Design and Analysis of High-Capacity Associative Memories Based on a Class of Discrete-Time Recurrent Neural Networks

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Abstract—This paper presents a design method for synthesizing associative memories based on discrete-time recurrent neural networks. The proposed procedure enables both hetero- and autoassociative memories to be synthesized with high storage capacity and assured global asymptotic stability. The stored patterns are retrieved by feeding probes via external inputs rather than initial conditions. As typical representatives, discrete-time cellular neural networks (CNNs) designed with space-invariant cloning templates are examined in detail. In particular, it is shown that a procedure herein can determine the input matrix of any CNN based on a space-invariant cloning template which involves only a few design parameters. Two specific examples and many experimental results are included to demonstrate the characteristics and performance of the designed associative memories.

Index Terms—Autoassociative memory, cellular neural networks (CNNs), cloning template, heteroassociative memory.

I. INTRODUCTION

ASSOCIATIVE memories refer to brainlike devices designed to store a set of prototype patterns such that the stored patterns can be retrieved with the recalling probes containing sufficient information about the contents of the patterns. When given a probe (e.g., noisy or corrupted version of a prototype pattern), the retrieval dynamics of the associate memory should converge to an equilibrium point representing the prototype pattern. There are two types of associative memory: autoassociative and heteroassociative. An autoassociative memory retrieves a previously stored pattern that closely resembles the recalling probe. In a heteroassociative memory, the retrieved pattern is generally different from the probe in content or format.

In the past 30 years, the analysis and design of associative memories have received much attention. A variety of neural networks are used to the models of associative memories. Associative memories have been a research subject since the early 1970s. The linear associator is one of the earliest and simplest studied associative memories. It is a type of one-layer feedforward network designed by using the outer product method and can be used as an autoassociator or heteroassociator. There are two shortcomings in linear associator: patterns must be mutually orthogonal, and it cannot recall the prototype pattern exactly when the corresponding input probe is noisy or corrupted. In 1977, Anderson et al. [1] proposed a discrete-time nonlinear dynamical system, called the Brain-State-in-a-Box (BSB) neural network, as a memory model based on neurophysiological considerations. Resurgent interests in associative memories were prompted by the inspiring work of Hopfield [2] in 1982, who showed how a discrete-time Hopfield network as an autoassociative memory can exhibit associative recall of stored binary patterns through collective computing. Similar to the BSB models, the Hopfield associative memory is a recurrent neural network with rich neurodynamic behaviors. In 1988, Kosko [3] extended the Hopfield associative memory to bidirectional associative memory by incorporating an additional layer to perform recurrent autoassociation or heteroassociation.

Since then, many methods have been proposed for designing associative memories based on several recurrent neural networks [4]. In particular, several methods were proposed for designing associative memories using both continuous-time cellular neural networks (CNNs) and discrete-time cellular neural networks (DTCNNs) (e.g., [5]–[13]). Discrete-time recurrent neural networks (DTRNNs) with linear saturation activation functions, including DTCNNs as a special case, to be implemented in digital hardware are suitable models of associative memories. For instance, CNNs were directly regarded as associative memories [4], [9]–[12].

In associative memories, the stored patterns are associated with their retrieval probes internally in a robust and fault-tolerant way. In general, there are two types of pattern-recalling modes in associative memories based on neural networks: 1) recalling probes are set as initial states and 2) probes are fed into input. In the first case, when a retrieval probe is set as an initial condition, it is necessary that the activation states converge to one of locally asymptotically stable equilibrium points [5]–[12]. As each equilibrium point has an attractive region, a pattern is associatively memorized if it is encoded with the corresponding equilibrium point and can be retrieved if the initial states are located within the attractive region. The outer
product method [2], the eigenstructure method [4], singular-value-decomposition techniques [8], and the pseudo-inverse technique [9] were developed for the design. The synthesis of CNNs with space-invariant cloning templates was discussed in [10]. A synthesis procedure for CNNs with space-invariant cloning templates was developed based on the well-known perceptron-training procedure [11], [12]. When the retrieval probes are fed via external inputs in the second mode rather than initial conditions in the first mode, it is necessary for the neural network to have a globally asymptotically stable equilibrium point in [14] and [15]. With assured global stability, CNNs can be designed to serve as hetero- or autoassociative memories. The global asymptotic stability of the equilibrium point via Lyapunov diagonally stable matrices [14] or frequency-domain stability criteria [15].

In addition to neural-network approaches, design methodology for associative memories based on cellular automata were also developed. In particular, the circuit design and error-correcting capability of associative memory based on cellular automata were investigated in [16], [17], and [18].

The existing approaches to associative-memory design suffer several shortcomings including low storage capacity with respect to the number of neuron or cell, increasing number of spurious states with respect to the ratio of number of patterns to the number of neurons/cells, and increasing number of design parameters with respect to the size of patterns. The aim in this paper is to present a new design procedure for high-capacity and high-performance associative memories based on DTRNNs stability results, e.g., [19]–[24], where the stored patterns are conveniently retrieved via input probes. The design procedure and designed associative memories have the following salient features: 1) high storage capacity: the capacity of the designed associative memories is made up to \(2^n\) patterns; 2) high degree of fault tolerance to noisy input probes; and 3) design of DTCNNs with a few design parameters using space-invariant cloning templates.

The remaining part of this paper consists of seven sections. Section II describes some preliminaries. Section III elucidates the theoretical groundwork for the design of hetero- and autoassociative memories. Specific results on DTCNN can be found in Section IV. In Section V, a design procedure for heteroassociative memories based on DTCNN is presented. In Section VI, two illustrative examples are given to demonstrate the effectiveness of the proposed approach. In Section VII, experiential results are reported to demonstrate the fault-tolerant capability of designed associative memories in the presence of distribution in input probes. Finally, concluding remarks are included in Section VIII.

II. PRELIMINARIES

A. Design Problem

Denote \(\{-1,1\}^n\) as the set of \(n\)-dimensional bipolar vectors, i.e.,

\[
\{-1,1\}^n = \{x \in \mathbb{R}^n, x = (x_1, x_2, \ldots, x_n)^T, x_i = 1 \text{ or } -1, i = 1, 2, \ldots, n\}
\]

as the product of the set of \(n\)- and \(m\)-dimensional bipolar vectors, i.e.,

\[
\{-1,1\}^n \times \{-1,1\}^m = \{(x,y) \in \mathbb{R}^{n+m}, x = (x_1, \ldots, x_n)^T, x_i = 1 \text{ or } -1, i = 1, 2, \ldots, n
\]

\[
y = (y_1, \ldots, y_m)^T, y_j = 1 \text{ or } -1, j = 1, 2, \ldots, m\}.
\]

Hence, \(\{-1,1\}^n \times \{-1,1\}^m\) is made up of \(2^{n+m}\) elements. We have the following design problem.

**Design Problem:** Given \(p\) (\(p \leq \min\{2^n, 2^m\}\)), pairwise vectors (paired patterns and probes) \((s^{(1)}, u^{(1)}), (s^{(2)}, u^{(2)}), \ldots, (s^{(p)}, u^{(p)})\), \((s^{(\ell)}, u^{(\ell)}) \in \{-1,1\}^n \times \{-1,1\}^m, \ell \in \{1, 2, \ldots, p\}\), design an associative memory based on a neural network such that if \(u^{(\ell)}(\ell \in \{1, 2, \ldots, p\})\) is fed to the associative memory from its input as a probe, then the output vector of neural network converges to corresponding pattern \(s^{(\ell)}\). When \(s^{(\ell)} \approx u^{(\ell)}, s^{(\ell)}\) is said to be autoassociatively memorized with \(u^{(\ell)}\) in the associative memory. Otherwise, \(s^{(\ell)}\) is said to be heteroassociatively memorized with \(u^{(\ell)}\).

B. Model

In this paper, we always assume that \(t \in \mathcal{N}\), where \(\mathcal{N}\) is the set of all natural numbers.

