

EXACT SOLUTION TO NON-LINEAR BIOLOGICAL POPULATION MODEL WITH FRACTIONAL ORDER

by

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In this paper, optimal homotopy asymptotic method has been extended to seek out the exact solution of fractional generalized biological population models. The time fractional derivatives are described in the Caputo sense. It optimal homotopy asymptotic method is a new approach for fractional models. The proposed approach presents a procedure by that we have transferred the model to a series of simpler problems which are solvable by hand work applying the Riemann-Liouville fractional integral operator and obtained exact solution of fractional the generalized biological population by adding the solutions of first three simple problems of the series of simpler problems. The new approach provides exact solution in the way of smoothly convergent series.

Key words: *optimal homotopy asymptotic, exact solution, fractional generalized biological population*

Introduction

In recent years, the non-linear fractional order PDE have been the center of attention of many studies due to their frequent applications in the fields of electromagnetic, electrochemistry, acoustics, material science, physics, viscoelasticity chemical processes, biology and engineering [1-4]. The solutions of non-linear fractional order PDE have great interest both in mathematics and in useful applications. Therefore, it is the main goal in the area of research of fractional models that how to develop a stable approach for to seek out the exact solution or approximate solution of fractional order PDE. Recently, several researchers have developed certain procedures for to seek out the analytical solution of non-linear fractional order PDE like the method of variable separation, the Laplace transform Mellin transform, Fourier transform and other techniques [5-9]. These are those fractional differential equations who has the exact analytical solutions are only a few easy cases and equivalent to a few special functions such as the hyperbolic geometric function and the Fox H function [10, 11] and the implementation of these methods for other strong non-linear fractional models are very complicated. Some time the provided methods are not capable to control non-linearity of fractional differential equation (FDE) to find out the solution of FDE. The reason of difficulties in finding exact solutions for most problems and the complexity of computing these special functions limit the applications of applied FDE in engineering and scientific fields. In some cases the exact solutions of non-linear

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fractional order PDE do not exist. Regarding these reasons, many of the scholars are searching and have developed numerical algorithms for approximate solution to solve the FDE instead to find the exact solution to achieve the goal. These methods are the finite difference method, spectral element method and finite element method, [12-16], *etc.* But it is the need of time to introduce a new method to achieve the goal which is ease in implementation and applicable for solution of all kind of non-linear FDE. Regarding this goal, we introduce a new recently developed method optimal homotopy asymptotic method (OHAM) that how to construct the exact solution of non-linear FDE. In OHAM procedure we use homotopy transformation in order to reduce non-linear FDE into simpler FDE and construct solution by using logical stipulation of OHAM which rapidly converges to the exact solution. It has many successful applications into the integer order differential equations so far. The OHAM has been used and introduced in many articles, see [17]. But in our work, the main focus is to extend a new approach OHAM for solving the exact solution of the non-linear fractional-order biological population model [10]:

$$\frac{\partial^\alpha u(x, y, t)}{\partial t^\alpha} = \frac{\partial^2 u^2(x, y, t)}{\partial x^2} + \frac{\partial^2 u^2(x, y, t)}{\partial y^2} + f[u(x, y, t)], \quad t > 0, \quad 0 < \alpha \leq 1 \quad (1)$$

subject to the initial condition:

$$u(x, y, 0) = f_0(x, y)$$

where $u(x, y, t)$ denotes the population density and $f[u(x, y, t)] = u(x, y, t)[1 - ru(x, y, t)]$ represents the population supply due to births and deaths. The functions $u(X, t)$ and $f(u)$ describe the diffusion of a biological species in a region R and are the functions of position $X = (x, y)$ in R and time t . The function $u(X, t)$ shows the number of individuals, per unit volume, at position X and time, t , its integral over any sub region shows the whole population of sub region at time, t . The function $f(u)$ shows the average rate at that individuals are supplied (per unit volume) straightforwardly at X by births and deaths. For $\alpha \rightarrow 1$, the three examples of constitutive equations for $f(u)$ are:

- $f(u) = c$, where c is a constant, it leads to Malthusian law,
- $f(u) = u(r_1 - r_2u)$, where r_1 and r_2 are positive constants, it leads to Verhulst law, and
- $f(u) = -ru^p$, ($r \geq 0$, $0 < p < 1$), it leads to porous media, see [10].

