another example of some of these parallel techniques, Perelman identifies an "entropy" that increases as the space flows and helps move the space toward geometrization. For the conformal Ricci flow, the strictly monotonically decreasing volume $\mathrm{vol}(M, g)$ naturally has the same effect (although the monotonicity is in the opposite direction). For example, globally, the strictly monotonically decreasing volume acts as Perelman's entropy and prevents periodic orbits in the conformal Ricci flow (Corollary 7.4). Potential advantages of this decreasing volume over entropy are that it occurs naturally in the conformal Ricci flow and is geometrical simpler.

Locally, and somewhat more subtly, the strictly monotonically decreasing *local* volume also has the potential to control the size and shape of collapsing regions under the conformal Ricci flow since, if a singularity is forming within a domain D of a manifold, the volume must still be decreasing on that domain. Thus one can reasonably hope to contain the regions undergoing singularity development in the conformal Ricci flow by using the decreasing local volume.

Thus, altogether, we believe that the conformal Ricci flow has the potential to further our understanding of geometrization of 3-manifolds.

Lastly, we remark that the use of evolution equations to understand and control changes in topology actually has a history longer than is usually recognized. Hamilton's program has a precursor in Wheeler's geometrodynamical formulation of general relativity ([30], reprinted in [42], and [43], pp. 279–281, 290–291) which dates back to 1957. In [42], p. 306, Misner and Wheeler ask *When the deterministic evolution*

of the metric with time leads at a certain moment to fission or coalescence of wormhole mouths or to any other change of topology, what new phenomena occur?

This question is partially answered in ([43], pp. 279–280) where it is recognized that space evolving according to the dynamical formulation of Einstein's equations can, quoting Wheeler, *signal its "intention" to change topology by developing somewhere a curvature that increases without limit ("gravitational collapse"). To go further with the analysis of the collapse phenomenon and treat changes in topology forces one to go outside the framework of classical theory.* A foundation for this new framework, quantum geometrodynamics (QGD), is then discussed ([43], p. 284), with the hope that one day QGD might provide the theoretical basis for the sequence of surgeries that an initial Riemannian manifold (M, g_0), together with an initial second fundamental form $k_0 \in S_2$, evolving under the Einstein evolution equations, must undergo to geometrize itself.

Acknowledgments

References

[1] Anderson, M (1997), *Scalar curvature and geometrization conjectures for 3-manifolds*, in *Comparison geometry* (Berkeley, CA 1993-94), Math. Sci. Res. Inst. Publ. **30**, Cambridge University Press, Cambridge.

[2] Andersson, L, and Moncrief, V (2003), *On the global evolution problem in $3 + 1$ dimensional general relativity*, to appear.

[3] Aubin, T (1976), *Equations différentielles non linéaires et probleme de Yamabé concernant la courbure scalaire*, J. Math. Pures Appl. **55**, 269–296.

[4] Aubin T (1976), *The scalar curvature*, in *Differential geometry and relativity*, Mathematical Phys. and Appl. Math., Vol. 3, Reidel, Dordrecht, 5–18.

[5] Berger, M, and Ebin, D (1969), *Some decompositions of the space of symmetric tensors on a Riemannian manifold*, J. Diff. Geom. **3**, 379-392.

[6] Cao, H, and Chow, B (1999), *Recent developments on the Ricci flow*, Bull. Am. Math. Soc. **36**, 59–74.

[7] DeTurck, D (1983), *Deforming metrics in the direction of their Ricci tensors*, J. Differential Geometry **18**, 157–162.

[8] DeTurck, D (2003), *Deforming metrics in the direction of their Ricci tensors (Improved version)*, in *Collected papers on Ricci flow*, H Cao, B Chow, S Chu, and S T Yau, editors, International Press, Somerville, Massachusetts, 163–165.

[9] Ebin, D (1970), *The space of Riemannian metrics*, Proc. Symp. Pure Math., Amer. Math. Soc. 15, 11–40.