Consider a DTRNN model

\[
\Delta x_i(t) = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} s_j(t) + \sum_{j=1}^{m} d_{ij} u_j(t) + v_i
\]

\[
s_i(t) = f(x_i(t))
\]

where \(i \in \{1, 2, \ldots, n\}\), \(\Delta x_i(t) = x_i(t+1) - x_i(t), x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n\) is the state vector, \(c_i\) is the self-connection weight, \(A = (a_{ij}) \in \mathbb{R}^{n \times n}\) is intraneuron connection weight matrix, and \(D = (d_{ij}) \in \mathbb{R}^{m \times m}\) is input matrix, \(\forall r \in \mathbb{R}\)

\[
f(r) = \frac{1}{2}(|r+1| - |r-1|) \quad (1)
\]

\(s = (s_1, \ldots, s_n)^T = (f(x_1(t)), \ldots, f(x_n(t)))^T \in \mathbb{R}^n\) is the output vector, \(u = (u_1, \ldots, u_m)^T \in \mathbb{R}^m\) is the input vector, and \(v = (v_1, \ldots, v_m)^T \in \mathbb{R}^m\) is the bias vector.

In many engineering applications and hardware implementations of neural networks, time delays, even time-varying delays in neuron signal transmission or processing, are often inevitable. Therefore, it is necessary to consider the following neural network with time-varying delays:

\[
\Delta x_i(t) = -c_i x_i(t) + \sum_{j=1}^{n} (a_{ij} f(x_j(t))) + b_{ij} f(x_j(t-\tau_j(t))) + \sum_{j=1}^{m} d_{ij} u_j(t) + v_i \quad (2)
\]

where \(B = (b_{ij}) \in \mathbb{R}^{n \times n}\) and \(0 \leq \tau_j(t) \leq \tau\) is a time delay.
Denote $Z$ as the set of all integers, $[a, b]^Z = \{a, a + 1, \ldots, b - 1, b\}$, where $a, b \in Z, a \leq b$. Let $C([t_0 - \tau, t_0]) \subset \mathbb{R}^n$ be the Banach space of functions mapping $[t_0 - \tau, t_0]$ into $\mathbb{R}^n$ with norm defined by $\|\phi\|_{t_0} = \max_{1 \leq s \leq n} \{\sup_{[t_0 - \tau, t_0]} |\phi_i(r)|\}$. Let $\phi(s) = (\phi_1(s), \phi_2(s), \ldots, \phi_n(s))^T$. Denote $\|\phi, \psi\|_{t_0} = \max_{1 \leq s \leq n} \{\sup_{[t_0 - \tau, t_0]} |\phi_i(r) - \psi_i(r)|\}$, where $\psi(s) = (\psi_1(s), \psi_2(s), \ldots, \psi_n(s))^T$. Denote $\|x\| = \max_{1 \leq s \leq n} \{|x_i|\}$ as the vector norm of the vector $x = (x_1, \ldots, x_n)^T$.

The initial value problem for DTRNN (2) requires the knowledge of initial data $\{x(-\tau), \ldots, x(0)\}$. This vector is called initial string. For every initial string and $t \geq 1$, there exists a unique solution $\{x(t)\}_{t \geq -\tau}$ of (2) that can be calculated by the explicit recursive formula

$$x_i(t + 1) = (1 - c_i)x_i(t) + \sum_{j=1}^n (a_{ij}f(x_j(t)) + b_{ij}f(x_j(t - \tau_j(t)))) + \sum_{j=1}^m d_{ij}u_j + v_i. \quad (3)$$

**Definition 1:** An equilibrium point $\bar{x}$ of (2) is said to be locally asymptotically stable in $D$ if there exist constants $\lambda_0 \in (0, 1)$ and $\beta(\phi) > 0$ which depends on $\phi$, such that $\forall t \geq t_0$

$$\|x(t; t_0, \phi) - \bar{x}\| \leq \beta(\phi)\lambda_0^{t-t_0}$$

where $x(t; t_0, \phi)$ is the state of DTRNN (2) with any initial string $\{\phi(t_0 - \tau), \phi(t_0 - \tau + 1), \ldots, \phi(t_0)\}$, $\phi(d) \in C([t_0 - \tau_0, t_0], D)$ and $D$ is said to be a locally attractive set of the equilibrium point $\bar{x}$. When $D = \mathbb{R}^n$, $\bar{x}$ is said to be globally asymptotically stable.

A vector $s$ is called as a (stable) memory vector (or simply, a memory) of a DTRNN if $s = f(\beta)$, where $\beta$ is an asymptotically stable equilibrium point of DTRNN (2).

Given $p (p \leq \min\{2^n, 2^m\})$ pairwise vectors $(s^{(1)}, u^{(1)}), (s^{(2)}, u^{(2)}), \ldots, (s^{(p)}, u^{(p)})$, where $(s^{(i)}, u^{(i)}) \in \{-1, 1\}^n \times \{-1, 1\}^m (i \in \{1, 2, \ldots, p\})$, if there exist the connection weight matrices $A, B, D$ and the bias vector $v$ such that $s^{(i)}$ is stable memory vector of DTRNN (2) with input vector $u^{(i)}$, $\forall i \in \{1, 2, \ldots, p\}$, then $p$ pairwise vectors $(s^{(1)}, u^{(1)}), (s^{(2)}, u^{(2)}), \ldots, (s^{(p)}, u^{(p)})$ are said to be associatively memorized by DTRNN (2).

**C. Notations**

Define the saturation region as follows:

$$\Omega = \left\{ \prod_{i=1}^n (-\infty, -1]^\delta^{(i)} \times [1, +\infty)^{1 - \delta^{(i)}} \delta^{(j)} = 1 \text{ or } 0, j = 1, 2, \ldots, n \right\}.$$ 

Hence, $\Omega$ is composed of $2^n$ elements.

We always assume that $2^n$ vectors $(s^{(1)}, s^{(2)}, \ldots, s^{(2^n)})$ in $\{-1, 1\}^n$ satisfy that $\forall i, j \in \{1, 2, \ldots, 2^n\}, i \neq j, s^{(i)} \neq s^{(j)}$ and that $2^m$ vectors $u^{(1)}, u^{(2)}, \ldots, u^{(2^m)}$ in $\{-1, 1\}^m$ satisfy that $\forall i, j \in \{1, 2, \ldots, 2^m\}, i \neq j, u^{(i)} \neq u^{(j)}$.

For $\ell = 1, 2, \ldots, p$, denote the desired memory patterns $s^{(\ell)} = (s^{(1)}_1, s^{(1)}_2, \ldots, s^{(1)}_n)^T$. For $i \in \{1, 2, \ldots, n\}, \ell \in \{1, 2, \ldots, p\}$ and constants $\sigma_i$, let

$$v_i^+(\ell) = c_i - \sum_{j=1}^n (a_{ij} + b_{ij}) s_j^{(\ell)} - \sum_{j=1}^m d_{ij} u_j^{(\ell)} \quad (4)$$

$$v_i^-(\ell) = -c_i - \sum_{j=1}^n (a_{ij} + b_{ij}) s_j^{(\ell)} - \sum_{j=1}^m d_{ij} u_j^{(\ell)} \quad (5)$$

$$c_{ij} = \left\{ \begin{array}{ll} a_{ij} + b_{ij} - \sigma_i, & i = j \\ a_{ij} + b_{ij}, & i \neq j \end{array} \right.$$ 

$$\mathcal{L}_i(\ell) = \left\{ \begin{array}{ll} (v_i^+(\ell), +\infty), & s_i^{(\ell)} = 1 \\ (-\infty, v_i^-(\ell)), & s_i^{(\ell)} = -1 \end{array} \right.$$

$$\mathcal{L}(\ell) = \prod_{i=1}^n \mathcal{L}_i(\ell)$$

**III. THEORETICAL RESULTS**

**A. Stability**

For $i, j \in \{1, 2, \ldots, n\}$, let

$$T_{ij} = \left\{ \begin{array}{ll} c_i - |a_{ij}| - |b_{ij}|, & i = j \\ -|a_{ij}| - |b_{ij}|, & i \neq j \end{array} \right.$$ 

Denote matrices

$$T = (T_{ij})_{n \times n}. \quad (6)$$

**Lemma 1 [22]:** If $\{x(t)\}_{t \geq -\tau}$ is a sequence of real vectors satisfying (2), and $\forall i \in \{1, 2, \ldots, n\}, c_i \in (0, 1)$, and $T$ in (6) is a nonsingular $M$-matrix, then DTRNN (2) has a unique equilibrium point $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)^T$, and there exist positive constants $\beta$ and $\lambda_0 \in (0, 1)$ such that $\forall i \in \{1, 2, \ldots, n\}, |x_i(t) - \bar{x}_i| \leq \beta \|x\| \lambda_0^t, t \geq 1$, i.e., DTRNN (2) is globally asymptotically stable.

