Algorithm for image reconstruction in multi-slice helical CT

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Efforts are being made to develop a new type of CT system that can scan volumes over a large range within a short time with thin slice images. One of the most promising approaches is the combination of helical scanning with multi-slice CT, which involves several detector arrays stacked in the z direction. However, the algorithm for image reconstruction remains one of the biggest problems in multi-slice CT. Two helical interpolation methods for single-slice CT, 360LI and 180LI, were used as starting points and extended to multi-slice CT. The extended methods, however, had a serious image quality problem due to the following three reasons: (1) excessively close slice positions of the complementary and direct data, resulting in a larger sampling interval; (2) the existence of several discontinuous changeovers in pairs of data samples for interpolation; and (3) the existence of cone angles. Therefore we have proposed a new algorithm to overcome the problem. It consists of the following three parts: (1) optimized sampling scan; (2) filter interpolation; and (3) fan-beam reconstruction. Optimized sampling scan refers to a special type of multi-slice helical scan developed to shift the slice position of complementary data and to acquire data with a much smaller sampling interval in the z direction. Filter interpolation refers to a filtering process performed in the z direction using several data. The normal fan-beam reconstruction technique is used. The section sensitivity profile (SSP) and image quality for four-array multi-slice CT were investigated by computer simulations. Combinations of three types of optimized sampling scan and various filter widths were used. The algorithm enables us to achieve acceptable image quality and spatial resolution at a scanning speed that is about three times faster than that for single-slice CT. The noise characteristics show that the proposed algorithm efficiently utilizes the data collected with optimized sampling scan. The new algorithm allows suitable combinations of scan and filter parameters to be selected to meet the purpose of each examination. © 1998 American Association of Physicists in Medicine. [S0094-2054(98)00204-1]

Key words: multi-slice CT, helical scan, image reconstruction, filter interpolation, optimized sampling scan

I. INTRODUCTION

Before the development of helical computed tomography (CT) around 1990,1–5 advances in CT technology focused mainly on faster scanning and on higher resolution in the axial plane. Helical CT enabled us to obtain volume data in a single breath hold, and opened the door to the age of volumetric imaging. Helical scan needs interpolation in the longitudinal direction (z direction) before fan-beam reconstruction. With the progress of helical scanning, various types of interpolation and reconstruction methods have been proposed. These include 360° linear interpolation1–4 (360LI), 180° linear interpolation5,6 (180LI), 180° linear extrapolation7 (180LE), nonlinear interpolation, a general approach,6 and a deconvolution process.7

Since higher-pitch helical scanning degrades the section sensitivity profile (SSP), we have to compromise between scan speed and spatial resolution in single-slice helical CT. There is, however, a need for faster scanning and higher resolution in volume in various clinical fields. CT scanners are expected to evolve into volume scanners with a small voxel size, large scanning range, and high speed. The use of a multi-slice detector, consisting of several detector arrays stacked in the z direction, is a promising approach to the development of volume scanners. It can also improve the time resolution of dynamic volume scans.

The lack of a suitable algorithm for image reconstruction is one of the biggest stumbling blocks in the development of multi-slice CT. The cone angle creates problems even in the case of small angles. Several algorithms using cone-beam backprojection have been proposed for helical scan.5–10 Cone-beam backprojection, however, requires complicated calculation relative to fan-beam backprojection. This is an important practical problem hindering implementation.

We have therefore developed a new algorithm for multi-slice CT using fan-beam backprojection. It consists of following three parts: (1) optimized sampling scan; (2) filter interpolation; and (3) fan-beam reconstruction. Optimized sampling scan refers to a special type of multi-slice helical scan technique developed in order to shift the slice position of complementary data and to acquire data with a significantly smaller sampling interval. Filter interpolation refers to a filtering process performed in the z direction using several data samples in order to reduce the effect of discontinuous changeovers. Subsequently, the normal fan-beam reconstruction technique is used. The proposed algorithm allows us to...
achieve both the image quality and the SSP levels sufficient for practical use.

In this paper we define the scan geometry and notation in Sec. II. In Sec. III, we describe two methods for single-slice CT, outline the extension of these methods for multi-slice CT, and discuss their shortcomings. We then propose a new algorithm in Sec. IV and use computer simulations in Sec. V to demonstrate that this approach overcomes the problems discussed in Sec. III. The discussion and conclusions are presented, respectively, in Sec. VI and Sec. VII.

