Outage Probability for Maximal Ratio Combining of Arbitrarily Correlated Faded Signals Corrupted by Multiple Rayleigh Interferers

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Abstract—Due to the functional complexity of its signal-to-interference ratio (SIR) ratio, the outage probability of maximum ratio combining is available only for certain independent and identically distributed (i.i.d.) fading environments. In this paper, we partially relax this restriction by allowing the signal channel-gain vector to follow a general fading distribution with arbitrary spatial correlation. The problem is tackled in a novel framework. The key step is to represent the probability density of the reciprocal of SIR, conditioned on signal vector, as the higher-order derivative of a simple exponential function in signal power whereby a generic formula for outage probability can be determined. The application of the generic formula to Rayleigh, Rician, and Nakagami faded signals is elaborated. Numerical results are also presented for illustration.

I. INTRODUCTION

Diversity reception with adaptive array processing is an effective technique to combat multipath fading and co-channel interference, which are two major sources of performance impairment in many cellular mobile radio systems. The optimal combining (OC) maximizes the signal-to-interference-noise ratio (SINR) at the combiner’s output thereby achieving the best performance in the presence of interference, at the cost of the need to estimate the channel-gain vectors for both signal and interferers. Maximal ratio combining (MRC), on the other hand, is suboptimal but requiring the channel-gain vector only for the desired user. Therefore, it is often a more feasible choice in practice.

Outage probability is an important performance measure for a wireless system to operate in a fading environment with co-channel interference. Outage performance of MRC in a multi-user environment has been analyzed in the literature but mainly focusing on some special cases. Though simpler in its structure, the outage analysis of MRC is much more difficult than its OC counterpart. The difficulty arises from the complicated expression of its instantaneous signal to interference ratio (SIR), which takes the ratio of two correlated order-4 statistics as opposed to a relatively simple ratio of two quadratic forms in OC. Only limited cases can be handled. One such case is a MRC system in which both desired and interfering signals are subject to spatially independent and identically distributed (i.i.d.) Nakagami fading [2], which is called the i.i.d Nakagami/i.i.d.Nakagami model in the sequel for ease of illustration. Another tractable case is the i.i.d. Rician/Rayleigh model for which the probability density function (pdf) of SIR is given in [3] whereby the outage probability is determined [4].

II. FORMULATION

Consider the uplink of a micro-cellular mobile radio system in which a base station employs an array of L antennas for the reception of a desired signal corrupted by M equal-powered Rayleigh faded co-channel interferers. We assume, throughout the paper, that the system performance is interference limited so that thermal noise is negligible and the received signal at the antenna array can be written as

\[ x = \sqrt{P_s} s_0 + \sqrt{P_t} \sum_{k=1}^{M} c_k b_k. \]  

(1)
Here $b_0$ and $b_k$ represent the desired and $k$th interfering complex transmit symbols using $q$-ary PSK so that $|b_i|^2 = 1, i = 0, 1, \cdots, q-1$. $L$-by-$1$ vectors $s$ and $c_k$ denote the complex channel-gain vectors for desired user and for the $k$th interferer, respectively. They have been normalized so that their entries, on average, have unit power and thus, $P_s$ and $P_I$ represent the average received signal and interference power, respectively. The Rayleigh fading assumption on interferers implies that $c_k$ follows the joint $L$-dimensional zero-mean complex Gaussian distribution with covariance matrix $R_c$. Symbolically, we can write $c_k \sim CN_L(0, R_c)$. In this paper, we require all entries of $c_k$ to be i.i.d. so that $R_c = I$ with $I$ signifying the identity matrix. However, $s$ is spatially correlated and can follow an arbitrary distribution.

The MRC correlates the received vector with the weights defined by the desired signal gain vector producing the output

$$y = \sqrt{P_s}s^\dagger b_0 + \sqrt{P_I} \sum_{k=1}^{M} s^\dagger c_k b_k,$$

where superscripts $T$ and $\dagger$ denote transposition and conjugate transposition, respectively. It follows that the SIR is given by

$$\gamma = \frac{P_s}{P_I} \frac{(s^\dagger s)^2}{s^\dagger As},$$

with matrix $A = \sum_{k=1}^{M} c_k c_k^\dagger$.

An outage event occurs whenever the SIR drops below a preset power protection ratio, say $\Lambda_0$. Accordingly, the outage probability is given by $P_{out} = \Pr\{\gamma < \Lambda_0\}$. To determine the outage probability, we need to find the pdf of $\gamma$. However, directly determining the pdf of $\gamma$ is very difficult and hence, we consider, instead, the reciprocal of the SIR $\gamma$, which is defined by

$$\xi = \frac{s^\dagger As}{(s^\dagger s)^2}.$$

Hence, by denoting $\Lambda = (P_s/P_I)\Lambda_0^{-1}$, we can re-formulate the outage probability as

$$P_{out} = \Pr\{\xi > \Lambda\}.$$

The pdf of $\xi$ is relatively easy to handle. First, following the independence assumption on the interfering vectors, we can assert that $A$ is complex Wishart distributed when $M \geq L$ [9], [10] and is complex pseudo-Wishart distributed when $M < L$ [11]. In either case, we will use the notation $A \sim CW_L(M, R_c)$ to denote the distribution of $A$. We next invoke the following lemma to obtain the pdf of $\xi$.

