A Connection Between Coding Theory and Polarized Partition Relations

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In this paper we establish a connection between polarized partition relations and the function $A_q(n, d)$ of coding theory, which implies some known results and new examples of polarized relations. A connection between $A_q(n, d)$ and the Zarankiewicz numbers is also discussed.

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1. INTRODUCTION

Finite polarized partition relations of dimension two can be defined as follows: given positive integers $q, t \leq a$, $j \leq b$, the relation symbol

$$(a, b) \rightarrow (i, j)_q$$

means that every matrix $a \times b$ with entries in the set $\{0, 1, \ldots, q - 1\}$ always contains a constant $i \times j$ submatrix. The negation of (1) is denoted by $(a, b) \not\rightarrow (i, j)_q$.

The problem of deciding whether (1) holds was posed by Erdős and Rado [7] (more generally and in a different context), which also can be restated in terms of graph theory or as a variant of the celebrated Ramsey problem. Several authors, for example in [6, 11, 12, 15, 20], have considered distinct approaches of this partition, either studying related problems, or investigating connections to other combinatorial structures. In particular, we remark that Hadamard matrices and orthogonal Latin squares yield classes of polarized partitions (see [1, 4]). Since such combinatorial concepts are also used to construct error-correcting codes (see [18, 19]), it seems interesting to investigate how codes and polarized partitions are related.

We are concerned with the above question. The main result (Theorem 1) is to establish a connection between polarized partition relation and a central problem in coding theory, described as follows: determine the maximum number $A_q(n, d)$ of codewords in a $q$-ary code of length $n$ and minimum distance $d$.

This connection constitutes a systematic way of constructing polarized partitions. In particular, bounds for the extremal problem $P_q(2, t) = \min\{n \in \mathbb{N} : (n, n) \rightarrow (2, t)_q\}$ are determined (Section 4), which imply previous results from [1, 4]. In contrast to the case where $q = 2$, our knowledge on exact values of $P_q(2, t)$ is rather poor when $q \geq 3$ and $t \geq 2$. Indeed, the topic is so short of construction that the only exact value known is $P_3(2, 2) = 11$, by Exoo [8]. However, making use of the table for $A_3(n, d)$ due to Vaessen et al. [21], we derive good bounds for $P_3(2, t)$ when $t$ is small.

Moreover, the method produces a relationship between $A_q(n, d)$ and the classical Zarankiewicz numbers (Section 5). As a consequence, the Plotkin bound is obtained from a result due to Hyltén-Cavallius [14].

2. PRELIMINARIES: THE FUNCTION $A_q(n, d)$

Given the set $\mathbb{Z}_q^n$ of all words with length $n$ and components $0, 1, \ldots, q - 1$ from the ring $\mathbb{Z}_q$, the Hamming distance $d_H(x, y)$ between $x, y \in \mathbb{Z}_q^n$ is defined as the number of components in which $x$ and $y$ differ. Thus, $A_q(n, d)$ is the maximal cardinality of a code $C$ in $\mathbb{Z}_q^n$ such that the minimum Hamming distance of $C$ is $d$, i.e., $d_H(x, y) \geq d$ for all $x, y \in C, x \neq y$. 

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The determination of the values of $A_q(n, d)$ is a central research topic in combinatorial coding theory. For the sake of our purposes, we recall the following well-known results on $A_q(n, d)$ (see [18]): Plotkin and Levenshtein showed that if there exists a Hadamard matrix of order $2(n - 1)$, with odd $n$, then

$$A_2(2n - 2, n - 1) = 4n - 4.$$  \hspace{1cm} (2)

Singleton [19] established the numbers: for a prime power $q$,

$$A_q(q + 1, q) = q^2.$$  \hspace{1cm} (3)

Others results and a table of lower and upper bounds for $A_2(n, d)$ appears in [18], while [21] presents a similar table for $A_3(n, d)$, with $n \leq 16$.