**Theorem 1:** If $\forall i \in \{1, 2, \ldots, n\}$

$$\sum_{j=1}^n (|a_{ij}| + |b_{ij}|) < c_i < 1 \quad (7)$$

then DTRNN (2) has a globally asymptotically stable equilibrium point $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)^T$. 

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Proof: If (7) holds, then $T$ in (6) is a nonsingular $M$-matrix. According to Lemma 1, DTRNN (2) has a globally asymptotically stable equilibrium point.

**B. Single Pattern**

**Theorem 2:** If there exists $\ell \in \{1, 2, \ldots, p\}$ such that $v = (v_1, v_2, \ldots, v_n)^T \in \mathcal{L}(\ell)$ and $T$ in (6) is a nonsingular $M$-matrix, then the output vector of the heteroassociative-memory-based DTRNN (2) is globally asymptotically stable at $s^{(\ell)}$ with the retrieval probe $u^{(\ell)}$ as its input vector.

**Proof:** For $i \in \{1, 2, \ldots, n\}$, let

$$x_i = \left( \sum_{j=1}^{n} (a_{ij} + b_{ij}) s_j^{(\ell)} + \sum_{j=1}^{m} d_{ij} u_j^{(\ell)} + v_i \right) / c_i.$$ 

(8)

Since $v \in \mathcal{L}(\ell)$, if $s_i^{(\ell)} = 1$, then

$$v_i > v_i^{+}(\ell) = c_i - \sum_{j=1}^{n} (a_{ij} + b_{ij}) s_j^{(\ell)} - \sum_{j=1}^{m} d_{ij} u_j^{(\ell)}.$$ 

Hence, when $s_i^{(\ell)} = 1$, from (8)

$$\bar{x}_i > 1, f(\bar{x}_i) = s_i^{(\ell)}.$$ 

(9)

Similarly, since $v \in \mathcal{L}(\ell)$, if $s_i^{(\ell)} = -1$, then

$$v_i < v_i^{-}(\ell) = -c_i - \sum_{j=1}^{n} (a_{ij} + b_{ij}) s_j^{(\ell)} - \sum_{j=1}^{m} d_{ij} u_j^{(\ell)}.$$ 

Hence, when $s_i^{(\ell)} = -1$, from (8)

$$\bar{x}_i < -1, f(\bar{x}_i) = s_i^{(\ell)}.$$ 

(10)

From (8), (9), and (10), $\forall i \in \{1, 2, \ldots, n\}$

$$\bar{x}_i = \left( \sum_{j=1}^{n} (a_{ij} + b_{ij}) f(\bar{x}_j) + \sum_{j=1}^{m} d_{ij} u_j^{(\ell)} + v_i \right) / c_i.$$ 

i.e., $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)^T$ is an equilibrium point of (2).

In addition, since $T$ in (6) is a nonsingular $M$-matrix, according to Lemma 1, DTRNN (2) has a globally asymptotically stable equilibrium point. Hence, $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)^T$ is a globally asymptotically stable equilibrium point of DTRNN (2). Sequentially, the output vector of the heteroassociative-memory-based DTRNN (2) is globally asymptotically stable at $s^{(\ell)}$ with the retrieval probe $u^{(\ell)}$ as its input vector.

**C. Heteroassociative Memories**

Let

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$ 

Hence

$$v_i^{+}(r) < v_i^{-}(q).$$
Define an operator
\[ \mathcal{D}_i = \begin{cases} \max_{t \in J_+^i} v_i^+(t), +\infty, & \mathcal{J}_i^- = \emptyset \\ -\infty, \min_{t \in J_-^i} v_i^-(t), & \mathcal{J}_i^+ = \emptyset \\ \{v_i^+(t), \min_{t \in J_-^i} v_i^-(t)\}, & \text{other cases} \end{cases} \]

where \( \emptyset \) is an empty set. \( v_i^+(r) < v_i^-(q) \) implies that \( \mathcal{D}_i \) is not empty. Since \( \forall i \in \{1, 2, \ldots, n\}, \forall t \in \{1, 2, \ldots, p\}, \mathcal{D}_i \subset \mathcal{L}(t) \)

\[ \prod_{i=1}^n \mathcal{D}_i \subset \bigcap_{\ell=1}^p \mathcal{L}(t). \]

Hence, \( \bigcap_{\ell=1}^p \mathcal{L}(t) \) is not empty.

**Remark 1:** According to the proof of Lemma 2, \( \forall i \in \{1, 2, \ldots, n\}, \) when \( \mathcal{J}_i^+ \) and \( \mathcal{J}_i^- \) are not empty, if
\[ \max_{t \in J_+^i} \{v_i^+(t)\} < \min_{t \in J_-^i} \{v_i^-(t)\} \]
then \( \bigcap_{\ell=1}^p \mathcal{L}(t) \) is not empty.

**Theorem 3:** If (7) and (11) or (15) hold, then there exists a region \( \bigcap_{\ell=1}^p \mathcal{L}(t) \) such that for any bias vector \( v \in \bigcap_{\ell=1}^p \mathcal{L}(t) \), the output vector of the heteroassociative-memory-based DTRNN (2) is globally asymptotically stable at \( s(t) \) with the retrieval probe \( u(t) \) as its input vector for \( \ell = 1, 2, \ldots, p \).

**Proof:** According to Lemma 2 or Remark 1, \( \bigcap_{\ell=1}^p \mathcal{L}(t) \) is not empty. Determine \( v \in \bigcap_{\ell=1}^p \mathcal{L}(t) \). Then, \( \forall t \in \{1, 2, \ldots, p\}, v \in \mathcal{L}(t) \). According to Theorem 2, the pair \((s(t), u(t))\) is associatively memorized by DTRNN (2). Because of the arbitrary of \( \ell \), the output vector of the heteroassociative-memory-based DTRNN (2) is globally asymptotically stable at \( s(t) \) with the retrieval probe \( u(t) \) as its input vector for \( \ell = 1, 2, \ldots, p \).

**Remark 2:** According to Remark 1 and Theorem 3, if (7) holds and \( \forall i \in \{1, 2, \ldots, n\}, \) when \( \mathcal{J}_i^+ \) and \( \mathcal{J}_i^- \) are not empty, \( \max_{t \in J_+^i} \{v_i^+(t)\} < \min_{t \in J_-^i} \{v_i^-(t)\} \), then there exists a region \( \bigcap_{\ell=1}^p \mathcal{L}(t) \) such that any bias vector \( v \in \bigcap_{\ell=1}^p \mathcal{L}(t) \); the output vector of the heteroassociative-memory-based DTRNN (2) is globally asymptotically stable at \( s(t) \) with the retrieval probe \( u(t) \) as its input vector for \( \ell = 1, 2, \ldots, p \). According to Theorem 3, the designed neural networks can be obtained by choosing the connection weight matrices \( A, B, D \), and a bias vector \( v \) such that (7) and (11) hold, where \( v = (v_1, v_2, \ldots, v_n) \in \prod_{i=1}^n \mathcal{D}_i \). Hence, the bias vector can be determined in a region without affecting the desired heteroassociative memories. When a noisy input probe is input into DTRNN (2), its equilibrium point may be displaced. To increase the fault-tolerance capability against noisy input probes, it is desirable to make the bias vector robust to noisy probes by offsetting the impact. For any \( v = (v_1, v_2, \ldots, v_n) \in \prod_{i=1}^n \mathcal{D}_i \), let
\[ \hat{x}_i^{(t)} = \left( \sum_{j=1}^n (a_{ij} + b_{ij}) s_j^{(t)} + \sum_{j=1}^m d_{ij} u_j^{(t)} + v_i \right) / c_i. \]

Then
\[ s_i^{(t)} = f \left( \hat{x}_i^{(t)} \right). \]

Let \( \Delta u_j^{(t)} = \hat{u}_j^{(t)} - u_j^{(t)} \) be the distortion in input probes, i.e., an input probe \( u_j^{(t)} \) becomes \( \hat{u}_j^{(t)} \), \( (j = 1, 2, \ldots, m) \). Let
\[ \hat{v}_i = \sum_{j=1}^m d_{ij} \Delta u_j^{(t)} + v_i \]
\[ \hat{x}_i^{(t)} = \left( \sum_{j=1}^n (a_{ij} + b_{ij}) s_j^{(t)} + \sum_{j=1}^m d_{ij} u_j^{(t)} + \hat{v}_i \right) / c_i. \]

Then
\[ s_i^{(t)} = f \left( \hat{x}_i^{(t)} \right). \]

From (17) and (18), the output vector of the heteroassociative-memory-based DTRNN (2) is globally asymptotically stable at \( s(t) \) with the distorted probe \( \hat{u}(t) \) as its input vector. Hence, it is highly favorable for the fault-tolerance capability of the designed associative memories that the bias vector can be determined in a region without affecting the desired heteroassociative memories.