Basic definitions

We need the following axially results throughout the paper. Here we remark that the derivative has been taken in Caputo sense in this manuscript, see [18].

Definition 1. A real function $f(x)$, $x > 0$, is said to be in space C_μ , $\mu \in \mathfrak{R}$, if there a real number $p > \mu$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C(0, \infty)$ and it is said to be in the space C_μ^m if only if $f^m \in C_\mu$, $m \in \mathbb{N}$.

Definition 2. The Riemann-Liouville fractional integral operator of order $\alpha > 0$ of a function $f \in C_\mu$, $\mu \geq -1$ is defined:

$$J_{a_0}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_{a_0}^x (x - \mu)^{\alpha-1} f(\mu) d\mu, \quad \alpha > 0, \quad x > 0 \quad (2)$$

$$J_{a_0}^0 f(x) = f(x) \quad (3)$$

The Riemann-Liouville integral or fractional derivative operator have very useful and interesting applications in the real world problems. But, here, we are going to define Caputo operators.

Definition 3. The fractional derivative of $f(x)$ in Caputo sense is defined:

$$D_{a_0}^\alpha f(x) = J_{a_0}^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_{a_0}^x (x-\eta)^{m-\alpha-1} f^m(\eta) d\eta \quad (4)$$

where $m-1 < \alpha \leq m$, $m \in N$, $x > 0$, $f \in C_{-1}^m$

Definition 4. If $m-1 < \alpha \leq m$, $m \in N$, and $f \in C_\mu^m$, $\mu \geq -1$, then:

$$D_{a_0}^\alpha J_{a_0}^\alpha f(x) = f(x), \quad D_{a_0}^\alpha J_{a_0}^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)} \frac{(x-a_0)^k}{k!}, \quad x > 0 \quad (5)$$

One can verify the operator J^α [1, 18] with the following useful and helpful properties:
 For $f \in C_\mu^m$, $\alpha, \beta > 0$, $\mu \geq -1$ and $\gamma \geq -1$.

$J_{a_0}^\alpha f(x)$ exist for almost every $x \in [a_0, a_1]$,

$$J_{a_0}^\alpha J_{a_0}^\beta f(x) = J_{a_0}^{\alpha+\beta} f(x)$$

$$J_{a_0}^\alpha J_{a_0}^\beta f(x) = J_{a_0}^\beta J_{a_0}^\alpha f(x) \quad (6)$$

$$J_{a_0}^\alpha (x-a_0)^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} (x-a_0)^{\alpha+\gamma}$$

Mathematical formulation of OHAM for fractional order models

Consider the non-linear fractional order biological population eq. (1):

$$\frac{\partial^\alpha u(x, y, t)}{\partial t^\alpha} = A[u(x, y, t)] + f(x, y, t), \quad \alpha > 0, \quad (x, y) \in \Omega, \quad t \in [a, b] \quad (7)$$

Subject to the initial condition:

$$B \left[u(x, y, t), \frac{\partial u(x, y, t)}{\partial t} \right] = 0, \quad t \in \{a, b\} \quad (8)$$

Here, $\partial^\alpha / \partial t^\alpha$ is described in the Caputo sense operator, $A[u(x, y, t)]$ is non-linear terms in fractional order differential eq.(7), $u(x, y, t)$ is unknown exact solution of eq. (7) and $f(x, y, t)$ is given known function, x, y , and t are spatial and temporal independent variables, respectively. The Ω is domain, B is boundary operator of the eq. (7).

