Polar and axial vectors versus quaternions

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Vectors and quaternions are quite different mathematical quantities because they have different symmetry properties. Gibbs and Heaviside created their vector system starting from the quaternion system invented by Hamilton. They identified a pure quaternion as a vector and introduced some changes in the product of two vectors defined by Hamilton without realizing that the scalar product and vector product cannot be interpreted as the scalar part and vector part of the quaternion product. Toward the end of the 19th century some authors realized that there was an incompatibility between the vector and quaternion formalisms, but the central problem was not altogether clear. This paper will show that the main difficulty arose from Hamilton's contradictory use of *i*, *j*, and *k* both as versors and as vectors. © *2002 American Association of Physics Teachers.*
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I. INTRODUCTION

The development of electromagnetic theory in the 19th century brought with it the concept of field, the new vector quantities related with it, and the requirement of a vector analysis to deal with so many vector quantities in a more practical way.[1] In the last decade of the 19th century there was a debate concerning the best mathematical formalism to represent the new vector quantities in three-dimensional space. On one side were Peter G. Tait, Cargil Knott, and Alexander MacFarlane, who defended the use of quaternion algebra as the best tool to deal with the new vector quantities. On the other side were Willard Gibbs and Oliver Heaviside, who defended the use of vector analysis. Both groups were influenced by James Clerk Maxwell, who used the coordinate calculus and the quaternion calculus in his *Treatise on Electricity and Magnetism*, published in 1873.[2] One of the most important aspects that makes this discussion interesting is the fact that all debaters were important physicists with mathematical interests.[3,4] The main question was whether one should use quaternions or vectors to represent the electromagnetic quantities.

A quaternion is a special mathematical entity containing four components. William Hamilton conceived it in 1843. It was born as a generalization of complex numbers ($z=a+bi$). A quaternion can be written as $q=a+bi+cj+zk$, where i, j, k are imaginary units that obey the following rules:

$$i^2=j^2=k^2=-1, \tag{1}$$

$$ij=k, \quad ji=-k, \quad jk=i, \quad kj=-i, \quad ki=j, \quad ik=-j. \tag{2}$$

A quaternion $q=a+bi+cj+zk$ contains a scalar part (a) and a vector part ($bi+cj+zk$). A quaternion of the form $q=bi+cj+zk$ is called a "pure quaternion" and looks like an ordinary vector. But as will be shown, this similarity is superficial.

The quaternion or complete product of two pure quaternions $\alpha=(ix+jy+kz)$ and $\beta=(iu+jv+kw)$ is obtained using the above rules for multiplication of i, j, k:

$$\alpha\beta=[-(xu+yv+zw)]+[i(yw-zv)+j(zu-xw)+k(xv-yu)]=S\alpha\beta+V\alpha\beta. \tag{3}$$

In quaternion theory the complete product of two pure quaternions is composed of two parts. The scalar part, $S\alpha\beta=-(xu+yv+zw)$, contains a negative sign, and the vector part, $V\alpha\beta=i(yw-zv)+j(zu-xw)+k(xv-yu)$, is a pure quaternion.

Quaternions were intensively studied and applied to physics during the second half of the 19th century and early 20th century.[5] Nowadays they are studied by mathematicians as an example of an interesting noncommutative algebra and physicists seldom use them. Instead of quaternions they prefer the matrix form of spinor calculus and the vector (or tensor) calculus.[6]

Vectors (in three-dimensional space) have three spatial components (X, Y, Z) and can be represented as $V=Xi+Yj+Zk$. It is possible to define addition and subtraction for vectors. There are two types of vector products that are analogous to the scalar and vector parts of the quaternion product. The division operation is not defined for vectors.

In spite of the recognition by some authors that a pure quaternion is different from a vector,[7] it is common practice to interpret a pure quaternion as a vector in R^3. Kuipers, for instance, defined a quaternion as "the sum of a scalar and a vector."[8] He regards it as obvious that we may identify a vector as a pure quaternion:

> How can a quaternion, which lives in R^4, operate on a vector, which lives in R^3? There is an answer to this question, which may seem obvious to some, and that is: A vector $\mathbf{v}\in R^3$ can simply be treated as though it were a quaternion $q\in R^4$ whose real part is zero.[9]

Of course, there is a one-to-one correspondence between the set of vectors and that of pure quaternions, because vectors and pure quaternions are both triplets. However, not every triplet can be regarded as a vector, because a vector is a triplet with some *specific properties*.