**lemma 1:** Suppose $A \sim CW_n(m, \Sigma)$. Then for a given $n \times k$ matrix $M$ of rank $k$ with $k \leq n$ and $k \leq m$, the product $M^\dagger A M$ also follows the Wishart distribution as shown by $M^\dagger A M \sim CW_k(m, M^\dagger \Sigma M)$.

The proof of this lemma is given in Appendix I. Application of the above lemma to $\xi$ results in

$$\xi \sim CW_1 \left( M, \frac{s^\dagger R_c s}{(s^\dagger s)^2} \right).$$

The Wishart distribution $CW_1(\cdot, \cdot)$ reduces to the gamma distribution. Explicitly, we can write

$$f_{\xi | \mathbf{s}}(z | \mathbf{s}) = \frac{\alpha^M}{\Gamma(M)} z^{M-1} e^{-\alpha z}.$$  \hspace{1cm} (7)

It involves both exponential and polynomial factors in $z$ making it difficult to proceed further. We therefore rewrite it in the form of differentiation yielding

$$f_{\xi | \mathbf{s}}(z | \mathbf{s}) = \frac{(-1)^M z^{M-1} d^M}{\Gamma(M)} d z e^{-\alpha z}$$

(8)

where $\Gamma(\cdot)$ denotes the gamma function and $\alpha$ is define by

$$\alpha = (s^\dagger s)^2 / (s^\dagger R_c s).$$

Finally, we average (8) over random signal vector $s$. Note that (8) depends on $s$ only through the exponential function $\exp(-\alpha z)$, and the expected value of $\exp(-\alpha z)$ simply equals the characteristic function (CHF) of $\alpha$ with transform variable $-z$. It follows that

$$f_{\xi}(z) = E_{s}[f_{\xi | s}(z)] = \frac{(-1)^M z^{M-1} d^M}{\Gamma(M)} d z \phi_\alpha(-z).$$

(10)

where $\phi_\alpha(s) = E[\exp(\alpha s)]$. By inserting (10) into (5) and simplifying, it also follows that

$$P_{out} = \Pr\{\gamma < \Lambda_0\} = \Pr\{\xi > \Lambda\} = \int_\Lambda^\infty f_{\xi}(z) \, dz = \sum_{k=0}^{M-1} \left[ (-1)^k \frac{1}{\alpha^k} \frac{d^k}{dz^k} \phi_\alpha(-z) \right]_\Lambda^\infty.$$  \hspace{1cm} (11)

Clearly, the outage probability in (11) depends on the faded signal vector only through the CHF of $\alpha$. It is therefore applicable to various faded signals and allows for an arbitrary branch correlation as long as their characteristic functions exist. The CHF of $\alpha$ is difficult to determine in general. However, with the i.i.d assumption on $c_k$, we have $R_c = I$ which enables us to simplify $\alpha$ to $\alpha = s^\dagger s$.

### III. THE RESULTS FOR VARIOUS FADED SIGNALS

In this section, we show how to use the generic formula (11) to various operational environments. In particular, we consider signals which suffer from correlated Rayleigh, Rician, or Nakagami fading. The outage probability for these fading environments are treated in order. The skill used for derivations is similar to that used in [8].

**A. Rayleigh Faded Signals**

For a general rayleigh faded signal, we can assume that $s \sim CN_L(0, R_s)$. The CHF of $\alpha$ is thus given by

$$\phi_\alpha(z) = \det(I - z R_s)^{-1}$$

(12)

which, when represented in terms of the eigenvalues $\lambda_i, i = 1, \cdots, L$ of $R_s$, gives

$$\phi_\alpha(z) = \prod_{i=1}^{L} (1 - \lambda_i z)^{-1}.$$  \hspace{1cm} (13)
Insert (13) into (10) and (refouage) and simplify yielding pdf of ξ

\[
f_\xi(z) = Mz^{M-1} \sum_{\tau(L,M)} \prod_{i=1}^{L} \frac{\lambda_j^{t_i}}{(1 + \lambda_j z)^{t_i+1}}
\]  
(14)

and outage probability

\[
P_{out} = \sum_{k=0}^{M-1} A_k \prod_{\tau(L,k)} \prod_{i=1}^{L} \frac{\lambda_j^{t_i}}{(1 + \lambda_j)^{t_i+1}}.
\]

(15)

Here \(\tau(L,k)\) is an L-tuple such that \(\tau(L,k) = \{(t_1, \ldots, t_L) : \sum_{i=1}^{L} t_i = k\} \), \(t_i \in \mathbb{N}_0\) with \(\mathbb{N}_0\) signifying the set of nonnegative integers.