Construct an optimal homotopy for unknown exact solution of the eq. (7):

$$\phi(x, y, t; p) : \Omega \times [0, 1] \rightarrow R$$

$$(1-p) \left[\frac{\partial^\alpha \phi(x, y, t)}{\partial t^\alpha} - f(x, y, t) \right] - H(t, p) \left(\frac{\partial^\alpha \phi(x, y, t)}{\partial t^\alpha} - \{A[\phi(x, y, t)] + f(x, y, t)\} \right) = 0 \quad (9)$$

here $p \in [0, 1]$ is an embedding parameter, $(x, y) \in \Omega$ and $H(t, p)$ is non-zero at $p \neq 0$ and $H(t, 0) = 0$. When p increase in $[0, 1]$, the homotopy ensures a rapid convergence of solution $\phi(x, y, t)$ to the exact solution of the eq. (7). The accurate implementation of the OHAM, which does not depend on a small or large parameters appear in the non-linear fractional order PDE, is purely based on the unbiased selection and true determination of the auxiliary function. The $H(t, p)$ is an arbitrary chosen auxiliary function for eq. (7). The region of fast convergence of

the OHAM solution to the exact solution of eq. (7) depends strictly on auxiliary function $H(t, p)$. Fundamentally, the expression of auxiliary function follows the terms appearing in non-linear part $A[u(x, y, t)]$ of the eq. (7) such that product of the auxiliary function and $A[u(x, y, t)]$ to be of the same shape.

The function $H(t, p)$ can be expressed:

$$H(t, p) = pk_1(t, C_i) + p^2k_2(t, C_i) + p^3k_3(t, C_i) + \dots \quad (10)$$

where $C_i, i = 1, 2, 3, \dots$ are auxiliary controller convergence constants and $k_i(t, C_i), i = 1, 2, 3, \dots$ is a function of t and C_i . But to choose $k_i(t, C_i)$ is purely on the basis of terms appear in non-linear part of the eq. (7). Regarding this focal and logical point, we choose $k_1(t, C_i) = C_1, k_2(t, C_i) = C_2,$ and $k_3(t, C_i) = C_3 \dots$ for non-linear fractional order biological population model because shape of the each solution of simpler problem will be same that is proved in simulation section.

By expanding $\phi(x, y, t; p, C_i)$ in Taylor's series about p , to seek out the exact solutions:

$$\phi(x, y, t; C_i) = u_0(x, y, t) + \sum_{k=1}^m u_k(x, y, t; C_i) p^k, \quad i = 1, 2, 3, \dots \quad (11)$$

The point is to be noted that eq. (11) converges to the exact solution of eq. (7) at $p = 1$:

$$\tilde{u}(x, y, t; C_i) = u_0(x, y, t) + \sum_{k=1}^{\infty} u_k(x, y, t; C_i), \quad i = 1, 2, 3, \dots \quad (12)$$

Generally speaking, we may truncate the eq. (12) into finite terms to get the exact or approximate solution of non-linear PDE.

Equating coefficient of the like powers of p by substituting eq. (11) into eq. (9), we get zero order, 1st order, 2nd order and high order problems:

$$\begin{aligned} p^0 : \frac{\partial^\alpha u_0(x, y, t)}{\partial t^\alpha} - f &= 0 \\ p^1 : \frac{\partial^\alpha u_1(x, y, t, C_1)}{\partial t^\alpha} - (1 + C_1) \frac{\partial^\alpha u_0(x, y, t)}{\partial t^\alpha} + (1 + C_1)f + C_1 A_0 &= 0 \\ p^2 : \frac{\partial^\alpha u_2(x, y, t, C_1, C_2)}{\partial t^\alpha} - C_2 \frac{\partial^\alpha u_0(x, y, t)}{\partial t^\alpha} - (1 + C_1) \frac{\partial^\alpha u_1(x, y, t, C_1)}{\partial t^\alpha} + \\ &+ C_2(f + A_0) + C_1 A_1 = 0 \\ p^3 : \frac{\partial^\alpha u_3(x, y, t, C_1, C_2, C_3)}{\partial t^\alpha} - C_3 \frac{\partial^\alpha u_0(x, y, t)}{\partial t^\alpha} - C_2 \frac{\partial^\alpha u_1(x, y, t, C_1)}{\partial t^\alpha} - \\ &-(1 + C_1) \frac{\partial^\alpha u_2(x, y, t, C_1, C_2)}{\partial t^\alpha} + C_3(f + A_0) + C_2 A_1 + C_1 A_2 = 0 \\ &\dots \end{aligned} \quad (13)$$