The notation for vectors and quaternions also helps to increase the confusion. In both cases one employs i, j, k, and this notation conjures up an identification between pure

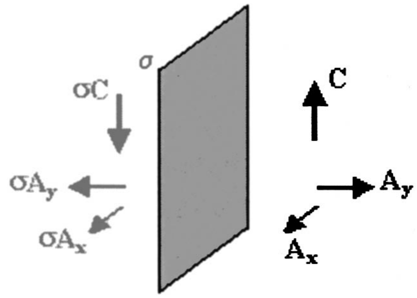

Fig. 1. The components of a polar vector **A** in which the *z* component vanishes. The vector **C** is an axial vector.

quaternions and vectors. However, for vectors, *i, j, k* are unit vectors in three perpendicular directions. In the case of quaternions, *i, j, k* are imaginary units.

Altmann has shown, from a modern point of view, how dangerous it is to identify uncritically a pure quaternion with a vector.[10] A pure quaternion and a vector do not have the same symmetry proprieties.

Historically, vector algebra arose from quaternion algebra. How did it happen that they are such different entities, and why do they look so similar in some respects? The chief aim of this paper is to analyze the origin of this misunderstanding from a historical point of view.

II. THE NATURE AND SYMMETRY OF VECTORS AND PURE QUATERNIONS

As was pointed out in Sec. I, pure quaternions and vectors have different symmetry proprieties. The pure quaternion behaves with respect to coordinate transformations like an *axial vector*, while the ordinary vector behaves as a *polar vector*.

By multiplying two vectors it is possible to generate new objects. For example, the vector product of two polar vectors is an axial vector.[11] To understand this point, let us analyze a simple case of the vector product $\mathbf{C}=\mathbf{A}\times\mathbf{B}$ in which A_z and B_z vanish. According to the usual rule for the vector product, in this case **C** has only one component, $C_z=A_xB_y-A_yB_x$.

Let us consider a reflection plane σ parallel to the *xz* plane (Fig. 1). The components A_x and B_x are symmetrical with regard to parallel reflection in σ, that is, $\sigma A_x=A_x$ and $\sigma B_x=B_x$. But the components A_y and B_y are antisymmetrical with regard to a perpendicular reflection in σ, that is, $\sigma A_y=-A_y$ and $\sigma B_y=-B_y$. Therefore, $\sigma\mathbf{C}=\sigma(A_xB_y-A_yB_x)=(-A_xB_y+A_yB_x)=-\mathbf{C}$, and hence the vector **C** behaves under reflection in the parallel plane σ like an axial vector.

When one represents a polar vector **A** as $A_xi+A_yj+A_zk$, the symbols *i, j,* and *k* are understood to be unit polar vectors and the components of **A**, that is, A_x, A_y, and A_z, are scalars. Now, if one attempts to represent the vector product of two polar vectors (or any other axial vector **C**) as $C_xi+C_yj+C_zk$, the symbols *i, j,* and *k* cannot be understood as unit *polar* vectors, because the addition of polar vectors produces polar vectors. It would be possible to regard *i, j,* and *k* as *axial* vectors. In that case, however, it would seem impossible, at a first sight, to represent a polar vector (such as position) as $\mathbf{r}=ai+bj+ck$.

There is a way out of the problem, however. The answer is in *a, b,* and *c*. If a real number *A* is a function of the position vector **r** in such a way that $A(\mathbf{r})=\pm A(-\mathbf{r})$, then *A* is either a scalar (+) or a pseudoscalar (−) depending on the sign. It is possible to represent an axial vector $\mathbf{C}=ai+bj+ck$, with *i, j,* and *k* as polar vectors, if *a, b, c* are pseudoscalars. In this case **C** is an axial vector, because it is a sum of pseudoscalars (*a, b, c*) multiplied by polar vectors (*i, j, k*).[12] The assumption that *i, j, k* are polar vectors and *a, b, c* are pseudoscalars in an axial vector is implicitly presupposed in the usual vector algebra used by physicists.