II. DEFINITION OF SCANNER GEOMETRY AND PARAMETERS

In this section, we define the scanner geometry and establish the notation used in this paper. For the following discussion we focus our attention on the third generation geometry for the case of four-array multi-slice CT and single-slice CT. The results can easily be extended to other geometries or to multi-slice CT with a different number of arrays. The geometry and coordinates are shown in Fig. 1. The multi-slice detector consists of four arrays \((N=4)\) stacked in the \(z\) direction along a cylindrical surface. The view and channel angles are denoted by \(\beta\) and \(\gamma\), respectively. The maximum channel angle is \(\gamma_m\). The focus-to-center distance is defined by \(f_{cd}\) and the focus-to-detector distance by \(f_{dd}\). BW refers to the beamwidth in the \(z\) direction for one detector array at the center of rotation. BW for single-slice CT is defined in Fig. 1(c). The slice position of each array is defined as the center of BW at the center of rotation. The normalized helical pitch \(P\) is given by

\[
P = \frac{L}{BW},
\]

where \(L\) denotes the \(z\) increment per rotation.

III. SINGLE-SLICE CT AND EXTENSION TO MULTI-SLICE CT

This section outlines and extends 360LI and 180LI, which are helical interpolation methods for single-slice CT. These two methods were chosen because of their complementarity. The characteristics of 360LI are adequate with respect to both noise and artifact elimination but inadequate with respect to the SSP while the converse is true for 180LI. The use of the above two algorithms for comparison will therefore be sufficient for evaluation of the proposed algorithm. The helical pitch was chosen as \(P=1\) for single-slice CT and \(P=4\) for multi-slice CT. \(P=N\) was chosen because it is equivalent to the total width of the beam at the rotation center and seems to be the simplest condition for the extension.

The direct and complementary data are defined as \(pd(\beta, \gamma)\) and \(pc(\beta, \gamma)\), respectively, and \(zd(\beta)\) and \(zc(\beta, \gamma)\) are the corresponding slice positions. Here, direct data refer to data obtained at the focus position at the current view angle. The data obtained at the opposite side are referred to as complementary data [Fig. 2(a)]. For the slice \(z = z_0\), 360LI is described by Eq. (2) and 180LI by Eq. (3):

\[
p(\beta, \gamma) = w(\beta) \times pd(\beta \pm 2\pi , \gamma) + (1 - w(\beta)) \times pd(\beta, \gamma),
\]

\[
w(\beta) = \frac{z_0 - zd(\beta)}{zd(\beta \pm 2\pi ) - zd(\beta)}.
\]
Figure 2 shows the helical scan geometry. Figure 2(a) is called the view diagram and Fig. 2(b) is called the scan diagram. The view diagram shows the relationship between the channel angle and the slice position at an arbitrary view angle, while the scan diagram shows the relationship between the view angle and the slice position for the whole helical scan. Note that the complementary data of only the central channel is drawn in the scan diagram. Since the distance between the complementary and direct data is small, 180LI gives sharper SSP than 360LI.

Figure 3 shows the helical scan geometry for multi-slice CT in the case of \( P = 4 \). The direct and complementary data are identified by the detector array number \( (n = 1, 2, 3, 4) \). The direct and complementary data are thus defined as \( pd(\beta, \gamma, n) \) and \( pc(\beta, \gamma, n) \), respectively, and \( zd(\beta, n) \) and \( zc(\beta, \gamma, n) \) are the corresponding slice positions. The extended 360° linear interpolation (ex-360LI) is given by Eq. (4) and extended 180° linear interpolation (ex-180LI) by Eq. (5):
where \( ph(\beta, \gamma, j) \) and \( ph(\beta, \gamma, j+1) \) are the lower and upper data sets adjacent to the slice \( z = z_0 \), respectively, and \( zh(\beta, \gamma, j) \) and \( zh(\beta, \gamma, j+1) \) are the corresponding slice positions. Figure 3 and Eq. (5) indicate that there are the following two serious problems in multi-slice helical CT: (1) excessively close slice positions for complementary and direct data, resulting in a larger sampling interval; (2) several discontinuous changeovers in pairs of data samples in the interpolated data sets obtained for one rotation. Because the slice position of the complementary data is exactly the same as that of the direct data of another array at the central channel (\( \gamma = 0 \)), the sampling distance between two data points used for interpolation is larger than that for 180LI. In addition, multi-slice CT needs several times the number of changeovers required by single-slice CT; which degrades the SSP and image quality.