For the special case of \(R_s = I\), we have \(\lambda_i = 1\), for \(i = 1, \ldots, L\). The pdf and outage probability are reduced to

\[
f_\xi(z) = M \left( \frac{L + M - 1}{M} \right) \frac{z^{M-1}}{(1 + z)^{M+L}}
\]

(16)

and

\[
P_{out} = \sum_{k=0}^{M-1} \left( \frac{L + k - 1}{k} \right) \frac{A_k}{(1 + \Lambda)^{L+k}}.
\]

(17)

These results are equivalent to those obtained in [4] but the restriction that the number of interferers must be larger than the number of antennas is removed.

B. Rician Faded Signals

For a general Rician faded signal, we assume that \(s \sim CN_z(L, \mu_s, R_s)\), for which the CHF of \(\alpha\) can be expressed explicitly as

\[
\phi_\alpha(z) = \frac{1}{\det(I - zR_s)} \exp(-\mu_s R_s^{-1} \mu_s)
\]

\[
\times \exp \left( \mu_s r_s^{\top} (I - zR_s)^{-1} r_s - \frac{1}{2} \mu_s r_s^{\top} \mu_s \right).
\]

(18)

Eigen-decompose \(R_s\) such that \(R_s = U \Sigma U^{\top}\) where \(U\) is the matrix of eigenvectors of \(R_s\) and \(\Sigma = \text{diag}[\lambda_1, \ldots, \lambda_L]\) is the diagonal matrix of its eigenvalues. By denoting \(v = U^{\top} r_s \mu_s\), the \(j\)th entry of \(v\) by \(v(j)\) and \(h_j = |v(j)|^2\), we can simplify (18) to obtain

\[
\phi_\alpha(z) = \frac{1}{\det(I - z\Sigma)} e^{-a} \exp(v^{\top}(I - z\Sigma)^{-1}v)
\]

\[
= e^{-a} \prod_{j=1}^{L} \left( 1 - z\lambda_j \right)^{-1} \exp \left( \frac{h_j}{1 - z\lambda_j} \right)
\]

(19)

where \(a = \mu_s r_s^{\top} \mu_s\). Inserting (19) into (10) and (11) and simplifying, we obtain

\[
f_\xi(z) = Mz^{M-1} e^{-a} \prod_{j=1}^{L} \left( 1 + z\lambda_j \right)^{-1} \exp \left( \frac{h_j}{1 + z\lambda_j} \right)
\]

\[
\times \sum_{l=1}^{M} \sum_{\pi(M,l)} \prod_{i=1}^{L} \frac{1}{t_i} \left( \sum_{j=1}^{L} \lambda_j^{t_i} \right)^{(L-1)}
\]

\[
\times \left( \frac{1}{i} + \frac{h_j}{1 + z\lambda_j} \right)^t_i
\]

\]

(20)

and

\[
P_{out} = e^{-a} \prod_{j=1}^{L} \left( 1 + \lambda_j \right)^{-1} \exp \left( \frac{h_j}{1 + \lambda_j} \right)
\]

\[
\times \left( 1 + \sum_{k=1}^{M-1} \Lambda_k \sum_{l=1}^{k} \prod_{i=1}^{L} \frac{1}{t_i} \right)
\]

\[
\times \left( \sum_{j=1}^{L} \lambda_j^{t_j} \left( 1 + \lambda_j \right)^{-1} \left( \frac{1}{i} + \frac{h_j}{1 + \lambda_j} \right)^{t_j} \right)
\]

(21)

where \(\pi(n,k)\) is an n-tuple defined by \(\pi(n,k) = \{(t_1, \ldots, t_n) : t_i \in \mathbb{N}_0; \sum_{i=1}^{n} t_i = k, \sum_{i=1}^{n} it_i = n\}\). In the derivation, we have used Faa di Bruno’s formula [13] for differentiation of a composite function.