where $A_{k-j}, j = 1, 2, 3, \dots, k$ in the general k^{th} -order governing problem is coefficient of the p^{k-j} with respect to the embed parameter p and:

$$A[\phi(x, y, t; p, C_1, C_2, \dots)] = A_0 + \sum_{i=1}^{\infty} A_i p^i$$

Applying the J^α operator on the eq. (13), we get series solutions:

$$\begin{aligned}
 u_0(r, t) &= J^\alpha [f] \\
 u_1(r, t; C_1) &= J^\alpha \left[(1 + C_1) \frac{\partial^\alpha u_0(x, y, t)}{\partial t^\alpha} - (1 + C_1)f - C_1 A_0 \right] \\
 u_2(r, t; C_1, C_2) &= J^\alpha \left[C_2 \frac{\partial^\alpha u_0(x, y, t)}{\partial t^\alpha} + (1 + C_1) \frac{\partial^\alpha u_1(x, y, t, C_1)}{\partial t^\alpha} - C_2(f + A_0) - C_1 A_1 \right] \\
 u_3(r, t; C_1, C_2, C_3) &= J^\alpha \left[C_3 \frac{\partial^\alpha u_0(x, y, t)}{\partial t^\alpha} + C_2 \frac{\partial^\alpha u_1(x, y, t, C_1)}{\partial t^\alpha} + (1 + C_1) \frac{\partial^\alpha u_2(x, y, t, C_1, C_2)}{\partial t^\alpha} - \right. \\
 &\quad \left. - C_3(f + A_0) - C_2 A_1 - C_1 A_2 \right] \\
 &\dots
 \end{aligned}$$

By substituting the previous solutions in eq. (12), one can get the exact solution $u(x, y, t; C_i)$ of the eq. (7).

The auxiliary convergence control constants C_1, C_2, \dots can be found by using least square method:

$$\begin{aligned}
 \chi(C_i) &= \int_0^t \iint_{\Omega} R^2(x, y, t; C_i) dx dy dt \\
 \frac{\partial \chi}{\partial C_1} &= \frac{\partial \chi}{\partial C_2} = \dots = \frac{\partial \chi}{\partial C_n} = 0
 \end{aligned}$$

Numerical simulation

In this portion, we apply OHAM to get exact solution of the three special cases of the non-linear fractional model eq. (1) to show the accuracy and appropriateness of the new approach for to solving exactly non-linear fractional order PDE.

Problem 1. Consider

$$\frac{\partial^\alpha u(x, y, t)}{\partial t^\alpha} = \frac{\partial^2 u^2(x, y, t)}{\partial x^2} + \frac{\partial^2 u^2(x, y, t)}{\partial y^2} + u(x, y, t)[1 - ru(x, y, t)], \quad t > 0, \quad 0 < \alpha \leq 1 \quad (14)$$

with the initial condition:

$$u(x, y, 0) = e^{\sqrt{\frac{x+y}{8}}} \quad (15)$$

The non-linear part of eq. (14)

$$A[u(x, y, t)] = \frac{\partial^2 u^2(x, y, t)}{\partial x^2} + \frac{\partial^2 u^2(x, y, t)}{\partial y^2} + u(x, y, t)[1 - ru(x, y, t)]$$

Using the OHAM formulation we can get the following simpler problems with initial conditions:

– zero-order problem

$$\frac{\partial^\alpha u_0(x, y, t)}{\partial t^\alpha} = 0, \quad u_0(x, y, 0) = e^{\sqrt{\frac{x+y}{8}}} \quad (16)$$

– first-order problem

$$\frac{\partial^\alpha u_1(x, y, t, C_1)}{\partial t^\alpha} - (1 + C_1) \frac{\partial^\alpha u_0(x, y, t)}{\partial t^\alpha} + C_1 A_0 = 0, \quad u_1(x, y, t) = 0 \quad (17)$$