IV. PROPOSED METHOD

This section introduces a new algorithm which consists of the following three parts: (A) optimized sampling scan; (B) filter interpolation; and (C) fan-beam reconstruction.

A. Optimized sampling scan

One problem in ex-180LI is the slice position of the complementary data at the central channel. The most effective solution is to change the helical pitch in order to shift the complementary data. Figure 4(a), (b), and (c) shows scan diagrams with a pitch of 2.5, 3.5, and 4.5, respectively (\( P = 2.5, 3.5, 4.5 \)). Complementary data of only the central channel is shown. Figure 4(d) shows the view diagram for a pitch of 2.5. Figure 4(a)–(d) shows that the sampling interval in the \( z \) direction decreases significantly to about one-fourth as compared to \( P = 4 \) (Fig. 3).
B. Filter interpolation

Another problem with the extended method is the existence of several discontinuous changeovers in pairs of data samples in the interpolated data sets obtained for one rotation. The most effective method of eliminating these changeover effects is to utilize more than two data sets for interpolation. ‘‘Filter interpolation’’ is a filtering process that is performed in the z direction with several data sets and is used instead of helical interpolation. The concept is shown in Fig. 5: A width is assumed in the z direction and is defined as the filter width (FW). As a result of the optimized sampling scan, several data points lie within FW. Filter interpolation can be divided into the following two steps as shown in Eq. (6): (1) resampling by linear interpolation using adjacent data points; and (2) filtering the resampled data set.

In the first step, the resampling positions are determined by the following two factors: (1) the number of resampling points, 2I + 1; and (2) the filter width, FW. Complementary data are used in resampling.

1. Step 1: Resampling by linear interpolation (for −I ≤ i ≤ I)

\[ p(\beta, \gamma, i) = w(\beta, \gamma, j(i)) \times ph(\beta, \gamma, j(i) + 1) + (1 - w(\beta, \gamma, j(i))) \times ph(\beta, \gamma, j(i)) \]

\[ w(\beta, \gamma, j(i)) = \frac{zf(i) - zh(\beta, \gamma, j(i))}{zh(\beta, \gamma, j(i) + 1) - zh(\beta, \gamma, j(i))} \]

\[ zf(i) = \Delta z = \frac{FW}{2 \times I + 1} \]

2. Step 2: Filtering

\[ p(\beta, \gamma) = \frac{\sum_{i=-I}^{I} w(i) \times p(\beta, \gamma, i)}{\sum_{i=-I}^{I} w(i)} \]

where \( ph(\beta, \gamma, j(i)) \) and \( ph(\beta, \gamma, j(i) + 1) \) are the lower and upper data sets adjacent to the resampling position \( z = zf(i) \), respectively, and \( zh(\beta, \gamma, j(i)) \) and \( zh(\beta, \gamma, j(i) + 1) \) are the corresponding slice positions. \( \Delta z \) is the resampling interval.

The shape and width of the filter are defined by \( wt(i) \), I, and FW. They can be chosen freely for the expected SSP, image quality, and noise. Some examples of filter shape are shown in Fig. 6: (a) smoothing; (b) edge enhancement; and (c) modifying the shape of SSP.

Filter interpolation, as given by Eq. (6), needs a large number of calculations. ‘‘Direct filtering’’, which is a much faster implementation method, is outlined in the Appendix. Note that filter interpolation corresponds to ex-180LI with optimized sampling scan when FW=0 mm.

C. Fan-beam reconstruction

The normal fan-beam reconstruction technique, which uses filtered backprojection, is applied to obtain slice images using the \( p(\beta, \gamma) \) data of Eq. (6).

V. COMPUTER SIMULATIONS

This section shows the results of computer simulations with three types of projection data: (1) coin phantom data for the SSP; (2) random noise data for the image noise; and (3) ball phantom data for the image quality. A program was written to generate line integrals through the phantoms.