C. Nakagami Faded Signals

For Nakagami faded signals, we assume that the desired signal is subject to Nakagami fading with power covariance matrix \(R_r\) and fading parameter \(m \geq 1/2\). \(R_r\) is the covariance of the power vector \(r\) where \(r = [s(1)]^2, \ldots, [s(L)]^2\) and \(s(k)\) is the \(k\)th entry of \(s\). The CHF of \(\alpha\) is then given by

\[
\phi_\alpha(z) = \det(I - z\Gamma_s)^{-m}
\]

(22)

where the \((i, j)\) entry of \(\Gamma_s\) is related to the \((i, j)\)th entry of \(R_r\) by \(\Gamma_s(i, j) = \sqrt{R_r(i, j)/m}\). Let \(\{\lambda_i, i = 1, \ldots, L\}\) denote the eigenvalues of \(\Gamma_s\). With these notations, the CHF of \(\alpha\) can be rewritten as

\[
\phi_\alpha(z) = \prod_{i=1}^{L} (1 - z\lambda_i)^{-m}
\]

(23)

Inserting it into (10) and (11) and simplifying result in

\[
f_\xi(z) = Mz^{M-1} \times \prod_{\tau(L,M)} \Gamma(m + t_i) \frac{\lambda_i^{t_i}}{(1 + \lambda_i)^{m+t_i}}
\]

(24)

and

\[
P_{out} = \sum_{k=0}^{M-1} A_k \prod_{\tau(L,k)} \Gamma(m + t_i) \frac{\lambda_i^{t_i}}{(1 + \lambda_i)^{m+t_i}}
\]

(25)

respectively. This is the outage probability formula of MRC for the correlated Nakagami/uncorrelated Rayleigh model.

IV. NUMERICAL RESULTS

As illustration, we present numerical results on the outage performance of MRC for different faded signals. We assume that the signal correlation is dictated by the exponential model. To be specific, the signal correlation between antennas \(p\) and \(q\) is given by \(C_{pq}(\rho) = \rho^{p-q} \exp(j[p - q] \times \pi/12)\), where \(\rho = \exp(-d)\) denotes the absolute value of correlation coefficient between two adjacent antennas, and \(d\) stands for the antenna separation normalized by the wavelength. The L-by-L covariance matrix is simply denoted by \(C(\rho) = [C_{pq}(\rho)]\). The signal is corrupted by \(M = 6\) co-channel i.i.d. Rayleigh faded
interferers, and the power protection ratio is set to be 12 dB, which is roughly the same as that used in the GSM system. Shown in Fig.1, Fig.2, and Fig.3 are the results for correlated Rayleigh, Rician and Nakagami faded signals, respectively. The outage performance is plotted as a function of average SIR for different diversity orders of $L = 4, 6$ and 8 and different spatial correlations of $\rho = 0, 0.3$ and 0.6. For the Rayleigh faded signal, its covariance matrix is given by $R_s = C(\rho)$. For the case of Rician fading, we assume that the signal vector follows the $L$-dimensional Gaussian distribution with mean $\sqrt{0.5/K/(K+1)}e$ and covariance matrix $(K + 1)^{-1}C(\rho)$. Here $K = 5$ is the Rician factor to be used in Fig.2 and vector $e = (1 + j)1$ with 1 denoting the all-one column vector. For Nakagami fading, the covariance matrix is given for the signal power where $C_{pq}(\rho) = \rho^{p-q}$ with Nakagami fading parameter $m = 0.75$. For all graphs, as expected, the outage performance improves with increased diversity order $L$ for a given number of interferers. It is also observed that the outage performance improves with decreased signal correlation.

V. Conclusion

In this paper, we have studied the outage performance of space diversity reception with maximal ratio combining in presence of co-channel interference. The channel model considered in this paper is more general than those treated in the literature allowing for the signal channel-gain vector to follow an arbitrary fading distribution with an arbitrary covariance structure, although multiple interferers are still required to be i.i.d Rayleigh faded. A generic and simple expression for the outage probability have been obtained. Its application to various faded signals is elaborated ending up with various results for Rayleigh, Rician, and Nakagami faded signals. Future effort is to remove the i.i.d. Rayleigh fading assumption on co-channel interferers to thoroughly solve the outage problem of maximum ratio combining.

Appendix I

Proof of Lemma 1

When $A$ is complex Wishart distributed, we can directly extend the result for real data by Muirhead ([10], p.95) to complete the proof. However, here we give a general proof for both Wishart and pseudo-Wishart distribution. By definition, we have $A = \sum_{i=1}^{m} h_i h_i^*$ where $n$-dimensional $\{h_i, \ i = 1, \ldots, m\}$ are independent and each follows the zero-mean complex Gaussian distribution with covariance matrix $\Sigma$, i.e., $h_i \sim CNc(0, \Sigma)$. Sample matrix $A$ follows the complex Wishart distribution when $m \geq n$, and is complex pseudo-Wishart distributed otherwise. Let $g_i = M^j h_i, \ i = 1, \ldots, m$, then $g_i \sim CNc(M^j \Sigma M^j)$. Since $M^j A M^j = \sum_{i=1}^{m} g_i g_i^*$ and $k \leq m$, we can assert that $M^j A M^j$ follows the complex Wishart distribution.
REFERENCES