NOTE

SOLUTION TO THE PROBLEM OF KUBESA

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Abstract

An infinite family of $T$-factorizations of complete graphs $K_{2n}$, where $2n = 56k$ and $k$ is a positive integer, in which the set of vertices of $T$ can be split into two subsets of the same cardinality such that degree sums of vertices in both subsets are not equal, is presented. The existence of such $T$-factorizations provides a negative answer to the problem posed by Kubesa.

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1. Introduction

Let $K_{2n}$ be the complete graph on $2n$ vertices and $T$ be its spanning tree. A $T$-factorization of $K_{2n}$ is a collection of edge disjoint factors $T_1, T_2, \ldots, T_n$ of $K_{2n}$, each of which being isomorphic to $T$.

At the workshop in Krynica in 2004 D. Fronček presented the following problem originally posed by M. Kubesa [2].

Problem. Suppose that there exists a $T$-factorization of $K_{2n}$. Is it true that the vertex set of $T$ can be split into two subsets, $V_1$ and $V_2$, such that $|V_1| = |V_2| = n$ and $\sum_{v \in V_1} \deg(v) = \sum_{v \in V_2} \deg(v)$?

Notice that there is no requirement on connectness or disconnectness of graphs induced by $V_1$ or $V_2$. 


Recently, N.D. Tan [3] solved the problem in the affirmative for two narrow classes of trees.

2. Constructions

A tree which becomes a star after removal of its pendant edges is called a snowflake. Its central vertex (ie. the central vertex of a star obtained in such a way) is called a root, whilst remaining vertices of degrees greater than one are called inner vertices.

We define a family of snowflakes $\tilde{T}_{2n}$ of order $2n = 56k$, for every positive integer $k$. There are 7 vertices of degrees: $28k - 18, 28k - 20, 11, 10, 8, 7, 7$, the remaining $56k - 7$ are leaves. The vertex of degree 11 is the root of $\tilde{T}_{2n}$.

Lemma 1. For every positive integer $k$, the complete graph $K_{56k}$ has $\tilde{T}_{56k}$-factorization.

Proof. The snowflake $\tilde{T}_{56k}$ is defined by listing its edges; we use the notation $u \prec u_1, u_2, \ldots, u_m$ if all the vertices $u_1, u_2, \ldots, u_m$ are adjacent to $u$. Consider two cases.

Case I. $k = 1$. Let $V(K_{56}) = U \cup X \cup Y \cup Z$, where $U = \{u_0, u_1, \ldots, u_{13}\}$, $X = \{x_0, x_1, \ldots, x_{13}\}$, $Y = \{y_0, y_1, \ldots, y_{13}\}$ and $Z = \{z_0, z_1, \ldots, z_{13}\}$. Edges of $K_{56}$ with both endvertices either in $U$ or $X$ or $Y$ or $Z$ are called pure edges; the remaining ones are mixed edges. To indicate a required $\tilde{T}_{56}$-factorization we prescribe 28 snowflakes split into two classes: $\{T_i : i = 0, 1, \ldots, 13\}$ and $\{T_i^* : i = 0, 1, \ldots, 13\}$, each $T_i$ and $T_i^*$ being isomorphic to $\tilde{T}_{56}$.

We construct the first class. The vertex $u_{12}$ of degree 11 is the root of $T_0$ and its inner vertices: $u_0, x_1, x_2, y_0, y_1, z_7$ have degrees $8, 8, 7, 10, 7, 10$, respectively. The remaining pendant edges are: $u_{12} \prec u_1, u_2, u_4, u_7, u_{11}; u_0 \prec x_8, x_9, x_{11}, x_{12}, x_{13}, y_4, y_5; x_1 \prec u_5, u_6, u_{10}, u_{13}; y_8, y_{10}; x_2 \prec x_3, x_4, x_5, x_6, x_7, x_{10}; y_0 \prec u_6, u_8, z_0, z_1, z_2, z_6, z_9; y_1 \prec y_2, y_6, y_7, y_{11}, y_{12}, y_{13}; z_7 \prec u_3, x_0, y_9, z_5, z_8, z_{10}, z_{11}, z_{12}, z_{13}$. Snowflakes $T_1, T_2, \ldots, T_{13}$ can be obtained from $T_0$ by applying the cyclic permutation $\varphi = (0, 1, \ldots, 13)$ in parallel on the indices of vertices in the sets $U$, $X$, $Y$ and $Z$. One can easily check that the lengths $1, 2, 3, 4, 5, 6$ of all pure edges in $K_{56}$ have been already covered, as well as the following lengths of mixed edges for types: $UX$: $2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13$; $UY$: $2, 3, 4, 5, 6, 8$; $UZ$: $4, 9$; $XY$: $2, 7, 9$; $XZ$: $7$; $YZ$: $0, 1, 2, 3, 4, 6, 9, 12$. 