**Theorem 4:** If \( T \) in (6) is a nonsingular \( M \)-matrix and there exists constant \( \lambda_i > 1 \) such that
\[ D\hat{U}_p = (AC - A - B)\hat{S}_p \]
then there exists a region \( \bigcap_{\ell=1}^p \mathcal{L}(t) \) such that, for any bias vector \( v \in \bigcap_{\ell=1}^p \mathcal{L}(t) \), the output vector of the heteroassociative-memory-based DTRNN (2) is globally asymptotically stable at \( s(t) \) with the retrieval probe \( u(t) \) as its input vector for \( \ell = 1, 2, \ldots, p \).

**Proof:** Since (19) holds, \( \forall i \in \{1, 2, \ldots, n\}, \forall t \in \{1, 2, \ldots, p\} \)
\[ \sum_{j=1}^m d_{ij} u_j^{(t)} = \lambda_i c_i s_i^{(t)} - \sum_{j=1}^n (a_{ij} + b_{ij}) s_j^{(t)}. \]

If \( r, q \in \{1, 2, \ldots, p\} \) and \( s_r^{(1)} = 1, s_q^{(1)} = -1 \), then from (20)
\[ \sum_{j=1}^m d_{ij} u_j^{(r)} + \sum_{j=1}^n (a_{ij} + b_{ij}) s_j^{(r)} = \lambda_i c_i \]
\[ \sum_{j=1}^m d_{ij} u_j^{(q)} + \sum_{j=1}^n (a_{ij} + b_{ij}) s_j^{(q)} = -\lambda_i c_i. \]
Hence, from \( \lambda_1 > 1 \)

\[
c_i - \sum_{j=1}^{n} (a_{ij} + b_{ij}) s_j^{(r)} - \sum_{j=1}^{m} d_{ij} u_j^{(r)} < -c_i - \sum_{j=1}^{n} (a_{ij} + b_{ij}) s_j^{(q)} - \sum_{j=1}^{m} d_{ij} u_j^{(q)}. \]

Similar to the proof of Lemma 2, \( \bigcap_{\ell=1}^{p} \mathcal{L}(\ell) \) is not empty. Determine \( v \in \bigcap_{\ell=1}^{p} \mathcal{L}(\ell) \). Then, \( \forall \ell \in \{1, 2, \ldots, p\} \), \( v \in \mathcal{L}(\ell) \). According to Theorem 2, the pair \( (s^{(\ell)}, u^{(\ell)}) \) is associatively memorized by DTRNN (2). Because of the arbitrary of \( \ell \), the output vector of the heteroassociative-memory-based DTRNN (2) is globally asymptotically stable at \( s^{(\ell)} \) with the retrieval probe \( u^{(\ell)} \) as its input vector for \( \ell = 1, 2, \ldots, p \).

D. Autoassociative Memories

\textbf{Theorem 5:} If \( T \) in (6) is a nonsingular \( M \)-matrix and \( \forall i \in \{1, 2, \ldots, n\} \)

\[
a_{ii} + b_{ii} + d_{ii} - \sum_{j=1,j \neq i}^{n} |a_{ij} + b_{ij} + d_{ij}| > c_i \quad (21)
\]

then there exists a region \( \bigcap_{\ell=1}^{p} \mathcal{L}(\ell) \) such that, for any bias vector \( v \in \bigcap_{\ell=1}^{p} \mathcal{L}(\ell) \), the output vector of the autoassociative-memory-based DTRNN (2) is globally asymptotically stable at \( s^{(\ell)} \) with the retrieval probe \( s^{(\ell)} \) as its input vector for \( \ell = 1, 2, \ldots, p \).

\textbf{Proof:} For any \( q \in \{1, 2, \ldots, p\} \), \( \forall i \in \{1, 2, \ldots, n\} \), if \( s_i^{(r)} = u_i^{(r)} \), \( s_j^{(q)} = u_j^{(q)} \), and \( s_i^{(r)} = 1, s_j^{(q)} = -1 \), then from (21)

\[
c_i - \sum_{j=1}^{n} (a_{ij} + b_{ij}) s_j^{(r)} - \sum_{j=1}^{m} d_{ij} u_j^{(r)} < -c_i - \sum_{j=1}^{n} (a_{ij} + b_{ij}) s_j^{(q)} - \sum_{j=1}^{m} d_{ij} u_j^{(q)}. \]

Similar to the proof of Lemma 2, \( \bigcap_{\ell=1}^{p} \mathcal{L}(\ell) \) is not empty. Determine \( v \in \bigcap_{\ell=1}^{p} \mathcal{L}(\ell) \). Then, \( \forall \ell \in \{1, 2, \ldots, p\} \), \( v \in \mathcal{L}(\ell) \). According to Theorem 2, the pair \( (s^{(\ell)}, u^{(\ell)}) \) is associatively memorized by DTRNN (2). Because of the arbitrary of \( \ell \), the output vector of the autoassociative-memory-based DTRNN (2) is globally asymptotically stable at \( s^{(\ell)} \) with the retrieval probe \( s^{(\ell)} \) as its input vector for \( \ell = 1, 2, \ldots, p \).

\textbf{Theorem 6:} If there exist constants \( \lambda_i \geq 2(c_i + 1) \), such that \( \forall i, j \in \{1, 2, \ldots, n\} \)

\[
d_{ii} = \lambda_i, \quad d_{ij} = 0, \quad i \neq j \quad (22)
\]

and (7) holds, then there exists a region \( \bigcap_{\ell=1}^{p} \mathcal{L}(\ell) \) such that, for any bias vector \( v \in \bigcap_{\ell=1}^{p} \mathcal{L}(\ell) \), the output vector of the autoassociative-memory-based DTRNN (2) is globally asymptotically stable at \( s^{(\ell)} \) with the retrieval probe \( s^{(\ell)} \) as its input vector for \( \ell = 1, 2, \ldots, p \).

\textbf{Proof:} For any \( \forall i \in \{1, 2, \ldots, n\} \), \( \forall \ell \in \{1, 2, \ldots, p\} \), \( s_i^{(\ell)} = u_i^{(\ell)} \), from (22), (11) holds. Since (7) holds, according to Theorem 3, the output vector of the autoassociative-memory-based DTRNN (2) is globally asymptotically stable at \( s^{(\ell)} \) with the retrieval probe \( s^{(\ell)} \) as its input vector for \( \ell = 1, 2, \ldots, p \).

\textbf{Theorem 7:} If \( T \) in (6) is a nonsingular \( M \)-matrix and there exists constant \( \lambda_i > 1 \) such that \( \forall i, j \in \{1, 2, \ldots, n\} \)

\[
d_{ii} = \lambda_i c_i - a_{ii} - b_{ii}, \quad d_{ij} = -a_{ij} - b_{ij}, \quad i \neq j \quad (23)
\]

then there exists a region \( \bigcap_{\ell=1}^{p} \mathcal{L}(\ell) \) such that, for any bias vector \( v \in \bigcap_{\ell=1}^{p} \mathcal{L}(\ell) \), the output vector of the autoassociative-memory-based DTRNN (2) is globally asymptotically stable at \( s^{(\ell)} \) with the retrieval probe \( s^{(\ell)} \) as its input vector for \( \ell = 1, 2, \ldots, p \).