Are there two types of quaternions, as there are two types of vectors? The answer is no. Quaternions are defined in such a way that all arithmetic operations involving two quaternions produce another quaternion with the same properties as the initial ones. How can that be? Let us consider a simple case.

Equation (3) shows that the product of two pure quaternions is a quaternion. A pure quaternion cannot be equivalent to a polar vector. If pure quaternions were equivalent to polar vectors, the vector part $V\alpha\beta$ of a quaternion product would be equivalent to an axial vector, and consequently a different kind of mathematical entity than a pure quaternion—and it would be impossible to build a closed quaternion algebra. Can the pure quaternion be equivalent to an axial vector? It can, provided that *i, j,* and *k* correspond to *axial* vectors. In this way, a pure quaternion is an axial vector, and the vector part of the product of two quaternions is also an axial vector. Hence, the symmetry properties of pure quaternions are different from those of polar vectors and a quaternion is not equivalent to a polar vector plus a scalar.

The analysis described above shows that it is incorrect to identify a pure quaternion as a vector as some present day authors do. Although quaternions and vectors are completely different mathematical objects, vectors originated from quaternions via a change of meaning of the pure quaternion and of the symbols *i, j, k*. When developing the vector system used nowadays by physicists, Gibbs and Heaviside interpreted the pure quaternion as a polar vector in Euclidean space and the imaginary units *i, j, k* as unit vectors. We now return to the 19th century to examine in more detail the meaning of the symbols *i, j, k* in the quaternion system and in vector algebra.

III. HAMILTON'S ORIGINAL VIEWS ON QUATERNIONS

The invention of quaternions is related to Hamilton's studies on complex numbers, their geometric representation, and associated ideas.[13] Wessel had shown that it was possible to represent a complex number $z=a+bi$ by means of an inclined line on a plane surface.[14] Hence, points in a two-dimensional space correspond to complex numbers, where the real part represents one direction and the imaginary part represents a perpendicular direction.

Hamilton thought that the analogous situation in three dimensions would be the representation of a point in space (or a straight line in space passing through the origin) by a triplet of numbers such as $a+bi+cj$. In this case, *j* would correspond to a third direction perpendicular to the first two directions. Hamilton attempted to build an algebraic system embodying those ideas, but for several years he was unsuccessful. So, Hamilton gave up the attempt to build an algebra of triplets, and introduced a third imaginary unit *k*, thus creating the quaternions.[15]

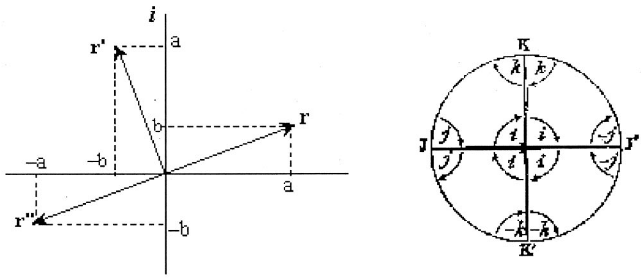

Fig. 2. The rotator character of *i, j, k*.

When Hamilton was studying the triplet $t=a+bi+cj$, its real part was interpreted as corresponding to one direction in space, and the two imaginary parts were regarded as corresponding to the other two perpendicular directions in three-dimensional space. Now, after introducing a third imaginary unit *k*, Hamilton changed his interpretation. In a quaternion $q=a+bi+cj+dk$, the real part has no geometrical meaning, and the three imaginary terms correspond to three perpendicular directions.

At this point one may wonder why Hamilton did not drop the scalar part of the quaternion. It seems that he could have just retained the vector part of the quaternion to represent a spatial quantity, as was done when vector algebra was created a few decades later. However, he did not take this step, and the reason can be found in the context of his research. He was looking for a generalization of complex numbers to represent geometrical quantities in Euclidian space. Therefore, the scalar part of a quaternion was essential to preserve the ''complex number'' character of the new object. Besides that, if an imaginary triplet $t^*=ai+bj+ck$ is multiplied by another imaginary triplet, the product will contain real terms and hence will not be an imaginary triplet. Therefore, if one requires the product of two quantities to be of the same kind as those quantities, it is impossible to build a suitable algebra with imaginary triplets.

IV. VERSORS AND UNITARY VECTORS IN THE SYSTEM OF QUATERNIONS

For complex numbers, *i* performs a double role: it could be regarded as representing a direction (perpendicular to the real axis), or it could be regarded as a $\pi/2$ counterclockwise rotation.

