Generalized exponents of primitive symmetric digraphs

Richard A. Brualdi a,*,1, Jia-yu Shao b,2

a Department of Mathematics, University of Wisconsin, Madison, WI 53706, USA
b Department of Applied Mathematics, Tongji University, Shanghai, China

Received 10 January 1996; revised 9 May 1996

Abstract

A strongly connected digraph $D$ of order $n$ is primitive (aperiodic) provided the greatest common divisor of its directed cycle lengths equals 1. For such a digraph there is a minimum integer $t$, called the exponent of $D$, such that given any ordered pair of vertices $x$ and $y$ there is a directed walk from $x$ to $y$ of length $t$. The exponent of $D$ is the largest of $n$ 'generalized exponents' that may be associated with $D$. If $D$ is a symmetric digraph, then $D$ is primitive if and only if its underlying graph is connected and is not bipartite. In this paper we determine the largest value of these generalized exponents over the set of primitive symmetric digraphs whose shortest odd cycle length is a fixed number $r$. We also characterize the extremal digraphs. Our results are common generalizations of a number of related results in the literature.

1. Introduction

The concept of generalized exponents for primitive matrices (and primitive digraphs) was first introduced in 1990 [1]. It is a common generalization of the traditional concepts of primitive exponents for nonnegative primitive matrices and ergodic indices for the transition matrices of finite homogeneous Markov chains. Primitive exponents and ergodic indices have been extensively studied. Generalized exponents also arise in the study of memoryless communication networks which we will explain after giving the definition of generalized exponents.

In this paper we will adopt the graph-theoretic version to define the generalized exponents and will use graph-theoretic methods to prove our main results.

A strongly connected digraph $D$ is primitive provided the greatest common divisor of the lengths of its directed cycles equals 1. It is well known (see e.g. [2]) that $D$ is primitive if and only if there exists an integer $k$ such that for each ordered pair of vertices $x$ and $y$ there is a directed walk from $x$ to $y$ of length $k$. The smallest such integer $k$ is the exponent of $D$, denoted by $\exp(D)$.

* Corresponding author. E-mail: brualdi@math.wise.edu.
1 Research partially supported by National Science Foundation Grant No. DMS-9123318.
2 Research supported by the National Natural Science Foundation of China.

0166-218X/97/$17.00 © 1997 Elsevier Science B.V. All rights reserved
PII S0166-218X(96)00077-7
Let $D$ be a primitive digraph with vertex set $V = V(D)$. The exponent of a primitive digraph can be broken down into more local exponents as follows. If $x$ and $y$ are vertices of $D$, let $\gamma_D(x, y)$ be the smallest integer $p$ such that there exists a directed walk from $x$ to $y$ of each length $t \geq p$. For each vertex $x$ of $D$, let $\gamma_D(x)$ be the smallest integer $q$ such that for each vertex $y$ of $D$, there is a directed walk of length $q$ from $x$ to $y$ (and thus a directed walk of length $t$ from $x$ to $y$ for each integer $t \geq q$). It follows that

$$\gamma_D(x) = \max \{ \gamma_D(x, y) \mid y \in V \}. \quad (1)$$

Let the vertices of $D$ be ordered as $v_1, v_2, \ldots, v_n$ so that

$$\gamma_D(v_1) \leq \gamma_D(v_2) \leq \cdots \leq \gamma_D(v_n).$$

Then $\gamma_D(v_k)$ is called in [1] the $k$th generalized exponent of $D$ and is denoted by $\exp_D(k)$, $1 \leq k \leq n$. We have

$$\exp_D(1) \leq \exp_D(2) \leq \cdots \leq \exp_D(n).$$

Clearly, the exponent of $D$ equals $\exp_D(n)$. Thus, the generalized exponents of $D$ are generalizations of the classical exponent of $D$. Also if $D$ is the digraph corresponding to the transition matrix of a finite homogeneous Markov chain, then $\exp_D(1)$ is just the ergodic index of the chain. Thus, the classical primitive index and the ergodic index are special cases of $k = n$ and $k = 1$ of our generalized exponents $\exp_D(1)$.