\textbf{Proof:} For any \( \forall i \in \{1, 2, \ldots, n\} \), \( \forall \ell \in \{1, 2, \ldots, p\} \), \( s_i^{(\ell)} = u_i^{(\ell)} \), from (23), (19) holds. Since \( T \) in (6) is a nonsingular \( M \)-matrix, according to Theorem 4, the output vector of the autoassociative-memory-based DTRNN (2) is globally asymptotically stable at \( s^{(\ell)} \) with the retrieval probe \( s^{(\ell)} \) as its input vector for \( \ell = 1, 2, \ldots, p \).

E. Capacity

\textbf{Proposition 1:} The storage capacity of heteroassociative memories based on DTRNN (2) is at least \( n \).

\textbf{Proof:} Let \( p = n \), and \( U_p \) is invertible, i.e., \( \text{Rank} \{u^{(1)}, u^{(2)}, \ldots, u^{(n)}\} = n \). For any \( \lambda_i \geq 2(c_i + 1) \), let \( D = \Lambda S_U U_p^{-1} \). Then, (11) holds. In addition, it is easy to determine \( A, B \) such that (7) holds. Hence, according to Theorem 3, the capacity of minimal memorized patterns is \( n \) in heteroassociative memories with DTRNN (2).

\textbf{Proposition 2:} If (7) and (21) or (22) or (23) hold, then the storage capacity of autoassociative memories based on DTRNN (2) is at least \( n \).

\textbf{Proof:} If (7) and (21) or (22) or (23) hold, then there exists a region \( \bigcap_{\ell=1}^{p} \mathcal{L}(\ell) \) such that, for any bias vector \( v \in \bigcap_{\ell=1}^{p} \mathcal{L}(\ell) \), \( s_i^{(1)}, s_i^{(2)}, \ldots, s_i^{(n)} \) can be autoassociatively memorized by DTRNN (2), i.e., the storage capacity of memorized patterns is at least \( n \) in heteroassociative memories with DTRNN (2).

According to Theorems 5–7, it is not necessary to restrict the desired memory patterns in an autoassociative memory. In other words, if the conditions of Theorems 5–7 hold, then any pattern can be autoassociatively memorized by a designed neural network. Hence, we have the following proposition.

\textbf{Proposition 3:} If (7) and (21) or (22) or (23) hold, then the storage capacity of autoassociative memories based on DTRNN (2) is at least \( 2^n \).

\textbf{Proof:} If (7) and (21) or (22) or (23) hold, then there exists a region \( \bigcap_{\ell=1}^{p} \mathcal{L}(\ell) \) such that, for any bias vector \( v \in \bigcap_{\ell=1}^{p} \mathcal{L}(\ell) \), \( s_i^{(1)}, s_i^{(2)}, \ldots, s_i^{(n)} \) can be autoassociatively memorized by DTRNN (2), i.e., the storage capacity of memorized patterns is at least \( 2^n \) in autoassociative memories with DTRNN (2).

Generally, an \( n \)-neuron CNN can have no more than \( 2^n \) stable isolated equilibrium points [20]. A pattern in an associative memory must correspond to an isolated equilibrium point. Hence, for a neural network with multiequilibrium points, the maximal storage capacity of the designed associative memories is \( 2^n \) patterns. In addition, when each trajectory converges to a unique equilibrium point depending on external input instead of initial state, the neural network with a unique equilibrium point can also have \( 2^n \) stable isolated equilibrium points.
corresponding to \(2^n\) external input probes \((n \leq m)\). Hence, for a neural network to have a unique stable equilibrium point with each external input probe, the storage capacity of the designed associative memories is also \(2^n\) patterns at most.

IV. DTCNNs DESCRIBED USING CLONING TEMPLATES

Consider a DTCNN described by a space-invariant template where the cells are arranged on a 2-D array composed of \(N\) rows and \(M\) columns. The dynamics of such a DTCNN are governed by the following:

\[
\Delta y_{ij}(t) = -\hat{c}_{ij}y_{ij}(t) + \sum_{k=k_1(i,r_1)}^{k_2(i,r_1)} \sum_{l=l_1(i,r_2)}^{l_2(i,r_2)} a_{k,l},f(y_{i+k,j+l}(t)) \\
+ \sum_{k=k_1(i,r_1)}^{k_2(i,r_1)} \sum_{l=l_1(i,r_2)}^{l_2(i,r_2)} b_{k,l},f(y_{i+k,j+l}(t - \tau_{ij}(t))) \\
+ \sum_{k=k_1(i,r_1)}^{k_2(i,r_1)} \sum_{l=l_1(i,r_2)}^{l_2(i,r_2)} b_{k,l}u_{i+k,j+l} + \hat{v}_{ij}
\]  

(24)

where, for \(i \in \{1, 2, \ldots, N\}\) and \(j \in \{1, 2, \ldots, M\}\), \(k_1(i,r_1) = \max\{1 - i, -r_1\}\), \(k_2(i,r_1) = \min\{N - i, r_1\}\), \(l_1(j,r_2) = \max\{1 - j, -r_2\}\), \(l_2(j,r_2) = \min\{M - j, r_2\}\), \(y_{ij}\) denotes the state of the cell located at the crossing between the \(i\)th row and \(j\)th column of the network, \(q\) and \(q\) denote neighborhood radius and are positive integers, \(A = (d_{kl}(x_{r1}+1) \times (x_{r2}+1))\) is the feedback cloning template defined by a \((2r_1 + 1) \times (2r_2 + 1)\) real matrix, \(B = (b_{kl}(x_{r1}+1) \times (x_{r2}+1))\) is the delay feedback cloning template defined by a \((2r_1 + 1) \times (2r_2 + 1)\) real matrix, \(D = (d_{kl}(x_{r1}+1) \times (x_{r2}+1))\) is the input cloning template defined by a \((2r_1 + 1) \times (2r_2 + 1)\) real matrix, \(\hat{c}_{ij} > 0\).

For \(i = 1, 2, \ldots, N\) and \(j = 1, 2, \ldots, M\), let \(c_{(i-1)M+j} = \hat{c}_{ij},\ n_{(i-1)M+j} = \bar{v}_{ij}\). Denote \(C = \text{diag}\{c_1, c_2, \ldots, c_{NM}\}\). An alternative expression for the state equation of DTCNN (24) is obtained by ordering the cells in some way (e.g., by rows or by columns) and by cascading the state variables into a state vector \(x = (x_1, x_2, \ldots, x_{NM})^T = (y_{11}, y_{12}, \ldots, y_{1M}, y_{21}, \ldots, y_{2M}, \ldots, y_{NM})^T\). The following compact form is then obtained:

\[
\Delta x(t) = -Cx(t) + Af(x(t)) + Bf^{(d)}(x(t)) + Du + v
\]  

(25)

where the coefficient matrices \(A, B, D\) are obtained through the templates \(A, B, D\), respectively, the vectors \(u\) and \(v\) are obtained through \(u_{ij}\) and \(v_{ij}\), respectively, vector-valued activation functions \(f(x(t)) = (f(x_1(t)), f(x_2(t)), \ldots, f(x_{NM}(t)))^T, f^{(d)}(x(t)) = (f(x_1(t) - \tau_1(t)), f(x_2(t) - \tau_2(t)), \ldots, f(x_{NM}(t) - \tau_{NM}(t)))^T\), and the delay \(\tau(t)\) is obtained through \(\bar{v}_{ij}(t)\). Denote \(n = NM\).