– second-order problem

$$\begin{aligned} \frac{\partial^\alpha u_2(x, y, t, C_1, C_2)}{\partial t^\alpha} - C_2 \frac{\partial^\alpha u_0(x, y, t)}{\partial t^\alpha} - (1 + C_1) \frac{\partial^\alpha u_1(x, y, t, C_1)}{\partial t^\alpha} + \\ + C_2 A_0 + C_1 A_1 = 0, \quad u_2(x, y, t) = 0 \end{aligned} \quad (18)$$

– third-order problem

$$\begin{aligned} \frac{\partial^\alpha u_3(x, y, t, C_1, C_2, C_3)}{\partial t^\alpha} - C_3 \frac{\partial^\alpha u_0(x, y, t)}{\partial t^\alpha} - C_2 \frac{\partial^\alpha u_1(x, y, t, C_1)}{\partial t^\alpha} - \\ - (1 + C_1) \frac{\partial^\alpha u_2(x, y, t, C_1, C_2)}{\partial t^\alpha} + C_3 A_0 + C_2 A_1 + C_1 A_2 = 0, \quad u_3(x, y, t) = 0 \end{aligned} \quad (19)$$

Respective solutions of the eqs. (16)-(19) are given:

$$u_0(x, y, t) = e^{\sqrt{\frac{t}{8}}(x+y)}$$

$$u_1(x, y, t, C_1) = -C_1 e^{\sqrt{\frac{t}{8}}(x+y)}$$

$$u_2(x, y, t, C_1, C_2) = -B_0 e^{\sqrt{\frac{t}{8}}(x+y)} \frac{t^\alpha}{\Gamma(\alpha+1)} + C_1^2 e^{\sqrt{\frac{t}{8}}(x+y)} \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

$$u_3(x, y, t, C_1, C_2, C_3) = -B_1 e^{\sqrt{\frac{t}{8}}(x+y)} \frac{t^\alpha}{\Gamma(\alpha+1)} + 2B_0 C_1 e^{\sqrt{\frac{t}{8}}(x+y)} \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - C_1^3 e^{\sqrt{\frac{t}{8}}(x+y)} \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$

...

In similar way, we can compute the solution of next simpler problems.

Finally, we can get the following expression:

$$\tilde{u}(x, y, t) = u_0(x, y, t) + u_1(x, y, t, C_1) + u_2(x, y, t, C_1, C_2) + u_3(x, y, t, C_1, C_2, C_3) \dots$$

$$\begin{aligned} \tilde{u}(x, y, t) = e^{\sqrt{\frac{t}{8}}(x+y)} - (C_1 + B_0 + B_1) e^{\sqrt{\frac{t}{8}}(x+y)} \frac{t^\alpha}{\Gamma(\alpha+1)} + (C_1^2 + 2C_1 B_0) e^{\sqrt{\frac{t}{8}}(x+y)} \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \\ - C_1^3 e^{\sqrt{\frac{t}{8}}(x+y)} \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \dots \end{aligned} \quad (20)$$

where

$$B_0 = C_2 + C_1(1 + C_1)$$

$$B_1 = C_3 + C_2 C_1 + C_2(1 + C_1) + C_1(1 + C_1)^2$$

After finding the residual, we used least square method and then obtained the values of auxiliary constants:

$$C_1 = -1.0000356780000000000, C_2 = -0.0000000004663021896, \text{ and} \\
 C_3 = 6.583711972000000 \cdot 10^{-14}$$

and by substituting the values of auxiliary constants into the eq. (20), we obtain the expression:

$$\tilde{u}(x, y, t) = e^{\sqrt{\frac{x}{8}(x+y)}} + e^{\sqrt{\frac{y}{8}(x+y)}} \frac{t^\alpha}{\Gamma(\alpha+1)} + e^{\sqrt{\frac{x}{8}(x+y)}} \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + e^{\sqrt{\frac{y}{8}(x+y)}} \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \dots \\
 \tilde{u}(x, y, t) = e^{\sqrt{\frac{x}{8}(x+y)}} \sum_{i=1}^{\infty} \frac{t^{i\alpha}}{\Gamma(i\alpha+1)} = e^{\sqrt{\frac{x}{8}(x+y)}} E_\alpha(t^\alpha)$$

If $\alpha = 1$ then we get exact solution of the eqs. (14) and (15):

$$\tilde{u}(x, y, t) = e^{\sqrt{\frac{x}{8}(x+y)+t}}$$

It is full agreed with the exact solution.