Let us consider the complex number $\mathbf{r}=a+bi$ that represents a straight line on a plane with a real component *a* and an imaginary component *b* [see Fig. 2(a)]. If we multiply $\mathbf{r}$ by *i*, we obtain $\mathbf{r}'=i\mathbf{r}=-b+ai$. If we multiply $\mathbf{r}'$ again by *i*, we obtain $\mathbf{r}''=i\mathbf{r}'=-a-ib=-\mathbf{r}$. The two multiplications by *i* produce $\mathbf{r}''=ii\mathbf{r}=-\mathbf{r}$, which is compatible with $i^2=-1$. The multiplication of a vector $\mathbf{r}$ (represented by a complex number) by *i* and i^2 produces a counterclockwise rotation of $\pi/2$ and π, respectively.

Because Hamilton regarded quaternions as an extension of complex numbers for four dimensions, he used the geometrical representation of a complex number on the plane *xy* (Ref. 16) to explain the meaning of $i^2=-1$, where *i* is the imaginary unit. The other new imaginary units were also regarded as corresponding to $\pi/2$ rotations, but around different axes.

Now, if the vector part of a quaternion contains imaginary units that represent rotations, this part of the quaternion might be interpreted as something that produces rotations, instead of something that corresponds to a straight line in space. Hamilton soon explored this interpretation. He used the word ''versor'' (meaning ''rotator'') and defined the concept of ''perpendicular versor'' as a quaternion which is able to rotate a vector of $\pi/2$ around a perpendicular direction.[17] If we consider three perpendicular axes, it is possible to associate a versor to each of them, and it will be able to rotate any of the other two axes so as to coincide with the third one.

Tait was the next mathematician who studied quaternions. He developed the quaternion analysis as a tool for physical research. Tait was a friend of Maxwell at Cambridge, and Maxwell's interest in quaternions is due to his relationship with him. He also associated versors with rotations and attributed to them the same meaning as Hamilton did. However, Tait changed the designation and replaced ''perpendicular versor'' by ''quadrantal versor.''[18] A quadrantal versor is an operator that produces a $\pi/2$ counterclockwise rotation of a vector without changing its length.

Let us suppose a system of three unitary and perpendicular vectors *I, J, K*. The operator that changes *J* into *K* is the quadrantal versor *i*, and the axis of *i* lies in the same direction as *I*. Thus $K=iJ$ or $i=K/J$. Analogously, we have $I=jK$ and $J=kI$ [see Fig. 2(b)]. Furthermore, there exist the inverse transformations: $-J/K=K/J=i$ or $-J=iK$, and $K=jI$ and $-I=kJ$. From those relations we have $-J=iK=i(iJ)=i^2J$ and therefore $i^2=-1$. In the same way, we obtain the relations $j^2=k^2=-1$.

As shown above, *I, J, K* and *i, j, k* have different meanings. *I, J, K* are unitary vectors and *i, j, k* are operators that produce rotations. According to the later nomenclature, *i, j, k* are axial vectors, and *I, J, K* are polar vectors.[19] Notwithstanding this difference, Tait used those concepts interchangeably.[20]

> Now the meanings we have assigned to *i, j, k* are quite independent of, and not inconsistent with, those assigned to *I, J, K*. And it is superfluous to use two sets of characters when one will suffice. Hence it appears that *i, j, k* may be substituted for *I, J, K*; in other words, *a unit-vector when employed as a factor may be considered as a quadrantal versor whose plane is perpendicular to the vector*. This is one of the main elements of the singular simplicity of the quaternion calculus.[21]

This indifferent use of *i, j, k* for *I, J, K* appears explicitly when Tait represented a quaternion using its coordinates:

> Because any vector whatever may be represented by $xi+yj+zk$, where *x, y, z* are numbers (or scalars), and *i, j, k* may be any three non-coplanar vectors—though they are usually understood as representing a rectangular system of unit-vectors—and because any scalar may be denoted by *w*; we may write, for any quaternion *q*, the expression $q=w+xi+yj+zk$.[22]

Although *i, j, k* and *I, J, K* are related to the same set of perpendicular axes, they are mathematical entities of different types, and it is inappropriate to use the same symbol to represent different entities.