3. The Connection

Before stating the main result, let us introduce some notation. For fixed $n \in \mathbb{N}$ and $c \in \mathbb{Z}_q$, denote by $\tau$ the vector in $\mathbb{Z}_q^n$ whose all components are $c$, i.e., $\tau = (c, c, \ldots, c)$ and let $\Delta = (0, 1, 2, \ldots, n - 1) \in \mathbb{Z}_q^n$. Given $z = (z_1, z_2, \ldots, z_n)$ in $\mathbb{Z}_q^n$, write $\Delta \ast z$ for the concatenation performed componentwise, i.e., $\Delta \ast z = ((0, z_1), (1, z_2), \ldots, (n - 1, z_n)) \in (\mathbb{Z}_n \times \mathbb{Z}_q)^n$. As usual, the symbol $+$ represents both the addition taken in $\mathbb{Z}_q$ and in $\mathbb{Z}_q^n$.

The following result has been motivated by [4, Theorem 5].

**Theorem 1.** For $q \geq 2$ and $1 \leq t \leq n$,

$$(A_q(n, n - t + 1), nq) \nrightarrow (2, t)_q.$$  \hspace{1cm}

**Proof.** Let $a = A_q(n, n - t + 1)$, $b = qn$ and $d = n - t + 1$. The codewords of an optimal code in $\mathbb{Z}_q^n$ with minimum distance $d$ are denoted by $L_x$, where $0 \leq x \leq a - 1$.

In order to prove the theorem it is sufficient to construct a matrix $M$ of order $a \times b$ filled with colors $0, 1, \ldots, q - 1$ which avoids a monochromatic $2 \times t$ submatrix. Let us define the matrix $M$ as follows: for each $x \in \mathbb{Z}_a$, $y \in \mathbb{Z}_n \times \mathbb{Z}_q$ and $c \in \mathbb{Z}_q$, we take $M(x, y) = c$ if and only if $y \in \Delta \ast (L_x + \tau)$.

The matrix $a \times b M$ is well defined. Indeed, for each fixed index $x$, note that $\mathbb{Z}_n \times \mathbb{Z}_q$ coincides with the disjoint union of the sets $\Delta \ast (L_x + \tau)$, where $0 \leq c \leq q - 1$.

Now, it remains to be proved that $M$ does not contain a monochromatic $2 \times t$ submatrix. Without loss of generality, suppose that there exists a $2 \times t$ submatrix of $M$ with color 0. Let $\{x_1, x_2\}$ and $\{y_1, y_2, \ldots, y_t\}$ be its index sets for the rows and columns, respectively.

By construction, note that $\{y_1, y_2, \ldots, y_t\} \subseteq (\Delta \ast L_{x_1}) \cap (\Delta \ast L_{x_2})$, and thus $\Delta \ast L_{x_1}$ agrees with $\Delta \ast L_{x_2}$ at least in $t$ elements. Since all components of $\Delta$ are distinct, the codewords $L_{x_1}$ and $L_{x_2}$ coincide at least in $t$ places. Hence $d_H(L_{x_1}, L_{x_2}) \leq n - t = d - 1$, which is a contradiction. \hfill \Box

4. Some Applications

In this section we focus on the numbers $P_q(2, t) = \min\{n \in \mathbb{N} : (n, n) \nrightarrow (2, t)_q\}$, which are closely related to the bipartite Ramsey numbers. Let us state an upper bound for this function, whose proof is similar to the one given in [4, Theorem 3.b].

**Proposition 2.** For every $q \geq 2$,

$$P_q(2, t) \leq q^2(t - 1) + q - 1.$$
This result improves the previous bounds due to Chvátal: $P_q(2, 2) \leq q^2 + q + 1$ [6, Theorem 2] and $(2^r - 1) + 2, 4t - 3 \to (2, t)_2$ [6, Theorem 7].

On the other hand, informations on $A_q(n, d)$ give us a way to fill the lower bound for $P_q(2, t)$, as follows.