<table>
<thead>
<tr>
<th>Scan parameters</th>
<th>Filter parameters</th>
</tr>
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<tbody>
<tr>
<td>( P )</td>
<td>( BW )</td>
</tr>
<tr>
<td>(1)</td>
<td>2.5</td>
</tr>
<tr>
<td>(2)</td>
<td>3.5</td>
</tr>
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<td>(3)</td>
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<td>(5)</td>
<td>1.0</td>
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focus, channel angle, and view angle are divided into a number of micro-parts and summed in order to avoid digitalization errors.

The values of all parameters are shown in Table I. We chose the following three scan modes from a number of combinations for discussion; (1) \( P = 2.5 \), (2) \( P = 3.5 \), and (3) \( P = 4.5 \). The filter shape was fixed as rectangular, and filter width \( \text{FW} \) was varied from 0 mm to 4 mm. The scan modes compared were (4) \( P = 4 \), multi-slice CT, and (5) \( P = 1 \), single-slice CT. The beamwidth (BW) and fcd were, respectively, fixed at 2 mm and at 600 mm in all the computer simulations.

A. Section sensitivity profile (SSP)

Coin phantoms were used to evaluate the SSP for various scan and filter parameters. The phantoms, which had a diameter of 20 mm and a thickness of 0.2 mm, were placed within the central slice (\( z = 0 \) mm) at four positions along a circle.

![Fig. 7. Section sensitivity profiles (SSPs) for a beam width (BW) of 2 mm for single-slice CT and extended multi-slice CT; (a) 360LI (\( P = 1 \)), (b) 180LI (\( P = 1 \)), (c) ex-360LI (\( P = 4 \)), and (d) ex-180LI (\( P = 4 \)). The profiles of the extended multi-slice methods are almost the same as that of 360LI.](image1)

![Fig. 8. Variation of (a) FWHM and (b) FWTM of the SSPs of the three proposed methods with \( \text{FW} \). BW is fixed at 2 mm. Proposed methods: (1) \( P = 2.5 \), (2) \( P = 3.5 \), (3) \( P = 4.5 \). These methods allowed better values than those of the 360LI single-slice method (\( P = 1 \)) and the extended methods (ex-360LI and ex-180LI with \( P = 4 \)) to be obtained when \( \text{FW} \leq 2 \) mm.](image2)

![Fig. 9. SSPs for the proposed method \( [P=2.5 \text{ (BW}=2 \text{ mm)}] \) for various filter widths (\( \text{FW}=0,1,2,3 \) mm). The profiles for single-slice CT (360LI and 180LI with \( P = 1 \)) are also shown. The profiles for the proposed method are similar to that of 180LI when \( \text{FW} \leq 1 \) mm and comparable to those of 180LI and 360LI when \( \text{FW} = 2 \) mm.](image3)

![Fig. 10. SSPs for the proposed scan modes \( \{P=2.5; \text{BW}=2 \text{ mm}\} \) with \( \text{FW}=2 \) mm and \( \text{BW}=2 \) mm. The profiles for single-slice CT (360LI and 180LI with \( P = 1 \)) are also shown. The profiles for the proposed methods are comparable to those for single-slice CT.](image4)
and 180LI until FW of 2 mm for single-slice CT and extended multi-slice CT; 360LI and 180LI with \( P = 1 \); (3) ex-360LI \( (P = 4) \), and (d) ex-180LI \( (P = 4) \). The profiles of the extended multi-slice methods are almost the same as that of 360LI.

Figure 7 shows the SSPs at the center of rotation for single-slice and extended multi-slice CT; 360LI and 180LI with \( P = 1 \); (3) ex-360LI \( (P = 4) \); and of single-slice CT \( (P = 1) \); (4) 360LI and (5) 180LI. FW is fixed at 2 mm. It can be seen that all the proposed methods give comparable profiles in spite of the thick FW.

Figure 10 shows the SSPs of the proposed methods; (1) \( P = 2.5 \), (2) \( P = 3.5 \), (3) \( P = 4.5 \); and of single-slice CT \( (P = 1) \); (4) 360LI and (5) 180LI. FW is fixed at 2 mm. It can be seen that the spatial variations of the SSP for the proposed methods are comparable to those for single-slice CT (Fig. 11). The spatial variations for \( P = 2.5 \) are particularly good.