V. VERSORS AND UNITARY VECTORS IN VECTOR ALGEBRA

In the early 19th century, before the development of quaternions, when physicists wanted to refer to the components of magnitudes such as force and velocity, they had to use different symbols for each component and to work out the mathematical equations using those symbols. There was no symbolic representation such as $\vec{R}=x\vec{i}+y\vec{j}+z\vec{k}$. Maxwell used quaternions in his treatise on electromagnetism,[23] but all relevant physical magnitudes were either pure quaternions or scalars. Hence, Maxwell used a simplified version of Hamilton's (and Tait's) theory of quaternions.

The vector algebra used nowadays was developed by Gibbs and Heaviside as a tool to deal with vectors in three-dimensional space. They both had studied quaternions as used by Maxwell in his electromagnetic theory and attempted to frame a new system that would be easier than Hamilton's.[24] The complete quaternion was never used in physical applications, and for that reason Gibbs and Heaviside chose to work with triplets (pure quaternions) representing directed magnitudes. They kept Hamilton's term "vector" to represent this part of the quaternion. Instead of the whole quaternion product (which has a scalar and a vector part), they defined two different types of product between two vectors (the "scalar"—or "direct product" and the "vector"—or "skew product").

Gibbs wrote a vector as $\alpha=xi+yj+zk$, where *i, j, k* compose a normal system of unit vectors, that is, three mutually perpendicular unit vectors. Their directions were chosen so that *k* is on the side of the $i-j$ plane from which a rotation from *i* to *j* (through one right angle) appears counterclockwise.[25]

The meaning of the symbols *i, j, k* in Gibbs's system is quite different from the original meaning in the quaternion system, although the symbols employed to represent them are the same. For Gibbs, *i, j, k* only represented unit vectors and are never associated with the concept of rotation. It is possible to produce rotations in vector algebra, but this is a property of the skew product, not a fundamental property of *i, j, k*. The operation $i\times j=k$ means that $i\times$ applied to *j* rotates it by $\pi/2$ around the *i* axis resulting in *k*. Notice that according to the analysis presented in Sec. II, the vector product of two polar vectors is an axial vector, and therefore, if *i, j, k* are polar vectors, *k* cannot be the product of $i\times j$. However, Gibbs did not use any distinction equivalent to this one.

Heaviside had an important role in the development and use of vector algebra. In his earlier work he explicitly used the same ideas of the quaternion system, but in his later work he denied any influence of quaternions on his vector system.

VI. THE PRODUCTS OF TWO VECTORS

As was mentioned above, the creation of vector algebra led to the introduction of new types of products. The study of the difference between the meaning and proprieties of the operations called "product" in the quaternion system and in the vector system shows that the identification of a pure quaternion as a vector is quite problematic because they have different multiplication proprieties.

In 1881 Gibbs introduced the notation used up to the present time. He dropped the complete product and maintained both the scalar part and vector part independently. He introduced a "direct product" $\alpha\cdot\beta$ of two vectors α and β that was similar to the scalar part of the complete product defined by Hamilton, and a "skew product" $\alpha\times\beta$ corresponding to the vector part of the quaternion product. The direct and skew products are related to the complete quaternion product as $\alpha\cdot\beta=-S\alpha\beta$ and $\alpha\times\beta=V\alpha\beta$. Note that the direct product is the scalar part of Hamilton's product with the opposite sign.

Starting from those definitions, the products of unitary vectors *i, j, k* were defined as[26] $i\cdot i=j\cdot j=k\cdot k=1$; $i\cdot j=j\cdot i=k\cdot i=i\cdot k=j\cdot k=k\cdot j=0$; $i\times i=j\times j=k\times k=0$; $i\times j=k$; $j\times i=-k$; $j\times k=i$; $k\times j=-i$; $k\times i=j$; $i\times k=-j=k$. Note that in quaternion algebra there is only one kind of product between *i, j, k*.

The direct product obeys the commutative rule, that is, $\alpha\cdot\beta=\beta\cdot\alpha$ but the skew product does not because $\alpha\times\beta=-\beta\times\alpha$. The complete product in quaternion system was not commutative also.