The dynamic rule of a DTCNN can be completely specified by its cloning templates. \(A, B, D\) in the DTCNN (25) depend on the established order among the cells and on the cloning templates and the delay cloning template. For example, for a 1-D space-invariant CNN with a neighborhood radius \(r = 1\), its cloning template is \(\hat{A} = [a_{-1}, a_0, a_1]\). Matrix \(A\), defined by (25), composed of the template has the circulant form

\[
\begin{bmatrix}
0 & a_1 & 0 & 0 \\
a_{-1} & a_0 & a_1 & 0 \\
0 & a_{-1} & a_0 & a_1 \\
\vdots & \vdots & \ddots & \vdots \\
a_{-1} & a_0 & a_1 & 0 \\
0 & a_{-1} & a_0 & a_1
\end{bmatrix}
\]

(26)

In addition, for example, for a 2-D space-invariant CNN with a neighborhood radius \(r\), its cloning template \(\hat{A}\) is a \((2r + 1) \times (2r + 1)\) real matrix, i.e.,

\[
\hat{A} = \\
\begin{bmatrix}
a_{-r,-r} & \cdots & a_{-r,0} & \cdots & a_{-r,r} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
a_{-r,-r} & \cdots & a_{r,0} & \cdots & a_{r,r}
\end{bmatrix}
\]

where

\[
\begin{bmatrix}
A^{(1)} & A^{(2)} & 0 & 0 & \cdots & 0 & 0 \\
A^{(3)} & A^{(1)} & A^{(2)} & 0 & \cdots & 0 & 0 \\
0 & 0 & A^{(3)} & A^{(1)} & A^{(2)} & \cdots & 0 \\
0 & 0 & 0 & A^{(3)} & A^{(1)} & A^{(2)} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & 0 & A^{(3)} & A^{(1)} \\
0 & 0 & 0 & 0 & 0 & 0 & A^{(3)} & A^{(1)}
\end{bmatrix}_{n\times n}
\]


\[
\begin{bmatrix}
a_{0,0} & a_{0,1} & 0 & \cdots & 0 & 0 \\
a_{0,-1} & a_{0,0} & a_{0,1} & 0 & \cdots & 0 \\
0 & a_{0,-1} & a_{0,0} & 0 & \cdots & 0 \\
0 & 0 & 0 & a_{0,0} & a_{0,1} & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\
0 & 0 & 0 & 0 & a_{0,0} & a_{0,1} \\
0 & 0 & 0 & 0 & 0 & a_{1,0} & a_{1,1} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & 0 & a_{1,0} & a_{1,1} \\
0 & 0 & 0 & 0 & 0 & 0 & a_{1,0} & a_{1,1} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
\end{bmatrix}_{M\times M}
\]

\[
\begin{bmatrix}
a_{1,0} & a_{1,1} & 0 & \cdots & 0 & 0 \\
a_{1,-1} & a_{1,0} & a_{1,1} & \cdots & 0 & 0 \\
0 & a_{1,-1} & a_{1,0} & a_{1,1} & \cdots & 0 \\
0 & 0 & a_{1,-1} & a_{1,0} & a_{1,1} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\
0 & 0 & 0 & a_{1,-1} & a_{1,0} & a_{1,1} \\
0 & 0 & 0 & 0 & a_{1,-1} & a_{1,0} & a_{1,1} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
\end{bmatrix}_{M\times M}
\]

\[
\begin{bmatrix}
a_{-1,0} & a_{-1,1} & 0 & \cdots & 0 & 0 \\
a_{-1,-1} & a_{-1,0} & a_{-1,1} & \cdots & 0 & 0 \\
0 & a_{-1,-1} & a_{-1,0} & a_{-1,1} & \cdots & 0 \\
0 & 0 & a_{-1,-1} & a_{-1,0} & a_{-1,1} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\
0 & 0 & 0 & a_{-1,-1} & a_{-1,0} & a_{-1,1} \\
0 & 0 & 0 & 0 & a_{-1,-1} & a_{-1,0} & a_{-1,1} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
\end{bmatrix}_{M\times M}.
\]

B, B^{(1)}, B^{(2)}, B^{(3)} \text{ and } D, D^{(1)}, D^{(2)}, D^{(3)} \text{ can be similarly defined.
A. One-Dimensional Space-Invariant DCNNs

For a 1-D space-invariant DTCNN (24) with a neighborhood radius \( r = 1 \), its cloning templates are \( \hat{A} = [a_{-1}, a_0, a_1] \), \( \hat{B} = [b_{-1}, b_0, b_1] \), \( \hat{D} = [d_{-1}, d_0, d_1] \). The matrix \( A \), defined by (25), composed of the template has the circulant form (26). \( B \) and \( D \) in (25) can be similarly defined.

For \( \ell \in \{1, 2, \ldots, p\} \), let

\[
\begin{align*}
U_+^{(\ell)} &= \begin{pmatrix} 0, u_1^{(\ell)}, u_2^{(\ell)}, \ldots, u_{n-1}^{(\ell)} \end{pmatrix}^T \\
U_-^{(\ell)} &= \begin{pmatrix} u_{n-1}^{(\ell)}, u_n^{(\ell)}, \ldots, u_2^{(\ell)}, 0 \end{pmatrix}^T \\
U_p &= \begin{pmatrix} u_1^{(1)}, u_1^{(2)}, u_1^{(3)}, u_1^{(4)} \\
\vdots & \vdots & \vdots & \vdots \\
u_p^{(1)}, u_p^{(2)}, u_p^{(3)}, u_p^{(4)} \end{pmatrix}_{(np) \times 3} \\
S_{+p} &= \begin{pmatrix} (s^{(1)})^T, (s^{(2)})^T, \ldots, (s^{(p)})^T \end{pmatrix}^T \\
\hat{\Lambda} &= \Delta C - A - B \\
\Lambda' &= \begin{pmatrix} \Lambda & 0 & \cdots & 0 \\
0 & \Lambda & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \Lambda \end{pmatrix}_{(np) \times (np)} \quad \text{Corollary 1: If}
\end{align*}
\]

\[
\frac{1}{n-1} \sum_{j=1}^{n} \left| a_j \right| + \left| b_j \right| \leq \min_{1 \leq \ell \leq n} \{c_\ell\} \leq \max_{1 \leq \ell \leq n} \{c_\ell\} < 1
\]

and there exist constants \( \lambda_i \geq 2(c_i + 1) \) (\( i = 1, 2, \ldots, n \)) such that

\[
U_p \begin{pmatrix} d_{-1} \\ d_0 \\ d_1 \end{pmatrix} = \Lambda' S_{+p}
\]

has a solution, then there exists a region \( \bigcap_{\ell=1}^{2^p} \mathcal{L}(\ell) \) such that, for any bias vector \( v \in \bigcap_{\ell=1}^{2^p} \mathcal{L}(\ell) \), the output vector of the heteroassociative-memory-based DTCNN (24) with the connection weight matrices \( A, B, \) and \( D \) is globally asymptotically stable at \( s^{(\ell)} \) with the retrieval probe \( u^{(\ell)} \) as its input vector for \( \ell = 1, 2, \ldots, p \).

**Proof:** From (29), according to the definition of \( D \), similar to \( A \) in (26), (11) holds.

In addition, since (28) holds, (7) holds. According to Theorem 3, the output vector of the heteroassociative memory based the designed DTCNN (24) is globally asymptotically stable at \( s^{(\ell)} \) with the retrieval probe \( u^{(\ell)} \) as its input vector for \( \ell = 1, 2, \ldots, p \).

**Remark 3:** If (29) has a solution, then the solution of (29) is \( U_p^+ \Lambda' S_{+p} \), where \( U_p^+ \) is the pseudoinverse of \( U_p \).

Corollary 2: If (28) holds and there exist constants \( \lambda_i > 1 \) (\( i = 1, 2, \ldots, n \)) such that

\[
U_p \begin{pmatrix} d_{-1} \\ d_0 \\ d_1 \end{pmatrix} = \Lambda' S_{+p}
\]

has a solution, then there exists a region \( \bigcap_{\ell=1}^{2^p} \mathcal{L}(\ell) \) such that, for any bias vector \( v \in \bigcap_{\ell=1}^{2^p} \mathcal{L}(\ell) \), the output vector of the heteroassociative-memory-based DTCNN (24) with the connection weight matrices \( A, B, \) and \( D \) is globally asymptotically stable at \( s^{(\ell)} \) with the retrieval probe \( u^{(\ell)} \) as its input vector for \( \ell = 1, 2, \ldots, p \).

**Proof:** Since (30) has a solution, according to the definition of \( D \), similar to \( A \) in (26), (19) holds.

In addition, since (28) holds, (7) holds. According to Theorem 4, the output vector of the heteroassociative-memory-based DTCNN (24) with the connection weight matrices \( A, B, \) and \( D \) is globally asymptotically stable at \( s^{(\ell)} \) with the retrieval probe \( u^{(\ell)} \) as its input vector for \( \ell = 1, 2, \ldots, p \).