To construct the second class we need the snowflak $T_0'$. Let the vertex $v_7$ of degree 11 be the root and $x_0, x_7, y_2, y_3, z_0, z_1$ be the inner vertices of degrees 8, 7, 6, respectively. The remaining pendant edges are: $u_6 \prec v_2, z_3, z_4, z_5, z_6, y_0 \prec x_7, y_0, y_1, y_2, y_3, x_1 \prec z_2, z_8, z_9, z_{10}, z_{11}, z_{12}, z_{13}, y_2 \prec x_1, x_{10}, x_{11}, x_{12}, x_{13}, y_3 \prec u_2, u_3, u_4, u_5, u_6, u_{10}; z_0 \prec u_8, u_9, u_{11}, u_{12}, u_{13}, y_1, y_6, y_7, z_7; z_1 \prec u_1, x_2, x_3, x_4, x_5, x_6, x_9, y_4, y_5$. Six snowflakes $T_i'$, for $i = 2, 4, \ldots, 12$, can be obtained from $T_0'$ by applying $i$th power of $\varphi$ in parallel on the sets $U, X, Y$ and $Z$. Thus the length 7 of all pure edges is covered completely and still remaining lengths of mixed edges, except the lengths 0 of type $UX$ and 5 of type $YZ$, are covered in a half. Seven remaining snowflakes $T_j'$ for $j = 1, 3, \ldots, 13$ are obtained from $T_0'$ by replacing the edges $u_0 u_7, x_0 x_7, y_2 y_9$ and $z_0 z_7$ with the edges $u_0 x_0, u_7 x_7, y_2 z_7$ and $y_9 z_0$, respectively, and then by applying the permutation $(\varphi)^j$ in parallel on the sets $U, X, Y$ and $Z$. Notice that such a replacement does not result in changing the structure of snowflake, i.e., all $T_j'$ are isomorphic to $T_0'$. In this way we cover all remaining lengths of mixed edges.

Case II. $k \geq 2$. Let $V(K_{56k}) = \bigcup_{l=1}^k (U^l \cup X^l \cup Y^l \cup Z^l)$, where $U^l = \{u_0^l, u_1^l, \ldots, u_{13}^l\}$, $X = \{x_0, x_1, \ldots, x_{13}\}$, $Y = \{y_0^l, y_1^l, \ldots, y_{13}^l\}$ and $Z = \{z_0^l, z_1^l, \ldots, z_{13}^l\}$, $l = 1, 2, \ldots, k$. In what follows subscripts should be read modulo 14.

In order to construct $28k$ factors, each isomorphic to $T_{56k}$, we proceed in the following way. First, for every snowflake $T_i$, $i = 0, 1, \ldots, 13$, in the $T_{56k}$-factorization of $K_{56}$ constructed in Case I we make $k$ copies $T_i^l$, $l = 1, 2, \ldots, k$, by copying every edge $s t$ of $T_i$ into $k$ edges $s^l t^l$, each being an edge of appropriate $T_i^l$, where $s, t \in U \cup X \cup Y \cup Z$. Moreover, for every $T_i^l$ among $14k$ trees obtained in this way, where $i = 0, 1, \ldots, 13$ and $l = 1, 2, \ldots, k$, we add $56(k-1)$ edges: $u_i^l \prec u_j^p, x_j^p, y_j^p, z_j^p, y_i^s \prec u_j^p, x_j^p, y_j^p, z_j^p$, where $l < p \leq k$, $1 \leq r < l$, $j = 0, 1, \ldots, 13$. Thus every $T_i^l$ is a snowflake with the root $u_{12+i}^l$ of degree 11, and six inner vertices $x_{1+i}^l, x_{2+i}^l, y_i^l, y_{1+i}^l, z_{7+i}^l$ of degrees $28k - 20, 8, 7, 28k - 18, 7, 10$, respectively.

Similarly, for every snowflake $T_i'$ constructed in Case I, $i = 0, 1, \ldots, 13$, we built $k$ copies $T_i^l'$, $l = 1, 2, \ldots, k$, by copying every edge $s t$ of $T_i'$ into $k$ edges $s^l t^l$, $s, t \in U \cup X \cup Y \cup Z$. Analogously to the above, for every $T_i^l'$ of $14k$ trees just obtained, $i = 0, 1, \ldots, 13$ and $l = 1, 2, \ldots, k$, new $56(k-1)$ edges are added: $x_i^l \prec u_j^p, x_j^p, y_j^p, z_j^p, z_i^l \prec u_j^p, x_j^p, y_j^p, z_j^p$, where $l < p \leq k$, $1 \leq r < l$, $j = 0, 1, \ldots, 13$. Every $T_i^l'$ obtained in this way is a snowflake.
with the root $u^k_{7+i}$ of degree 11, and six inner vertices $x^l_{7+i}$, $x^l_{8+i}$, $y^l_{2+i}$, $y^l_{3+i}$, $z^l_{i}$, $z^l_{1+i}$ of degrees $28k - 20$, $8$, $7$, $7$, $28k - 18$, $10$, respectively.

**Lemma 2.** For every set $V \subset V(\overline{T}_{56k}) = V(K_{56k})$ such that $|V| = 28k$, $\sum_{v \in V} \deg(v) \neq 56k - 1$.

**Proof.** One can check that there are only four sequences of length $28k$ whose terms are degrees of $\overline{T}_{56k}$ and whose sum of terms is $56k - 1$:

1. $28k - 18$, $10$, $10$, $1$, $1$, $\ldots$, $1$,
2. $28k - 18$, $7$, $7$, $7$, $1$, $1$, $\ldots$, $1$,
3. $28k - 20$, $11$, $11$, $1$, $1$, $\ldots$, $1$,
4. $28k - 20$, $8$, $8$, $7$, $1$, $1$, $\ldots$, $1$.

None of these sequences is a subsequence of degree sequence of $\overline{T}_{56k}$. Thus the assertion holds.

Notice that every of the sequences (1)–(4) indeed appears as a set of degrees for some vertex in factors of $\overline{T}_{56k}$-factorization of $K_{56k}$. It is easily seen that all terms of (1) are degrees of the vertex $z^k_i$ in $\overline{T}_{56k}$-factorization, similarly (2) is a set of degrees for $y^k_i$, (3) for $u^k_i$ and (4) for $x^k_i$, $i = 0, 1, \ldots, 13$, $l = 1, 2, \ldots, k$.

It is still possible that a similar example for the order $2n < 56$ exists. Nevertheless, a computer was used to check that in that case $2n$ cannot be smaller than 38.

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**References**