B. Image noise

Random noise data were used to evaluate image noise for various scan modes and filter widths. The images were re-
constructed and the standard deviation (s.d.) of the CT value in the ROI was measured at the same five positions as the SSP.

Figure 13 shows the s.d.’s at the five positions for the proposed methods ($P=2.5$, 3.5, and 4.5 with FW = 0, 2, and 4 mm) and single-slice CT (360LI and 180LI with $P=1$). The s.d.’s of the proposed methods are similar when FW is 0 mm, that is, the noise characteristics are similar even though the data sampling interval is different. This result indicates that only some data samples are used in helical interpolation and the others are wasted.

The s.d.’s, however, decrease with the increase in FW, because the number of data used in the filter interpolation increases. The ratio of the s.d.’s can be calculated as the square root of the number of data used when the filter shape used is a rectangle. The values of $\sqrt{2/5}$ and $\sqrt{2/9}$ are 0.63 and 0.47, respectively, and thus match the simulation result.

The higher the helical scanning pitch, the less the number of data sampled in the same filter width. Thus a difference between the s.d.’s for the three scan modes appears when the FW increases.

Note that the s.d. of image noise for the proposed method
is smaller than that for single-slice CT when FW is equivalent to BW. Thus the same noise level can be achieved using noisier projection data.

The spatial variation of the image noise for our methods was also smaller than that for single-slice CT: \(\pm 5\%\) maximum \((P=2.5\) with FW=0, 2 mm and \(P=3.5\) with FW=2 mm\) against \(\pm 13\%\) for 180LI and \(\pm 8\%\) for 360LI.

C. Image quality

Ball phantoms were used to evaluate image quality with various scan modes and filter widths. Eight balls \((\phi = 20\text{ mm})\) were set at the central plane \((z = 0\text{ mm})\) along a circle \((\phi = 200\text{ mm})\). The contrasts of the ball and the background were 0 HU and \(-1000\text{ HU}\), respectively. The scan parameters were the same as those for the SSP. Slice images near the edges of the balls \((z = 8\text{ mm})\) were reconstructed and compared with each other. The window level and window width were fixed at \(-1000\) and 80, respectively.

Images for single-slice and extended multi-slice CT (360LI, 180LI, ex-360LI, and ex-180LI) are shown in Fig. 14. Severe artifacts appear for both ex-180LI and ex-360LI. These artifacts will clearly be a serious problem in clinical use.

Images obtained by the three proposed methods are shown in Fig. 15; (1) \(P=2.5\), (2) \(P=3.5\), and (3) \(P=4.5\) with FW=0, 2, 4 mm. When FW=0 mm, these methods correspond to ex-180LI with optimized sampling scan. It can be seen that both optimized sampling scan and filter interpolation improve the image quality, and that the image quality improves markedly with the increase of FW. Note that the image quality with a pitch of 2.5 or 3.5 is comparable to that of 180LI when FW=2 mm (cf. Fig. 14). As long as the artifact level is almost equal to that of single-slice CT (360LI or 180LI), the results are acceptable.

For further quantitative analysis of the artifact level, the profiles across one ball phantom are shown in Fig. 16. The \((x,y)\) coordinates of the center of the phantom was \((0\text{ mm}, 100\text{ mm})\). The profiles in the horizontal direction \((x)\) are shown in (a) \(P=2.5\) with various filter widths, (b) FW=2 mm with various scan modes, and (c) single-slice (360LI and 180LI with \(P=1\)) and extended multi-slice CT (ex-360LI and ex-180LI with \(P=4\)). The corresponding profiles in the vertical direction \((y)\) are shown in Figs. 16(d)–(f). Either an undershoot or an overshoot near the background indicates an artifact. Filter interpolation improves...
both when FW is 2 mm [Fig. 16(a) and (d)]. It can also be seen that the artifact levels with the proposed methods \( P = 2.5 \) and 3.5 are comparable to that with 180LI.

VI. DISCUSSION

The computer simulations show that the extended methods (ex-360LI and ex-180LI) have severe problems as discussed in Sec. III. Both the image quality and the SSP are seriously degraded. This is an inherent problem of two-point interpolation in multi-slice CT. The problem level was so severe that a new method leading to drastic improvement was needed.