[10] Ebin, D, and Marsden, J (1970), *Groups of diffeomorphisms and the motion of an incompressible fluid*, Annals of Mathematics, **92**, 102-163.

[11] Fischer, A (2004), *Solutions to the Ricci flow equation in the H^s-setting*, in preparation.

[12] Fischer, A, and Marsden, J (1972), *The Einstein equations of evolution - a geometric approach*, Journal of Mathematical Physics, **28**, 1-38.

[13] Fischer, A, and Marsden, J (1975), *Deformations of the scalar curvature*, Duke Mathematical Journal **42**, 519–547.

[14] Fischer, A, and Moncrief, V (1994), *Reducing Einstein's equations to an unconstrained Hamiltonian system on the cotangent bundle of Teichmüller space*, in *Physics on Manifolds, Proceedings on the International Colloquium in honour of Yvonne Choquet-Bruhat*, edited by M Flato, R Kerner, and A Lichnerowicz, Kluwer Academic Publishers, Boston, 111–151.

[15] Fischer, A, and Moncrief, V (1996), *The structure of quantum conformal superspace*, in *Global Structure and Evolution in General Relativity*, edited by S Cotsakis and G W Gibbons, Springer-Verlag, Berlin, 111–173.

[16] Fischer, A, and Moncrief, V (1997), *Hamiltonian reduction of Einstein's equations of general relativity*, Nuclear Physics B (Proc. Suppl.) **57**, 142–161.

[17] Fischer, A, and Moncrief, V (1999), *The Einstein flow, the σ-constant and the geometrization of 3-manifolds*, Classical and Quantum Gravity, **16**, L79–L87.

[18] Fischer, A, and Moncrief, V (2000), *Hamiltonian reduction, the Einstein flow, and collapse of 3-manifolds*, Nuclear Physics B (Proc. Suppl.) **88**, 83-102.

[19] Fischer, A, and Moncrief, V (2001), *The reduced Einstein equations and the conformal volume collapse of 3-manifolds*, Classical and Quantum Gravity, **18**, 4493–4515.

[20] Fischer, A, and Moncrief, V (2002), *Collapse of 3-manifolds and the reduced Einstein flow* in *Geometry, Mechanics, and Dynamics: Volume in Honor of the 60th Birthday of J. E. Marsden*, edited by P Holmes, P K Newton, A Weinstein, Springer-Verlag, 463–522.

[21] Fischer, A, and Moncrief, V (2002), *Hamiltonian reduction and perturbations of continuously self-similar $(n+1)$-dimensional Einstein vacuum spacetimes*, Class. Quantum Grav. **19**, 5557–5589.

[22] Fischer, A, and Moncrief, V (2004), *Hidden structures in closed cosmological models*, to appear.

[23] Fischer, A, and Tromba, A (1984), *On a purely "Riemannian" proof of the structure and dimension of the unramified moduli space of a compact Riemann surface*, Mathematische Annalen **267**, 311–345.

[24] Hamilton, R (1982), *Three manifolds with positive Ricci curvature*, Jour. Diff. Geometry **17**, 255–306.

[25] Hamilton, R (1995), *The formation of singularities in the Ricci flow*, Surveys in Differential Geometry **2**, 7–136, International Press, Boston.

[26] Hamilton, R (1999), *Non-singular solutions of the Ricci flow on three-manifolds*, Commun. Anal. Geom. **7**, 695–729.

[27] Isenberg, J, and Jackson, M (1992), *Ricci flow of locally homogeneous geometries on closed manifolds*, J. Differential Geometry **35**, 723-741.

[28] Lang, S (1995), *Differentiable and Riemannian manifolds*, Springer-Verlag, New York.

[29] Marsden, J, Ebin, D, and Fischer, A (1972), *Diffeomorphism groups, hydrodynamics, and relativity*, in *Proceedings of the Thirteenth Biennial Seminar of the Canadian Mathematical Congress on Differential Topology; Differential Geometry and Applications, Volume* **1**, edited by J R Vanstone, Canadian Mathematical Society, Montreal, Canada, pp. 135–279.