Problem 2. Consider

$$\frac{\partial^\alpha u(x, y, t)}{\partial t^\alpha} = \frac{\partial^2 u^2(x, y, t)}{\partial x^2} + \frac{\partial^2 u^2(x, y, t)}{\partial y^2} + hu(x, y, t), \quad t > 0, \quad 0 < \alpha \leq 1 \quad (21)$$

Subject to the initial condition:

$$u(x, 0) = \sqrt{xy} \quad (22)$$

The non-linear part of eq. (21):

$$A[u(x, y, t)] = \frac{\partial^2 u^2(x, y, t)}{\partial x^2} + \frac{\partial^2 u^2(x, y, t)}{\partial y^2} + hu(x, y, t)$$

Using the OHAM formulation we can get the following simpler problems with initial conditions:

– zero order problem

$$\frac{\partial^\alpha u_0(x, y, t)}{\partial t^\alpha} = 0, \quad u_0(x, y, 0) = \sqrt{xy} \quad (23)$$

– first-order problem

$$\frac{\partial^\alpha u_1(x, y, t, C_1)}{\partial t^\alpha} - (1 + C_1) \frac{\partial^\alpha u_0(x, y, t)}{\partial t^\alpha} + C_1 A_0 = 0, \quad u_1(x, y, t) = 0 \quad (24)$$

– second-order problem

$$\frac{\partial^\alpha u_2(x, y, t, C_1, C_2)}{\partial t^\alpha} - C_2 \frac{\partial^\alpha u_0(x, y, t)}{\partial t^\alpha} - (1 + C_1) \frac{\partial^\alpha u_1(x, y, t, C_1)}{\partial t^\alpha} + \\
 + C_2 A_0 + C_1 A_1 = 0, \quad u_2(x, y, t) = 0 \quad (25)$$

– third-order problem

$$\frac{\partial^\alpha u_3(x, y, t, C_1, C_2, C_3)}{\partial t^\alpha} - C_3 \frac{\partial^\alpha u_0(x, y, t)}{\partial t^\alpha} - C_2 \frac{\partial^\alpha u_1(x, y, t, C_1)}{\partial t^\alpha} - (1 + C_1) \frac{\partial^\alpha u_2(x, y, t, C_1, C_2)}{\partial t^\alpha} + C_3 A_0 + C_2 A_1 + C_1 A_2 = 0, \quad u_3(x, y, t) = 0 \quad (26)$$

The respective solutions of the eqs. (23)-(26) are given:

$$\begin{aligned} u_0(x, y, 0) &= \sqrt{xy} \\ u_1(x, y, t, C_1) &= -C_1 \sqrt{xy} \\ u_2(x, y, t, C_1, C_2) &= -B_0 \sqrt{xy} \frac{t^\alpha}{\Gamma(\alpha + 1)} + C_1^2 \sqrt{xy} \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} \\ u_3(x, y, t, C_1, C_2, C_3) &= -B_1 \sqrt{xy} \frac{t^\alpha}{\Gamma(\alpha + 1)} + 2C_1 B_0 \sqrt{xy} \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} - C_1^3 \sqrt{xy} \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)} \\ &\dots \end{aligned}$$

In similar way, we can compute the solution of next simpler problems.