The numbers $\exp_D(k)$ have an interpretation in an application model of a “memoryless communication network” associated with $D$: Suppose that at time $t = 0$, $k$ of the vertices of $D$ each hold one bit of information with no two of the information bits identical. At time $t = 1$ each vertex with some information passes its information to each of its neighbors (those reachable by a walk of length 1) and then forgets its information. The vertex may however receive information from another vertex. The system continues this way for each discrete values of $t$. Then $\exp_D(k)$ is the smallest time such that every vertex simultaneously holds all $k$ bits of the information.

We would like to mention here that $\exp_D(k)$ can also be defined by using matrices. In fact, if $A$ is the adjacency matrix of the primitive digraph $D$, then $\exp_D(k)$ is the smallest power of $A$ for which there are $k$ rows with no zero entries. From this point of view, all the results in this paper can be expressed in matrix language.

In [1, 3, 4] Brualdi and Liu derived upper bounds for the generalized exponents of primitive digraphs and for primitive digraphs of special type (symmetric digraphs, tournaments, and minimally strong digraphs). In this paper we consider only symmetric digraphs, that is, digraphs such that for any vertices $x$ and $y$, $(x, y)$ is an arc if and only if $(y, x)$ is an arc. Let $G$ be the underlying graph of a symmetric digraph $G$. Then $D$ is strongly connected if and only if $G$ is connected, and since a symmetric digraph with at least one arc contains a directed cycle of length 2, $D$ is primitive if and only if $G$ is connected and has at least one cycle of odd length. We identify the digraph $D$ with its underlying graph $G$ and thus deal with connected nonbipartite graphs. We call
a graph *primitive* provided it is the underlying graph of a primitive symmetric digraph, that is, provided it is a connected nonbipartite graph.

For each positive odd integer $r \leq n$, let $S_n(r)$ denote the set of all connected graphs with vertex set \{1, 2, \ldots, n\} having a cycle of length $r$ but no cycle of any odd length less than $r$. In this paper we determine an explicit expression for the maximum $k$th generalized exponent of a graph in $S_n(r)$. We also characterize the extremal graphs which attain the maximum. In Theorem 3.1 we determine the generalized exponents of a special graph $G_{n,r}$ pictured in Fig. 1. In Theorem 5.3 we show that for each integer $k$ with $1 \leq k \leq n$, the $k$th generalized exponent of the graph $G_{n,r}$ is at least as large as the $k$th generalized exponent of each graph in $S_n(r)$. If $n, k$ and $r$ satisfy certain conditions, then there may be other graphs in $S_n(r)$ with the same $k$th generalized exponent as $G_{n,r}$. These graphs are characterized by Theorems 4.1, 6.1, 7.3, 8.2 and 8.4.

Let $S^*_n(r)$ denote the set of all connected graphs with vertex set \{1, 2, \ldots, n\} having at least one cycle of odd length $r$ (but possibly cycles of odd length less than $r$). In Theorem 5.4 we show that the $k$th generalized exponent of $G_{n,r}$ is also the maximum $k$th generalized exponent of graphs in $S^*_n(r)$.

Many of our results are generalizations of results in [1, 4, 6, 5, 7, 8].

2. Preliminary results

In this section we discuss some lemmas which will be useful in obtaining our main results. The following two lemmas are contained in [1] and [6], respectively.

**Lemma 2.1.** Let $D$ be a primitive digraph with $n$ vertices. Then

\[
\exp_D(k+1) \leq \exp_D(k) + 1 \quad (1 \leq k \leq n - 1).
\]

**Lemma 2.2.** Let $x$ and $y$ be vertices of a primitive graph $G$ such that there exist walks of length $k_1$ and $k_2$, respectively, of different parity between $x$ and $y$. Then