The product of two vectors in the quaternion system obeys the associative law. On the other hand, this property is not valid in the Gibbs–Heaviside system. The associative property is preserved in the quaternion system because the quadrantal versors obey the multiplication rules $i^2=j^2=k^2=-1$. To see this, we may consider the multiplication $i(i+j)j$, taking into account the multiplication rules $ij=k=-ji$, $jk=i=-kj$ and $ki=j=-ik$:

$$i(i+j)j=(ii+ij)j=(i^2+k)j=i^2j+kj=i^2j-i=-j-i, \tag{4}$$

$$i(i+j)j=i(ij+jj)=i(k+j^2)=ik+ij^2=-j+ij^2=-j-i. \tag{5}$$

In the case of the scalar product in the vector system, where *i, j, k* are unitary vector, we have

$$i\cdot(i+j)\cdot j=(i\cdot i+i\cdot j)\cdot j=(1+0)\cdot j \tag{6}$$

and

$$i\cdot(i+j)\cdot j=i\cdot(i\cdot j+j\cdot j)=i\cdot(1+0). \tag{7}$$

In both cases, the last operation represented by the dot symbol is impossible because the scalar product is an operation involving two vectors, and not a vector and a number.

In the case of the vector product the associative property is not valid either, as exemplified below,

$$i\times(i+j)\times j=(i\times i+i\times j)\times j=(0+k)\times j=-i, \tag{8}$$

$$i\times(i+j)\times j=i\times(i\times j+j\times j)=i\times(k+0)=j. \tag{9}$$

When Gibbs identified the scalar part of the complete product of the quaternion system and adopted the positive sign, he did not realize the implicit significance of this change. In fact, the sign is related to intrinsic proprieties of the product such as the associative rule and the meaning of the symbols *i, j, k* as explained in Secs. IV and V.

In the same way as Gibbs, Heaviside[27] associated the symbols *i, j, k* with "rectangular vectors with *unit* lengths parallel to x, y, z."[28] Following Hamilton, Heaviside also interpreted *i, j, k* as versors that produce rotations by $\pi/2$ and using this interpretation he explained the multiplication rules $i^2=j^2=k^2=-1$ and $ij=k=-ji$, etc.

> The meaning of the rules $ij=k$, etc., may be interpreted thus: *j* signifying a unit vector parallel to y, and *k* another parallel to z, let *i* be a unit taken to mean, not a unit vector parallel to x, but a rotation

through an angle of 90° about x as an axis. Then because j rotates 90° about de x-axis is turned to coincide with k, we have $ij=k$. (...) As for the squares, we may verify $i^2=-1$ thus: Rotate j a second time through 90° about the axis of x. The first rotation brought j into coincidence with k, the second brings it to the same line as at first, but pointing the other way.[29]

In this same paragraph, Heaviside criticized the confusion brought about by using the same symbol to represent lines and rotations because ''this double use of the same symbols makes it difficult to establish the elementary parts of quaternions in an intelligible manner.'' However, he did not introduce different symbols to represent those concepts.

In 1885 Heaviside changed his mind and began to adopt an approach that conflicted with the quaternion theory. He criticized the negative sign of the scalar part of the product in Hamilton's system because using this sign meant that it was difficult to pass from the Cartesian formulas to vector formulas. Therefore he defined a scalar product without the negative sign to avoid this difficulty.[30] Notice that he was not attempting to solve the mixing of unitary vectors and versors.

Now, when i, j, k are regarded as rotations, this interpretation leads naturally to $i^2=j^2=k^2=-1$. On the other hand, when i, j, k are regarded as unit vectors, one is led to $i^2=j^2=k^2=+1$. Therefore, a change from minus to plus in the sign of the scalar product corresponds to a shift of interpretation of i, j, k from versors to unit vectors.

In 1892 Heaviside emphasized the difference between versor and unitary vector and criticized the quaternionists because the unit vectors were identified with versors, with the consequence that the square of every vector should be a negative scalar.[31]

The confusion between versors and unit vectors was also discussed by MacFarlane during the controversy concerning vectors versus quaternions in the 1890s.[32] MacFarlane's attitude was intermediate—between the position of the defenders of the Gibbs–Heaviside system and that of the quaternionists.[33] He supported the use of the complete quaternionic product of two vectors, but he accepted that the scalar part of this product should have a positive sign. According to MacFarlane the equation $jk=i$ was a convention that should be interpreted in a geometrical way, but he did not accept that it implied the negative sign of the scalar product.