The computer simulations show that the optimized sampling scan improves SSPs. This can be attributed to the significantly smaller sampling interval.

The simulations also demonstrate that filter interpolation produces better image quality with larger FW and sharper SSPs with smaller FW. This is because the use of multiple data sampled over a smaller interval within FW can eliminate the effects of discontinuous changeovers.

In helical scan with a pitch of 2.5 or 3.5, both the image quality and the SSPs are comparable to those of single-slice CT when FW is equivalent to BW. When the helical pitch is higher, both the SSP and the image quality are degraded. Improvement using half-backprojection or other methods is required, especially for helical pitch values higher than 4.5.

In clinical practice, we have two choices with multi-slice CT: (1) scanning faster over a greater length with equivalent image quality and SSP; or (2) obtaining better data without partial volume effects by summing several narrow beam scans.
width data sets. We can also change the spatial resolution in the \( z \) direction by adjusting the filter parameters, as we can do presently in the transaxial plane. In short, we can choose a suitable combination of scan and filter parameters to meet the purpose of the examination.

The scan parameters are the normalized helical pitch (\( P \)) and the beam width for one detector array at the center of rotation (BW). The filter parameters are the width (FW) and the shape (rectangular, etc.) of the filter in filter interpolation.

Some algorithms for cone-beam backprojection have a patient dose problem, because one data is selected and the other is thrown away in overlapped areas. Although there are overlapped areas in our proposed method, all the data are efficiently utilized in filter interpolation. According to the results of our investigation on the image noise, the tube current for our method can be reduced while keeping the total patient dose and image noise at the same levels as those for single-slice CT. This is advantageous in practical systems for clinical use.

It should be noted here that the proposed algorithm can also be applied to multi-slice CT with a different number of detector arrays.

**VII. CONCLUSION**

We have presented a new reconstruction algorithm for multi-slice helical CT based on the combination of optimized sampling scan, filter interpolation, and fan-beam reconstruction. We have presented the results of computer simulations with various scan and filter parameters for four-array multislice CT. The algorithm enables us to achieve acceptable image quality and spatial resolution at a scanning speed that is about three times faster than that for single-slice CT. The noise characteristics show that the proposed algorithm efficiently utilizes the data collected with optimized sampling scan. A suitable combination of scan and filter parameters can be chosen according to the purpose of the examination.

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**APPENDIX: DIRECT FILTERING METHOD (FASTER FILTER INTERPOLATION TECHNIQUE)**

In the Appendix we propose an implementation method for filter interpolation that requires a shorter processing time. This method is based on integral calculus.

The following explanation assumes that a rectangular filter is used. Figure 17 shows the sampling data points in the \( z \) direction at an arbitrary view and channel angle. The number of data points used in this process is defined as \( J \). The \( j \)th data value is denoted by \( ph(j) \) and the corresponding slice position by \( zh(j) \). FW is the filter width and \( z0, zL, \) and \( zH \) are the \( z \) positions of the center, bottom edge, and top edge of the filter, respectively.

![Figure 17. Parameters for "direct filtering" (a faster filter interpolation).](image)

\[ \text{(1-1) Outside case \((j=1\) or \(J))} \]

\[
w(j) = \frac{1}{FW} \left[ \int_0^{zh(j+1)-zL} \frac{u}{zh(j+1)-zh(j)} \, du \right]
\]

\[
= \frac{1}{FW} \left[ \frac{(zh(j+1)-zL)^2}{2 \cdot (zh(j+1)-zh(j))} \right] \quad [j=1]. \tag{A1}
\]

\[
w(j) = \frac{1}{FW} \left[ \int_0^{zh(j+1)-zh(j)} \frac{t}{zh(j)-zh(j-1)} \, dt \right]
\]

\[
= \frac{1}{FW} \left[ \frac{(zh(j)-zh(j-1))}{2 \cdot (zh(j)-zh(j-1))} \right] \quad [j=J]. \tag{A2}
\]

\[ \text{(1-2) Inside case \((j=2\) or \(J-1))} \]