\[
\gamma_G(x,y) \leq \max\{k_1,k_2\} - 1.
\]
We will make use of the following notation. Let $G$ be a graph. If $W$ is a walk in $G$, then $|W|$ denotes the length of $W$. If $P$ is a path (i.e., a walk without repeated vertices) in $G$ and $x$ and $y$ are vertices of $P$, then $xPy$ denotes the subpath of $G$ joining $x$ and $y$. If $C$ is a cycle of odd length and $x$ and $y$ are vertices of $C$, then $C$ contains two walks joining $x$ and $y$, and these walks are of different length since $|C|$ is odd. We denote these walks by $xC'y$ and $xC''y$ where $|xC'y| < |xC''y|$. Note that if $x = y$, then $|xC'y| = 0$ and $xC''y = C$. A cycle of length $k$ is called a $k$-cycle. We denote the distance between two vertices $x$ and $y$ of $G$ by $d(x, y)$. The concatenation of a walk $W_1$ from a vertex $x$ to a vertex $y$, and a walk $W_2$ from $y$ to a vertex $z$ is denoted by $W_1 + W_2$.

**Lemma 2.3.** Let $G$ be a primitive graph, and let $x$ and $y$ be vertices of a cycle $C$ of odd length $r$. Then

$$
\gamma_G(x, y) \leq r - 1. 
$$

(2)

If, in addition, $z$ is a vertex of $G$, then

$$
\gamma_G(x, z) \leq |xC''y| + d(y, z) - 1.
$$

(3)

**Proof.** First we apply Lemma 2.2 to the two walks $xC'y$ and $xC''y$ of different parity and conclude that

$$
\gamma_G(x, y) \leq \max\{|xC'y|, |xC''y|\} - 1 = |xC''y| - 1 \leq r - 1.
$$

Now let $z$ be any vertex of $G$ and let $P$ be a shortest path between $y$ and $z$. Then $xC'y + P$ and $xC''y + P$ are two walks between $x$ and $z$ whose lengths are of different parity. Thus, by Lemma 2.2 we have

$$
\gamma_G(x, z) \leq |xC''y + P| - 1 = |xC''y| + |P| - 1 = |xC''y| + d(y, z) - 1. 
$$

\[ \square \]

**Lemma 2.4.** Let $G$ be a primitive graph of order $n$ and let $C$ be a cycle of odd length. Then $\gamma_G(x) \leq n - 1$ for each vertex $x$ of $C$.

**Proof.** Let $z$ be any vertex of $G$ and let $y$ be the vertex of $C$ which is closest to $z$. Then $d(y, z) \leq n - |C|$ and by (3) of Lemma 2.3, we have

$$
\gamma_G(x, z) \leq |xC''y + d(y, z)| - 1 \leq |C| + (n - |C|) - 1 = n - 1.
$$

Since $z$ is an arbitrary vertex of $G$, the result now follows by (1). \[ \square \]

3. A special graph in $S_n(r)$

In this section we investigate the graph $G_{n, r}$ in the class $S_n(r)$ of all primitive symmetric graphs of order $n$ with shortest odd cycle length equal to $r$. This graph is
drawn in Fig. 1 and consists of a path $P$ of length $n - r$ and an $r$-cycle $C$ which meets $P$ exactly at one of $P$'s endvertices. We determine the generalized exponents of $G_{n,r}$.

**Theorem 3.1.** Let $r$ be an odd integer with $1 \leq r \leq n$. Then

$$\exp_{G_{n,r}}(k) = \begin{cases} 
(n - 1) + (k - r) & \text{if } r \leq k \leq n, \\
\max\{n - \frac{r + 1}{2} + \left\lfloor \frac{k + 1}{2} \right\rfloor - 1, r - 1\} & \text{if } 1 \leq k \leq r - 1.
\end{cases}$$

**Proof.** We first assume that $r \leq k \leq n$. Suppose that there is a directed walk $W$ of length $2n - r - 2$ from vertex 1 to itself. Since a closed walk of odd length contains a cycle of odd length, $W$ contains the cycle $C$ of $G_{n,r}$ and hence

$$|W| \geq 2|P| + |C| = 2n - r > 2n - r - 2,$$

a contradiction. Hence,

$$\exp_{G_{n,r}}(n) \geq \gamma_{G_{n,r}}(1,1) \geq 2n - r - 1,$$

and hence by Lemma 2.1,

$$\exp_{G_{n,r}}(k) \geq \exp_{G_{n,r}}(n) - (n - k) \geq n - 1 + (k - r).$$

By Lemma 2.4 we also have

$$\exp_{G_{n,r}}(r) \leq n - 1.$$

Since $k \geq r$, Lemma 2.1 now implies that

$$\exp_{G_{n,r}}(k) \leq \exp_{G_{n,r}}(r) + (k - r) \leq n - 1 + (k - r).$$

Combining (5) and (6), we obtain the first part of (4).