MacFarlane credited the controversy concerning the sign of the scalar product to the conceptual mixture done by Hamilton and Tait. He made clear that the negative sign came from the use of the same symbol to represent both a quadrantal versor and a unitary vector. His view was that different symbols should be used to represent those different entities.

It is also necessary to stress the difference between a vector quantity and its graphical representation. In order to represent both polar and axial vector quantities graphically, we usually use arrows, that is, we use the same symbol to represent two different things.

Several years were to elapse before the need for different symbols was met. Woldemar Voigt proposed in 1910 the use of the symbols represented in Fig. 3 in order to represent polar and axial vectors.[34]

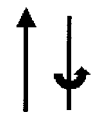

Fig. 3. Woldemar Voigt's representation of a polar and an axial vector (see Ref. 34).

In 1912 Paul Langevin proposed a bent arrow to identify axial vectors, and a straight arrow to identify polar vectors:[35] $\vec{E}$ and $\overset{\curvearrowright}{B}$. This proposal was never accepted, however, and this day physicists still use the same symbol to represent those different mathematical entities.

The traditional notation of arrows to represent the polar and axial vectors makes it difficult for students to realize that the electric field is a physical quantity with polar symmetry whereas the magnetic field has axial symmetry. This tradition began in the late 19th century with the invention of actual vector system by Gibbs and Heaviside from a quaternion system.

VII. CONCLUSION

An analysis of symmetry properties shows that it is wrong to identify a pure quaternion as a common (polar) vector, as Gibbs and Heaviside did when they developed their vector algebra and as some authors do nowadays.

A quaternion q is a mathematical object formed by four numbers which can be written as $q=a+bi+cj+zk$. Hamilton arrived at quaternions starting from the analysis of complex numbers, so that i, j, k are imaginary units that obey the rules $i^2=j^2=k^2=-1$. Hamilton interpreted a quaternion as made up of a scalar plus a ''vector'' part.

In accordance with the geometrical interpretation of complex numbers, Hamilton interpreted i, j, k as ''versors'' that produce a $\pi/2$ rotation when applied to another vector, but he also interpreted i, j, k as unit vectors, that is, the same meaning that was used afterwards in Gibbs–Heaviside vector algebra. He gave the name ''pure quaternion'' or ''vector'' to a quaternion without a scalar part, of the form $q=bi+cj+zk$. The pure quaternion is a vector in the current sense (that is, a polar vector) only if i, j, k are regarded as unit polar vectors, but not if they are interpreted as versors.

Hamilton did not drop the scalar part of the quaternion because pure quaternions did not fulfill the properties he wanted to ascribe to his system. The scalar part of a quaternion is essential to keep up the ''complex number'' character of the new object and to ensure that the product of two quaternions is also a quaternion.

Gibbs and Heaviside developed the contemporary vector algebra in R^3 starting from the quaternion system, although they later denied any influence of the quaternions over their system. They interpreted a pure quaternion as a common (polar) vector and introduced new definitions of the product of two vectors by abandoning the complete quaternion product and replacing it by two separate types of products, the scalar product and vector product. Besides, they changed the sign of the scalar product, because they interpreted i, j, k as unit vectors, not as versors.

A pure quaternion is not equivalent to a polar vector in R^3 because i, j, k have different meanings for pure quaternion and for a vector. Their products obey different rules and they exhibit different symmetry proprieties. Within the quaternion theory, the units i, j, k are versors (that is, axial vectors) and so is the pure quaternion. But the common vector in Euclidian space is a polar vector.

Besides the differences in symmetry proprieties of a pure quaternion and a vector, there are differences in the proprieties of the product of two pure quaternions and two vectors. The product of quaternions does not obey the commutative property, but obeys the associative property because $i^2=j^2=k^2=-1$. The scalar product of two vectors is commutative but is not associative and the vector product is neither commutative nor associative.

Finally, the root of the misunderstanding between pure quaternions and common vectors can be found in the two conflicting meanings ascribed to i, j, k by Hamilton and Tait and in the use of the same symbol to represent what nowadays we call a polar vector and an axial vector. The lack of different symbols is one of the sources of student's difficulty in understanding the difference between those widely different mathematical concepts.

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