[30] Misner, C W, and Wheeler, J A (1957), *Classical physics as geometry*, Ann. of Phys. **2**, 525–603.

[31] Omori, H (1970), *On the group of diffeomorphisms of a compact manifold*, Proc. Symp. Pure Math., Amer. Math. Soc. **15**, 167-183.

[32] Palais, R (1961), *Letter to Serge Lang*, dated November 10, 1961, unpublished.

[33] Palais, R (1965), *Seminar on the Atiyah-Singer index theorem*, Annals of Mathematics Studies, Number 57, Princeton University Press, Princeton, New Jersey.

[34] Palais, R (1968), *Foundations of global non-linear analysis*, W. A. Benjamin, Inc., New York.

[35] Perelman, G (2002), *The entropy formula for the Ricci flow and its geometric applications*, http://arxiv.org/abs/math/0211159, 1–39.

[36] Perelman, G (2003), *Ricci flow with surgery on three-manifolds*, http://arxiv.org/abs/math/0303109, 1–22.

[37] Perelman, G (2003), *Finite extinction time for the solutions of the Ricci flow on certain three-manifolds*, http://arxiv.org/abs/math/0307245, 1–7.

[38] Protter, M, and Weinberger, H (1984), *Maximum Principles in Differential Equations*, Springer-Verlag, New York.

[39] Schoen, R (1984), *Conformal deformation of a Riemannian metric to constant scalar curvature*, J. Differential Geom. **20**, 479–495.

[40] Taylor, M (1996) *Partial Differential Equations III, Nonlinear Equations*, Springer, New York.

[41] Thurston, W (1997), *Three-dimensional geometry and topology*, volume 1, edited by S Levy, Princeton University Press, Princeton, New Jersey.

[42] Wheeler, J A (1962), *Geometrodynamics*, Academic Press, New York.

[43] Wheeler, J A (1968), *Superspace and the nature of quantum geometrodynamics*, in *Battelle Rencontres – 1967 Lectures in Mathematics and Physics*, C. DeWitt and J. A. Wheeler, editors, W. A. Benjamin, Inc., New York.

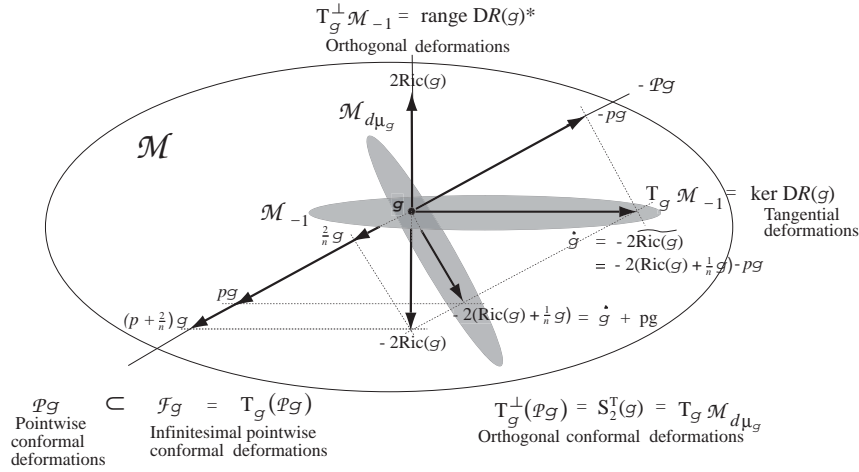[44] Ye, R (1993), *Ricci flow, Einstein metrics and space forms*, Trans. Amer. Math. Soc. **338**, 871–896.