B. Two-Dimensional Space-Invariant DCNNs

For a 2-D space-invariant DTCNN (24) with a neighborhood radius \( r = 1 \), its cloning templates are

\[
\hat{A} = \begin{pmatrix} a_{-1,-1} & a_{-1,0} & a_{-1,1} \\
0 & a_{0,0} & a_{0,1} \\
0 & a_{1,0} & a_{1,1} \end{pmatrix} \\
\hat{B} = \begin{pmatrix} b_{-1,-1} & b_{-1,0} & b_{-1,1} \\
b_{0,-1} & b_{0,0} & b_{0,1} \\
b_{1,-1} & b_{1,0} & b_{1,1} \end{pmatrix} \\
\hat{D} = \begin{pmatrix} d_{-1,-1} & d_{-1,0} & d_{-1,1} \\
d_{0,-1} & d_{0,0} & d_{0,1} \\
d_{1,-1} & d_{1,0} & d_{1,1} \end{pmatrix}
\]

The matrix \( A \), defined by (25), composed of the template \( \hat{A} \), has the circulant form (27). \( B \) and \( D \) in (25) can be similarly defined.

For \( \ell \in \{1, 2, \ldots, p\} \), \( i \in \{1, 2, \ldots, N\} \), let

\[
\Xi^{(\ell)} = \begin{pmatrix} 0, u_1^{(\ell)}, u_2^{(\ell)}, \ldots, u_{M-1}^{(\ell)}, u_M^{(\ell)} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0, u_1^{(\ell)}, u_2^{(\ell)}, \ldots, u_{M-1}^{(\ell)}, u_M^{(\ell)} \end{pmatrix}_{M \times 3}
\]

\[
\Xi^{(i)} = \begin{pmatrix} \Xi^{(1)} \\
\Xi^{(2)} \\
\vdots \\
\Xi^{(N-1)} \\
\Xi^{(N)} \end{pmatrix}_{(NM) \times 9}
\]
Let \( L_D = (d_{-1,-1}, d_{-1,0}, d_{-1,1}, d_{0,-1}, d_{0,0}, d_{0,1}, d_{1,-1}, d_{1,0}, d_{1,1})^T \).  

**Corollary 3:** If
\[
\sum_{i=-1}^{1} \sum_{j=-1}^{1} (|a_{i,j}| + |b_{i,j}|) < \min_{1 \leq i \leq n} \{c_i\} \leq \max_{1 \leq i \leq n} \{c_i\} < 1 \tag{31}
\]
and there exist constants \( \lambda_i \geq 2(c_i + 1) \) \( (i = 1, 2, \ldots, n) \) such that
\[
\Xi L_D = \Lambda S_p
\]
has a solution, then there exists a region \( \bigcap_{\ell=1}^{p} \mathcal{L}(\ell) \) such that, for any bias vector \( v \in \bigcap_{\ell=1}^{p} \mathcal{L}(\ell) \), the output vector of the heteroassociative-memory-based DTCNN (24) with the connection weight matrices \( A, B, \) and \( D \) is globally asymptotically stable at \( s^{(\ell)} \) with the retrieval probe \( u^{(\ell)} \) as its input vector for \( \ell = 1, 2, \ldots, p \).

**Proof:** Since (32) has a solution, according to the definition of \( D \), similar to \( A \) in (27), (11) holds.

In addition, since (31) holds, (7) holds. According to Theorem 3, the output vector of the heteroassociative-memory-based DTCNN (24) with the connection weight matrices \( A, B, \) and \( D \) is globally asymptotically stable at \( s^{(\ell)} \) with the retrieval probe \( u^{(\ell)} \) as its input vector for \( \ell = 1, 2, \ldots, p \).

**Corollary 4:** If (31) holds and there exist constants \( \lambda_i > 1 \) \( (i = 1, 2, \ldots, n) \) such that
\[
\Xi L_D = \Lambda S_p
\]
has a solution, then there exists a region \( \bigcap_{\ell=1}^{p} \mathcal{L}(\ell) \) such that, for any bias vector \( v \in \bigcap_{\ell=1}^{p} \mathcal{L}(\ell) \), the output vector of the heteroassociative-memory-based DTCNN (24) with the connection weight matrices \( A, B, \) and \( D \) is globally asymptotically stable at \( s^{(\ell)} \) with the retrieval probe \( u^{(\ell)} \) as its input vector for \( \ell = 1, 2, \ldots, p \).

**Proof:** Since (33) has a solution, according to the definition of \( D \), similar to \( A \) in (27), (19) holds.

In addition, since (31) holds, (7) holds. According to Theorem 4, the output vector of the heteroassociative-memory-based DTCNN (24) with the connection weight matrices \( A, B, \) and \( D \) is globally asymptotically stable at \( s^{(\ell)} \) with the retrieval probe \( u^{(\ell)} \) as its input vector for \( \ell = 1, 2, \ldots, p \).  

\section{V. Design Procedure With Space-Invariant DTCNNs}

\subsection{A. Design Procedure}

Step 1) Denote patterns and probes \((s^{(1)}, u^{(1)}), (s^{(2)}, u^{(2)}), \ldots, (s^{(p)}, u^{(p)})\), where \( \forall \ell \in \{1, 2, \ldots, p\} \), \( s^{(\ell)} \in \{-1, 1\}^n \), \( u^{(\ell)} \in \{-1, 1\}^n \), \( p \) is the number of patterns, \( n \) is the dimension of the patterns and number of neurons of the designed DTCNN, and \( m \) is the dimension of probes.

Step 2) Determine cloning templates \( \hat{A} \) and \( \hat{B} \) such that (28) or (31) holds. Compute the connection weights \( A \) and \( B \).

Step 3) Determine cloning template \( \hat{D} \) such that (15) holds. Or, choose input matrix \( D \) such that (11) or (19) holds.

Step 4) Compute \( v_1^{(\ell)} \) and \( v_2^{(\ell)} \) \((i \in \{1, 2, \ldots, n\}, \ell \in \{1, 2, \ldots, p\}\) according to (4) and (5).

Step 5) For \( i \in \{1, 2, \ldots, n\} \), let
\[
v_i^{+} = \begin{cases} +\infty, & \mathcal{J}_i^{+} = \Theta \vspace{.1in} \\text{or other cases} \\
\min_{\mathcal{J}_i^{+}} \{v_i^{+}(\ell)\}, & \mathcal{J}_i^{+} = \Theta \\vspace{.1in} \\text{or other cases}
\end{cases}
\]
\[
v_i^{-} = \begin{cases} -\infty, & \mathcal{J}_i^{-} = \Theta \\
\max_{\mathcal{J}_i^{-}} \{v_i^{-}(\ell)\}, & \mathcal{J}_i^{-} = \Theta
\end{cases}
\]
where \( \Theta \) is an empty set.

Step 6) Compute the mean value of the bias vector \( v = (v_1, v_2, \ldots, v_n)^T \in \prod_{\ell=1}^{p} (v_i^{+}, v_i^{-}) \), for \( i \in \{1, 2, \ldots, n\} \), i.e.,
\[
v_i = \begin{cases} \frac{v_i^{+} + v_i^{-}}{2}, & v_i^{+} \neq -\infty \text{ and } v_i^{-} \neq +\infty \\
2 \max_{\ell} \{v_i^{+}(\ell)\} - 2 \max_{\ell} \{v_i^{-}(\ell)\}, & v_i^{+} \neq -\infty \text{ and } v_i^{-} = +\infty \\
0, & v_i^{-} = -\infty \text{ and } v_i^{+} \neq +\infty \\
-2 \min_{\ell} \{v_i^{-}(\ell)\}, & v_i^{+} = +\infty \text{ and } v_i^{-} \neq -\infty
\end{cases}
\]

Step 7) Synthesize the DTCNN with the connection weight matrices \( A, B, C, D, \) and bias vector \( v \).

An equilibrium point \( \bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)^T \) of (2) satisfies
\[
\dot{x}_i = \left( \sum_{j=1}^{n} (a_{ij} + b_{ij}) f \left( \frac{\bar{x}_j}{c_i} \right) + \sum_{j=1}^{m} d_{ij} \delta_j u_j^{(\ell)} + v_i \right) / c_i.
\]

In the presence of distortion in input probes, i.e., an input probe \( u^{(\ell)} \) becomes \( \delta_i u^{(\ell)} \), where \( \delta_i \) is a distortion factor, \( 0 \leq \delta_i \leq 1 \) \((i = 1, 2, \ldots, n)\).