Variables \( t \) and \( u \) for the case of \( j=2 \) are shown in Fig. A1:

\[
w(j) = \frac{1}{FW} \left[ \int_0^{zh(j+1)-zh(j)} \left( 1 - \frac{t}{zh(j+1)-zh(j)} \right) \, dt \right]
\]

\[
+ \int_0^{zh(j)-zL} \left( 1 - \frac{u}{zh(j)-zh(j-1)} \right) \, du \]

\[
= \frac{1}{FW} \left[ \frac{(zh(j+1)-zh(j))}{2} + (zh(j)-zL) \right.
\]

\[
- \frac{(zh(j)-zL)^2}{2 \cdot (zh(j)-zh(j-1))} \quad [j=2]; \tag{A3}
\]

A. Calculating the weights for the data

Depending on the relative \( z \) position, the corresponding equation is selected from among Eqs. (7)–(11). Weights for all data are calculated using these equations. Note that the weights are normalized against FW.
\[ w(j) = \frac{1}{FW} \left[ \int_0^{zh(1)-zh(j)} \frac{t}{zh(j+1)-zh(j)} dt \right. \\
+ \int_{zh(j)-zh(j-1)}^{zh(j)} \frac{u}{zh(j)-zh(j-1)} du] \\
= \frac{1}{FW} \left[ \frac{(zh(j)-zh(j-1))}{2} + (zh-h(j)) \right] \\
- \frac{(zh-h(j))^2}{2 \cdot (zh(j+1)-zh(j))} \quad [j = J - 1]. \quad (A4) \]

**1(3) Internal case (3 ≤ j ≤ J - 2)**

Variables \( t \) and \( s \) for the case of \( j = 3 \) are shown in Fig. 17:

\[ w(j) = \frac{1}{FW} \left[ \int_0^{zh(j)-zh(j-1)} \frac{t}{zh(j)-zh(j-1)} dt \right. \\
+ \int_{zh(j)-zh(j-1)}^{zh(j)} \frac{s}{zh(j+1)-zh(j)} ds] \\
= \frac{zh(j+1)-zh(j-1)}{2 \cdot FW} \quad [3 \leq j \leq J - 2]. \quad (A5) \]

**B. Direct filtering using the calculated weights**

All the data are assigned weights and summed according to the following equation in order to obtain the data \( z = z0 \):

\[ p(zc) = \sum_{j=1}^{j} [w(j) \cdot ph(j)]. \quad (A6) \]

When \( J = 3 \) with small FW, (1-2) is modified to (1-2)** In-side case (\( j = J - 1 \))

\[ w(j) = \frac{1}{FW} \left[ \int_0^{zh(j)-zh(j-1)} \frac{t}{zh(j+1)-zh(j)} dt \right. \\
+ \int_{zh(j)-zh(j-1)}^{zh(j)-zh(j-1)} \frac{u}{zh(j)-zh(j-1)} du] \\
= 1 - \frac{1}{FW} \left[ \frac{(zh-h(j))}{2 \cdot (zh(j+1)-zh(j))} \\
+ \frac{(zh-h(j)-zh(j-1))}{2 \cdot (zh(j+1)-zh(j-1))} \right] \quad [j = 2] \\
\cdot [zh-h-L = FW]. \quad (A7) \]

When \( J = 2 \) with thin FW, this becomes a simple linear interpolation with respect to \( z = z0 \).

**1(4) Case of J=2**

\[ w(2) = \frac{1}{FW} \left[ \int_0^{zh(1)-zh(1)} \frac{t}{zh(2)-zh(1)} dt \right. \\
= \frac{zh(2)-zh(1)}{zh(2)-zh(1)}, \quad (A8) \]

\[ w(1) = 1 - w(2) = \frac{zh(2)-zh(1)}{zh(2)-zh(1)}. \]

This method can easily be modified for a nonrectangular filter \( w(z) \). For example, Eqs. (A5) and (A6) will be modified to

\[ w(j) = \frac{1}{FW} \left[ \int_0^{zh(j)-zh(j-1)} \frac{t}{zh(j)-zh(j-1)} \cdot wt(t) \cdot dt \right. \\
+ \int_{zh(j)-zh(j-1)}^{zh(j)} \left( 1 - \frac{s}{zh(j+1)-zh(j)} \right) \cdot wt(s) \cdot ds; \quad (A9) \]

\[ p(zc) = \frac{\sum_{j=1}^{J}[w(j) \cdot ph(j)]}{\sum_{j=1}^{J}[w(j)]}. \quad (A10) \]