$T_g^\perp \mathcal{M}_{-1} = $ range $DR(g)^*$
Orthogonal deformations

$2\mathrm{Ric}(g)$

$\mathcal{M}_{d\mu_g}$

$\mathcal{M}$

$- \mathcal{P}g$
$-pg$

$\mathcal{M}_{-1}$ g $T_g \mathcal{M}_{-1} = $ ker $DR(g)$
$\frac{2}{n}g$ Tangential deformations

$\dot{g} = -2\widetilde{\mathrm{Ric}(g)}$
$= -2(\mathrm{Ric}(g) + \frac{1}{n}g) - pg$

pg

$(p + \frac{2}{n})g$ $2(\mathrm{Ric}(g) + \frac{1}{n}g) = \dot{g} + pg$

$-2\mathrm{Ric}(g)$

$\mathcal{P}g \quad \subset \quad \mathcal{F}g \quad = \quad T_g(\mathcal{P}g)$ $T_g^\perp(\mathcal{P}g) = S_2^{\mathrm{T}}(g) = T_g \mathcal{M}_{d\mu_g}$
Pointwise   Infinitesimal pointwise   Orthogonal conformal deformations
conformal   conformal deformations
deformations

Figure 1. A depiction of the four "vector" sums

$$-2\,\widetilde{\mathrm{Ric}(g)} \quad = \ -2(\mathrm{Ric}(g) + \tfrac{1}{n}g) - pg = \frac{\partial g}{\partial t}$$

$$-2\,\mathrm{Ric}(g) \quad = \ -2\,\widetilde{\mathrm{Ric}(g)} + (p + \tfrac{2}{n})g$$

$$-2\,\mathrm{Ric}(g) \quad = \ -2(\mathrm{Ric}(g) + \tfrac{1}{n}g) + \tfrac{2}{n}g$$

$$-2(\mathrm{Ric}(g) + \tfrac{1}{n}g) = \ -2\widetilde{\mathrm{Ric}(g)} + pg\,,$$

where $p = 2L_g^{-1}(|\mathrm{Ric}(g)|^2) - \frac{2}{n}$. The tangential direction of $\mathcal{M}_{-1}$ is $T_g\mathcal{M}_{-1} \approx \ker DR(g)$ and the L_2-orthogonal direction is $T_g^\perp \mathcal{M}_{-1} \approx$ range $DR(g)^*$. The infinitesimal conformal direction is $T_g(Pg) \approx \mathcal{F}g$ with pointwise orthogonal direction $T_g^\perp(Pg) \approx S_2^T(g)$. Note that the "vector" $-2\mathrm{Ric}(g) \in$ range $DR(g)^*$ is orthogonal to $\mathcal{M}_{-1}$, that $-2(\mathrm{Ric}(g) + \frac{1}{n}g)$ is orthogonal to the conformal orbit $\mathcal{P}g$, and that $-2\mathrm{Ric}(g)$ and $-2(\mathrm{Ric}(g) + \frac{1}{n}g)$ have the same tangential component $-2\widetilde{\mathrm{Ric}(g)} \in \ker DR(g)$. The first equation above resolves the inertial "vector" $\frac{\partial g}{\partial t} \in \ker DR(g)$ into orthogonal directions so that the constraint force $-pg$ compensates for the conformal component pg of the nonlinear restoring force $-2(\mathrm{Ric}(g) + \frac{1}{n}g)$ in the non-orthogonal splitting. The resulting flow then lies in $\mathcal{M}_{-1}$.

Lastly, we note that since $\mathrm{Ric}(g)$ is orthogonal to $\mathcal{M}_{-1}$ (see Example 2.2), the classical unnormalized Ricci flow (1.14), at a metric $g \in \mathcal{M}_{-1}$, is orthogonal to $\mathcal{M}_{-1}$, whereas the conformal Ricci flow is tangential to $\mathcal{M}_{-1}$. Thus in some sense the classical unnormalized Ricci flow acts like a Riemannian gradient as it is orthogonal to the level hypersurface $\mathcal{M}_{-1}$, whereas the conformal Ricci flow acts like a symplectic gradient as it is tangent to the level hypersurface $\mathcal{M}_{-1}$.