**Theorem 3.** For every $q \geq 2$ and $t \geq 2$,

$$P_q(2, t) > \max\{\min\{A_q(n, n - t + 1), nq\} : t \leq n \leq q(t - 1) + 1\}.$$

**Proof.** Let $l = q(t - 1) + 1$ and $z_n = \min\{A_q(n, n - t + 1), nq\}$ for all $n$. Theorem 1 combined with the monotonicity of polarized relations imply that $(z_n, z_n) \not\sim (2, t)_q$, and thus $P_q(2, t) > \max\{z_n : t \leq n\}$. By Proposition 2, note that $z_n = A_q(n, n - t + 1)$ for all $n \geq 1$. Thus the constraint $n \leq l$ follows from the elementary property: $A_q(n, d) \geq A_q(n + 1, d + 1)$. Hence $P_q(2, t) > \max\{z_n : t \leq n\} = \max\{z_n : t \leq n \leq l\}$. 

We discuss now some applications of the previous results. We first derive classes of bounds for $P_q(2, t)$.

**Corollary 4.** (a) If there exists a Hadamard matrix of order $2(t - 1)$, with odd $t$, then $P_2(2, t) = 4t - 3$; (b) for a prime power $q$, $q^2 + 1 \leq P_q(2, 2) \leq q^2 + q - 1$.

**Proof.** An immediate application of (2), (3), Theorem 3 and Proposition 2.

The item (a) also follows from a result due to Beineke and Schwenk [1]; while (b) give us [4, Theorem 5].

Using existence theorems for Hadamard matrices, according to [2, 13], the following exact values hold by Corollary 4.

**Example 5.** Consider $k_i$ a number of the form: (i) either $k_i = p_i^{n_i} + 1$, where $p_i$ is an odd prime such that $p_i^{n_i} + 1 \equiv 0 \pmod{4}$; (ii) or $k_i = 2(p_i^{n_i} + 1)$, where $p_i$ is an odd prime such that $p_i^{n_i} \equiv 1 \pmod{4}$. For all $t$ such that $4t - 4 = 2^m k_1 k_2 \ldots k_i$, $m \geq 1$, and for odd $t$ with $3 \leq t \leq 133$, we have $P_2(2, t) = 4t - 3$.

Another construction establishes a new class of optimal bounds for $P_2(2, t)$, in particular, $P_2(2, t) = 4t - 3$ when $4t - 3$ is a prime power, see [5, Theorem 11]. Nevertheless, Theorem 3 only yields $4t - 5 \leq P_2(2, t)$, for $t = 4, 8$, which illustrates that our lower bound is not tight for these instances.

**Corollary 6.** For $t$ such that $2 \leq t \leq 5, 9t - 8 \leq P_3(2, t) \leq 9t - 7; 37 \leq P_3(2, 6) \leq 47$; and $49 \leq P_3(2, 7) \leq 56$.

**Proof.** Application of Theorem 3, Proposition 2 and the values $A_3(4, 3) = 9, A_3(6, 4) = 18, A_3(9, 6) = 27, A_3(12, 8) = 36, A_3(12, 7) \geq 44$ and $A_3(16, 10) \geq 54$ (see table in [21]).

## 5. Zarankiewicz Numbers and $A_q(n, d)$

In 1951 Zarankiewicz raised the following problem: determine the smallest integer $z = Z_{i, j}(a, b), \text{if } i \leq a, j \leq b, 2 \leq j \leq a \leq b$, such that every $0 - 1$ matrix of order $a \times b$ containing $z$ ones must have a $i \times j$ submatrix whose all entries are 1. See [3, 11] for overview, and
the recent contributions [9, 10]. We now mention only the Hyltén-Cavallius upper bound [14, Theorem 1]:

\[ Z_{2,t}(m, n) \leq 1 + \frac{n}{2} + \left[ (t - 1)nm(m - 1) + \frac{n^2 t \gamma^{1/2}}{4} \right]. \tag{4} \]

Polarized partitions are closely related to the Zarankiewicz problem, according to [11, 15]. We remark that the construction in Theorem 1 also implies the following connection between Zarankiewicz numbers and \( A_q(n, d) \).

**Theorem 7.** For \( q \geq 2 \) and \( a = A_q(n, n - t + 1) \),

\[ Z_{2,t}(a, qn) > na. \]

As an application, we derive the Plotkin bound (see [17, Theorem 5.2.4]) as given in the following corollary.