Let
\[
\dot{\bar{x}}_i = \left( \sum_{j=1}^{n} (a_{ij} + b_{ij}) f \left( \frac{\bar{x}_j}{c_i} \right) + \sum_{j=1}^{m} d_{ij} \delta_j u_j^{(\ell)} + v_i \right) / c_i.
\]

If
\[
f \left( \bar{x}(\ell) \right) = f \left( \bar{x}(\ell) \right) = u^{(\ell)}
\]
then, from (2), the corresponding pattern \( s^{(\ell)} \) can be retrieved with a distorted probe \( \delta_i u^{(\ell)} \).

In (3), if \( a_{ij} = b_{ij} = d_{ij} = v_i = 0 \), then \( x_i(t+1) = (1 - c_i(t+1) x_i(1) \). Hence, roughly speaking, the closer the value of \( c_i \) to one, the smaller are the values of \( |a_{ij}| \) and \( |b_{ij}| \), the faster is the convergence.

If \( f(\bar{x}(\ell)) = f(\bar{x}(\ell)) = u^{(\ell)} \), then \( \dot{\bar{x}}(\ell) \geq 1 \) or \( \dot{\bar{x}}(\ell) \leq -1 \). From (36), we can see that the smaller the value of \( c_i \) and the bigger the value of \( |a_{ij}|, |b_{ij}|, |d_{ij}| \), the higher is the degree of fault tolerance with respect to distorted input probes. Therefore, the degree of fault tolerance can be improved by increasing \( |d_{ij}| \).

In theory, any \( v = (v_1, v_2, \ldots, v_n)^T \in \prod_{i=1}^{n} (v_i^{-}, v_i^{+}) \) can be the bias vector. For example, the bias vector can be determined by the mean value of \( (v_i^{-}, v_i^{+}) \).  

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From (4) and (5), if $\forall i \in \{1, 2, \ldots, n\}$

$$d_{ii} > c_i + \left| \sum_{j=1}^{n} (a_{ij} + b_{ij}) s_j^{(l)} + \sum_{j=1, j \neq i}^{m} d_{ij} v_j^{(l)} \right|$$

(37)

then $(0, 0, \ldots, 0)^T \in \prod_{l=1}^{n} (v_i^-(\ell), v_i^+(\ell))$. Hence, when (37) holds, the bias vector can be the zero vector.

**VI. ILLUSTRATIVE EXAMPLES**

**Example 1:** Consider the same example introduced in [15]. The input probes and the output patterns are represented by four pairs of $(5 \times 5)$-pixel images (black pixel = −1, white pixel = 1). Specifically, the input probes compose the word “CMOS,” while the patterns to be memorized to constitute the word “VLSI.”

Let a DTCNN with $5 \times 5$ neurons ($n = 25$) behave as a heteroassociative memory. According to the design procedure in Section V, we can determine the cloning templates $\hat{A}$ and $\hat{B}$ such that

$$\hat{A} + \hat{B} = \begin{pmatrix} -0.01 & -0.01 & -0.01 \\ -0.01 & 0.9 & -0.01 \\ -0.01 & -0.01 & -0.01 \end{pmatrix}.$$ 

Let $c_i = 0.96$. Then, (31) holds.

Let $\lambda_i = 4, D = 4S_p (\hat{U}_p^T \hat{U}_p)^{-1} \hat{U}_p^T$. Then, (11) holds.

According to (4) and (5), for $i \in \{1, 2, \ldots, 25\}, \ell \in \{1, 2, 3, 4\}$, compute $v_i^-(\ell)$ and $v_i^+(\ell)$.

Let $v^- = (v_1^-, v_2^-, \ldots, v_n^-)^T, v^+ = (v_1^+, v_2^+, \ldots, v_n^+)^T$. According to (34) and (35), $v^- = (-3.91, -3.93, -3.89, -3.89, 3.91, -3.89, -3.92, -3.86, -3.86, -3.89, -3.89, -3.92, -3.90, -3.86, -3.89, -3.89, -3.92, -3.90, -3.92)$, $v^+ = (3.95, 3.95, 3.95, 3.95, 3.95, 3.95, 3.95, 3.95, 3.95, 3.95, 3.95, 3.95, 3.95, 3.95, 3.95, 3.95, 3.95, 3.95)$.

Simulation results show that the designed DTCNN with bias vector $v = (0, 0, \ldots, 0)^T$ and the input probes (CMOS) are able to recall the output patterns (VLSI) effectively and efficiently, starting from many random initial conditions (one case is shown in Fig. 1).

**Example 2:** Let us design a DTRNN to behave as an autoassociative memory. The input probes and the to-be-stored patterns are represented by $18 \times 3$-pixel images (black pixel = −1; white pixel = 1), as shown in Fig. 2.

Let $c_i = 0.9, a_{ij} + b_{ij} = 0.05, d_{ii} = 3.8, d_{ij} = 0$ ($i \neq j$), $v_i = 0, i, j \in \{1, 2, \ldots, 15\}$. Synthesize the DTRNN with $A$, $B$, $D$, and bias vector $v$. Then, according to Theorem 6, for any
bounded time-delays \( \tau_{ij}(t) \), the designed DTRNN can associate the 18 paired probes and patterns.

Simulation results with six random initial values and distorted input probes being \( 0.3u(1) \) (\( \delta_i = 0.3, i = 1, 2, \ldots, 15 \) and \( 0.3u(8) \) are shown in Figs. 3 and 4, respectively. It can be seen that, when the probe is \( 0.3u(1) \) or \( 0.3u(8) \), the network output is able to converge to the corresponding pattern correctly. But when \( \delta_i \) is too small or negative, the network output is unable to converge to the stored pattern \( u(1) \) or \( u(8) \).

VII. EXPERIMENTAL RESULTS

Fault tolerance is a very important issue in associative memories. To demonstrate the fault-tolerance capability of the designed associative memories, this section reports the statistics of successful retrievals of designed associative memories of various sizes and capacities with distorted input probes.

In each experiment, \( p (= 2n) \) bipolar patterns of various dimensions were generated randomly under binomial distribution. For each element of the retrieval probe \( u_i \), a random distribution factor \( \delta_i \) was generated following the uniform distribution on \([0.001, 1]\) \((i = 1, 2, \ldots, n)\). In the experiments, the randomly distorted probes \( \delta_0 u_i \) \((i = 1, 2, \ldots, n)\) were used. In DTCNN (24), let the cloning templates \( \hat{A} = [a_{-1}, d_0, a_1] = [0.1, 0.1, 0.1], \hat{B} = [b_{-1}, b_0, b_1] = [0.1, 0.1, 0.1], \hat{D} = [d_{-1}, d_0, d_1] = [0.7n + 15, 0], \) and \( c_i = 0.9, v_i = 0(i \in \{1, 2, \ldots, n\}) \).

By using the Monte Carlo method, the percentage of successful retrievals (successful rates in percent) of the designed associative memories based on DTCNNs from 100 random initial conditions within 100 iterations were recorded in Table I for recalling the random patterns in the presence of random probe distortion.

By using the design procedure presented in this paper, the network is unable to retrieve the correct pattern in the case where some bits in the pattern are changed in sign, because in the associative memories, the store patterns are retrieved via external input instead of initial state. When a retrieval probe is set as an initial condition, it is necessary that the activation states converge to one of locally asymptotically stable equilibrium points. In this case, it is important to memorize some given patterns and avoid that other unwanted patterns are memorized.

VIII. CONCLUDING REMARKS

In this paper, a synthesis procedure for designing associative memories based on DTRNNs is presented. The global convergence of the DTRNNs designed by the procedure herein is guaranteed via the stability analysis. The storage capacity of the designed associative memories is as high as \( 2^n \) patterns. The fault-tolerance capability of the designed associative memories is demonstrated by the statistics of successful retrieval rates of designed associative memories of various sizes and capacities with distorted input probes. In addition, by using the new design procedure, the input matrices in designed DTCNNs can be obtained with space-invariant cloning templates which involves a few design parameters only. Both theoretical and experimental results show the effectiveness and efficiency of the proposed approach.

The design procedure presented in this paper is limited to associative memories with bipolar patterns (or binary patterns) only. The present method may be extended to design associative memories for multivalued patterns, and it would be an interesting and important research topic for further investigation.

REFERENCES


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