We now assume that $1 \leq k \leq r - 1$. By Lemma 2.4, $\gamma_{G_{n,r}}(x) \leq n - 1$ for each of the $r$ vertices $x$ of the cycle $C$ of $G_{n,r}$. Combining this fact with what we have proved above, we conclude that $\exp_{G_{n,r}}(r + 1) = n - 1 + ((r + 1) - r) = n$. Thus, we need only compute the vertex exponents $\gamma_{G_{n,r}}(x)$ of the vertices $x$ of $C$ and arrange them in nondecreasing order.

We now denote the unique common vertex of $P$ and $C$ by $u$, and we denote by $y$ an arbitrary vertex of $C$. Let

$$|yC'u| = d \quad \text{and} \quad |yC''u| = r - d \quad (d < r - d).$$

Our goal is to prove that

$$\gamma_{G_{n,r}}(y) = m(d) := \max\{n - d - 1, r - 1\}.$$  

(7)

By Lemma 2.3, $\gamma_{G_{n,r}}(y,x) \leq r - 1 \leq m(d)$ for each vertex $x$ of $C$, and

$$\gamma_{G_{n,r}}(y,x) \leq |yC''u| + d(u,x) - 1 \leq r - d + (n - r) - 1 = n - d - 1 \leq m(d)$$


for all vertices $x$ of $P$. Hence, we have

$$\gamma_{G_n,r}(y) = \max \{ \gamma_{G_n,r}(y,x) : x \in V(G) \} \leq m(d) \quad (y \in V(C)). \quad (8)$$

We next show that $\gamma_{G_n,r}(y) \geq m(d)$ for each vertex $y$ of $C$. First we observe that there is no walk of the odd length $r - 2$ from $y$ to itself so that

$$\gamma_{G_n,r}(y) \geq \gamma_{G_n,r}(y,y) \geq r - 1. \quad (9)$$

Suppose there is a walk $W$ of length $n - d - 2$ from $y$ to vertex 1. Then

$$|W| = n - d - 2 < n - d = (r - d) + (n - r) = |yC'u| + |P|$$

implies that $W$ consists of the path $yC'u + P$ of length $d + (n - r)$ and cycles of lengths 2 and $r$ (the only cycle lengths for $G_{n,r}$). Hence, there exist nonnegative integers $a$ and $b$ such that

$$n - d - 2 = d + (n - r) + 2a + rb,$$

implying that $b = 0$ and that $r$ is even, a contradiction. We conclude that

$$\gamma_{G_n,r}(y) \geq \gamma_{G_n,r}(y,1) \geq n - d - 1. \quad (10)$$

Combining (9) and (10), we have

$$\gamma_{G_n,r}(y) \geq \max \{ n - d - 1, r - 1 \} = m(d). \quad (11)$$

Now (7) follows from (8) and (11).

The function $m(d)$ is a nonincreasing function, and for each $d$ with $1 \leq d \leq (r - 1)/2$, there are exactly two vertices $v_d$ and $w_d$ of $C$ for which $|v_dC'u| = |w_dC'u| = d$ and hence $\gamma_{G_n,r}(v_d) = \gamma_{G_n,r}(w_d) = m(d)$. Therefore, the smallest $r - 1$ vertex exponents of $G_{n,r}$ in nondecreasing order are

$$m \left( \frac{r - 1}{2} \right), m \left( \frac{r - 1}{2} \right), \ldots, m(2), m(2), m(1), m(1).$$

In other words,

$$\exp_{G_{n,r}}(2t - 1) = \exp_{G_{n,r}}(2t) = m \left( \frac{r + 1}{2} - t \right) = \max \left\{ n - \frac{r + 1}{2} + t - 1, r - 1 \right\}$$

for each integer $t$ with $1 \leq t \leq \frac{1}{2}(r - 1)$, and this is equivalent to the second part of (4). □