Finally we can get the following expression:

$$\tilde{u}(x, y, t) = u_0(x, y, t) + u_1(x, y, t, C_1) + u_2(x, y, t, C_1, C_2) + u_3(x, y, t, C_1, C_2, C_3) \dots$$

$$\begin{aligned} \tilde{u}(x, y, t) &= \sqrt{xy} - (C_1 + B_0 + B_1) \sqrt{xy} \frac{(ht)^\alpha}{\Gamma(\alpha + 1)} + (C_1^2 + 2C_1 B_0) \sqrt{xy} \frac{(ht)^{2\alpha}}{\Gamma(2\alpha + 1)} \\ &\quad - C_1^3 \sqrt{xy} \frac{(ht)^{3\alpha}}{\Gamma(3\alpha + 1)} \dots \end{aligned} \quad (27)$$

where

$$\begin{aligned} B_0 &= C_2 + C_1(1 + C_1) \\ B_1 &= C_3 + C_2 C_1 + C_2(1 + C_1) + C_1(1 + C_1)^2 \end{aligned}$$

After finding the residual, we used least square method and then obtained the values of auxiliary constants:

$$C_1 = -1.0003095680000000, \quad C_2 = 0.00000002526788734, \quad \text{and} \quad C_3 = 2.35014404700 \cdot 10^{-11}$$

and by substituting the values of auxiliary constants into the eq. (27), we obtain the expression:

$$\tilde{u}(x, y, t) = \sqrt{xy} + \sqrt{xy} \frac{(ht)^\alpha}{\Gamma(\alpha + 1)} + \sqrt{xy} \frac{(ht)^{2\alpha}}{\Gamma(2\alpha + 1)} + \frac{(ht)^{3\alpha}}{\Gamma(3\alpha + 1)} \dots$$

$$\tilde{u}(x, y, t) = \sqrt{xy} \sum_{i=1}^{\infty} \frac{(ht^\alpha)^i}{\Gamma(i\alpha + 1)} = \sqrt{xy} E_\alpha(ht^\alpha)$$

If $\alpha = 1$ then we get exact solution of the eqs. (21) and (22):

$$\tilde{u}(x, y, t) = \sqrt{xy}e^{ht}$$

It is full agreed with the exact solution.

Problem 3. Consider the non-linear time-fractional Advection equation:

$$\frac{\partial^\alpha u(x, y, t)}{\partial t^\alpha} = \frac{\partial^2 u^2(x, y, t)}{\partial x^2} + \frac{\partial^2 u^2(x, y, t)}{\partial y^2} + u(x, y, t), \quad t > 0, \quad 0 < \alpha \leq 1 \quad (28)$$

with initial condition as:

$$u(x, y, 0) = \sqrt{\sin(x)\sinh(y)} \quad (29)$$

The non-linear part of eq. (28):

$$A[u(x, y, t)] = \frac{\partial^2 u^2(x, y, t)}{\partial x^2} + \frac{\partial^2 u^2(x, y, t)}{\partial y^2} + u(x, y, t)$$

Using the OHAM formulation we can get the following simpler problems with initial conditions:

– zero-order problem

$$\frac{\partial^\alpha u_0(x, y, t)}{\partial t^\alpha} = 0, \quad u_0(x, y, 0) = \sqrt{\sin(x)\sinh(y)} \quad (30)$$

– first-order problem

$$\frac{\partial^\alpha u_1(x, y, t, C_1)}{\partial t^\alpha} - (1 + C_1) \frac{\partial^\alpha u_0(x, y, t)}{\partial t^\alpha} + C_1 A_0 = 0, \quad u_1(x, y, t) = 0 \quad (31)$$

– second-order problem

$$\begin{aligned} \frac{\partial^\alpha u_2(x, y, t, C_1, C_2)}{\partial t^\alpha} - C_2 \frac{\partial^\alpha u_0(x, y, t)}{\partial t^\alpha} - (1 + C_1) \frac{\partial^\alpha u_1(x, y, t, C_1)}{\partial t^\alpha} + \\ + C_2 A_0 + C_1 A_1 = 0, \quad u_2(x, y, t) = 0 \end{aligned} \quad (32)$$