**Corollary 8.** If \( n > q(t - 1) \), then

\[ A_q(n, n - t + 1) \leq \frac{(n - t + 1)q}{n - (t - 1)q}. \]

**Proof.** Let \( a = A_q(n, n - t + 1) \). Since \( a \) and \( Z_{2,t}(a, qn) \) are positive integers, Theorem 7 yields \( na \leq Z_{2,t}(a, qn) - 1 \). This inequality combined with (4) implies that

\[ an - \frac{qn}{2} \leq \left[ (t - 1)qna(a - 1) + \frac{q^2 n^2 t \gamma^{1/2}}{4} \right]. \]

Performing simple algebraic manipulation, we obtain the required bound. \(\square\)

On the other hand, (3), (4) and Theorem 7 give us an extension of a result due to Kövári et al. [16] as presented in the following corollary.

**Corollary 9.** For a prime power \( q \), \( Z_{2,2}(q^2, q^2 + q) = q^3 + q^2 + 1 \).

Consequently, the bound of Theorem 7 is tight at least in this case. Moreover, by using (4), Theorem 7 and tables in [18, 21], some new Zarankiewicz numbers can be completely determined, as for example: \( Z_{2,6}(12, 22) = 133, Z_{2,7}(8, 28) = 113, Z_{2,8}(16, 30) = 241 \).

6. **Final Remarks**

Table 1 below summarizes applications of the present method for some values of \( A_q(n, d) \), when \( q = 2 \) or \( 3 \). The numbers \( A_q(n, d) \) (first column) are presented in [18, 21]. For these numbers, the corresponding polarized partitions (second column) and lower bounds for Zarankiewicz function (third column) are obtained by Theorems 1 and 7, respectively.

The polarized partitions marked with the symbol + can be found in [11, 15]. In the last column, the symbol * indicates that the corresponding lower bound is optimal; otherwise the exact values for the remaining cases appear in round parenthesis, according to tables in [11].

We conclude this paper with some remarks. Our results show that particular values obtained by Theorems 3 and 7 are optimal or near optimal for several instances. Extensions or improvements on tables of \( A_q(n, d) \) could considerably improve the results. For example, the table in [21] shows that \( A_3(12, 7) \geq 44 \) and \( 22 \leq A_3(15, 10) \leq 45 \), which imply \( 37 \leq P_3(2, 6) \leq 47 \). If the plausible value \( A_3(15, 10) = 45 \) holds, then the previous bound can be improving to \( 46 \leq P_3(2, 6) \leq 47 \).
Table 1.

<table>
<thead>
<tr>
<th>A_2(3, 2) = 4</th>
<th>(4, 6) \not\rightarrow (2, 2)_2^+</th>
<th>13 \leq Z_{2,2}(4, 6)^*</th>
</tr>
</thead>
<tbody>
<tr>
<td>A_2(6, 3) = 8</td>
<td>(8, 12) \not\rightarrow (2, 4)_2</td>
<td>49 \leq Z_{2,4}(8, 12)</td>
</tr>
<tr>
<td>A_2(8, 5) = 4</td>
<td>(4, 16) \not\rightarrow (2, 4)_2</td>
<td>33 \leq Z_{2,4}(4, 16)</td>
</tr>
<tr>
<td>A_2(6, 4) = 4</td>
<td>(4, 12) \not\rightarrow (2, 3)_2^+</td>
<td>25 \leq Z_{2,3}(4, 12)^*</td>
</tr>
<tr>
<td>A_2(9, 6) = 4</td>
<td>(4, 18) \not\rightarrow (2, 4)_2^+</td>
<td>37 \leq Z_{2,4}(4, 18)^*</td>
</tr>
<tr>
<td>A_3(4, 3) = 9</td>
<td>(9, 12) \not\rightarrow (2, 2)_3^+</td>
<td>37 \leq Z_{2,2}(9, 12)^*</td>
</tr>
<tr>
<td>A_3(5, 4) = 6</td>
<td>(6, 15) \not\rightarrow (2, 2)_3^+</td>
<td>31 \leq Z_{2,2}(6, 15)^*</td>
</tr>
<tr>
<td>A_3(6, 4) = 4</td>
<td>(4, 18) \not\rightarrow (2, 2)_3</td>
<td>25 \leq Z_{2,2}(4, 18)^*</td>
</tr>
</tbody>
</table>

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