– third-order problem

$$\begin{aligned} \frac{\partial^\alpha u_3(x, y, t, C_1, C_2, C_3)}{\partial t^\alpha} - C_3 \frac{\partial^\alpha u_0(x, y, t)}{\partial t^\alpha} - C_2 \frac{\partial^\alpha u_1(x, y, t, C_1)}{\partial t^\alpha} - \\ - (1 + C_1) \frac{\partial^\alpha u_2(x, y, t, C_1, C_2)}{\partial t^\alpha} + C_3 A_0 + C_2 A_1 + C_1 A_2 = 0, \quad u_3(x, y, t) = 0 \end{aligned} \quad (33)$$

The respective solutions of eqs. (30)-(33) are given:

$$u_0(x, y, 0) = \sqrt{\sin x \sinh y}$$

$$u_1(x, y, t) = -C_1 \sqrt{\sin x \sinh y}$$

$$u_2(x, y, t) = -B_0 \sqrt{\sin x \sinh y} \frac{t^\alpha}{\Gamma(\alpha + 1)} + C_1^2 \sqrt{\sin x \sinh y} \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)}$$

$$u_3(x, y, t) = -B_1 \sqrt{\sin x \sinh y} \frac{t^\alpha}{\Gamma(\alpha+1)} + 2C_1 B_0 \sqrt{\sin x \sinh y} \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \\ - C_1^3 \sqrt{\sin x \sinh y} \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$

...

In similar way, we can compute the solution of next simpler problems. Finally, we can get the following the expression:

$$\tilde{u}(x, y, t) = u_0(x, y, t) + u_1(x, y, t, C_1) + u_2(x, y, t, C_1, C_2) + u_3(x, y, t, C_1, C_2, C_3) \dots$$

$$\tilde{u}(x, y, t) = \sqrt{\sin x \sinh y} - (C_1 + B_0 + B_1) \sqrt{\sin x \sinh y} \frac{t^\alpha}{\Gamma(\alpha+1)} + \\ + (C_1^2 + 2C_1 B_0) \sqrt{\sin x \sinh y} \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - C_1^3 \sqrt{\sin x \sinh y} \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \dots \quad (34)$$

where

$$B_0 = C_2 + C_1(1 + C_1)$$

$$B_1 = C_3 + C_2 C_1 + C_2(1 + C_1) + C_1(1 + C_1)^2$$

After finding the residual, we used least square method and then obtained the values of auxiliary constants:

$$C_1 = -0.999912725500000000, \quad C_2 = -0.000000002378853737, \quad \text{and}$$

$$C_3 = -1.241770350000 \cdot 10^{-13}$$

By substituting the values of auxiliary constants into the eq. (34), we obtain:

$$\tilde{u}(x, y, t) = \sqrt{\sin x \sinh y} + \sqrt{\sin x \sinh y} \frac{t^\alpha}{\Gamma(\alpha+1)} + \sqrt{\sin x \sinh y} \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \\ + \sqrt{\sin x \sinh y} \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \dots$$

$$\tilde{u}(x, y, t) = \sqrt{\sin x \sinh y} \sum_{i=1}^{\infty} \frac{(t^\alpha)^i}{\Gamma(i\alpha+1)} = \sqrt{\sin x \sinh y} E_\alpha(t^\alpha)$$

If $\alpha = 1$ then we get the exact solution of the eqs. (21) and (22):

$$\tilde{u}(x, y, t) = \sqrt{\sin x \sinh y} e^t$$

Which is full agreed with the solution.

Conclusion

Exact solution of non-linear fractional order biological population model with time fractional derivatives has been obtained by successful application of new approach OHAM. In

the fields of applied mathematics generally in the recent emergence of non-linear FDE, it is necessary to study various methods to obtain the solutions of such non-linear fractional order PDE. That are why we expect that our work is a pace towards to obtain the exact solution of physical problems devoted to non-linear fractional order PDE. In simulation section of our work, it has been proved that the exact solutions of non-linear fractional order biological population model obtained by extended new approach are in admirable agreement with the exact solutions of non-linear fractional order biological population model [10]. The application of this new approach into solving every kind of non-linear time-fractional and space-fractional partial or ODE will be our further consideration.

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