4. The maximum $k$th generalized exponent for $r \leq k \leq n$ and the extremal graphs

Let $e(n,r,k)$ denote the maximum value of $\exp_G(k)$ for $G$ in $S_n(r)$. In this section, for each integer $k$ with $r \leq k \leq n$, we determine $e(n,r,k)$ and characterize those graphs $G$ in $S_n(r)$ for which $\exp_G(k) = e(n,k,r)$.
Theorem 4.1. Let \( r \) be an odd integer with \( 1 \leq r \leq n \), and let \( k \) be an integer with \( r \leq k \leq n \). Then
\[
e(n, r, k) = n - 1 + (k - r).
\] (12)

If \( G \) is a graph in \( S_n(r) \), then \( \exp_G(k) = e(n, r, k) \) if and only if
\begin{enumerate}[(i)]  
  \item \( k \geq 2 \) and \( G \) is isomorphic to \( G_{n, r} \), or  
  \item \( k = 1 \) (and so \( r = 1 \)) and either \( G \) is isomorphic to \( G_{n, 1} \) (a path with a loop at one of its endvertices) or \( G \) is isomorphic to the graph \( G'_{n, 1} \) obtained from \( G_{n, 1} \) by adding a loop to its other endvertex.
\end{enumerate}

Proof. Let \( G \) be a graph in \( S_n(r) \), and let \( C \) be a cycle of length \( r \) in \( G \). It follows from Lemma 2.4 that
\[
\exp_G(r) \leq \max\{\gamma_G(x) : x \text{ a vertex of } C\} \leq n - 1,
\]
and hence by Lemma 2.1 that
\[
\exp_G(k) \leq \exp_G(r) + (k - r) \leq n - 1 + (k - r) \quad (r \leq k \leq n).
\]
Eq. (12) now follows, since by Theorem 3.1 we have \( \exp_{G_{n, r}}(k) = n - 1 + (k - r) \) for \( r \leq k \leq n \). If \( k = 1 \) and \( G \) is the graph \( G_{n, 1} \) in (ii), then it is easy to see that \( \exp_G(1) = n - 1 = e(n, 1, 1) \).

Now let \( G \) be a graph in \( S_n(r) \) and assume that \( \exp_G(k) = n - 1 + (k - r) \) for some integer \( k \) with \( r \leq k \leq n \). First suppose that \( k \geq 2 \). By Lemma 2.1, we have
\[
\exp_G(r) \geq \exp_G(k) - (k - r) = n - 1.
\]
Let \( C \) be an \( r \)-cycle of \( G \). Then
\[
\max\{\exp_G(x) : x \text{ a vertex of } C\} \geq \exp_G(r) \geq n - 1,
\]
and hence by (1) there exists a vertex \( x \) of \( C \) and a vertex \( y \) of \( G \) such that \( \gamma_G(x, y) \geq n - 1 \). Let \( z \) be a vertex of \( C \) which is nearest to \( y \) and let \( P \) be a shortest path between \( y \) and \( z \). We have \( |P| \leq n - r \). Since \( xC'z + P \) and \(xC''z + P \) are walks of different parity between \( x \) and \( y \), it follows from Lemma 2.2 that
\[
n - 1 \leq \gamma_G(x, y) \leq |xC''z| + |P| - 1 \leq (n - r) + r - 1 = n - 1.
\]
Hence, \( |xC''z| = r \) (and thus \( z = x \)) and \( |P| = n - r \), and since \( P \) is a shortest path between \( y \) and \( z \), \( G \) contains a spanning subgraph \( G^* \) isomorphic to \( G_{n, r} \).

We now show that \( G^* \) equals \( G \). Since \( C \) is a shortest odd length cycle of \( G \), \( C \) is an induced subgraph of \( G \). Since \( P \) is a shortest path between \( x \) and \( z \), \( P \) is also an induced subgraph of \( G \). Suppose that there is an edge \( e \) of \( G \), but not of \( G^* \), which joins a vertex \( u \) of \( P \) and a vertex \( v \) of \( C \), where \( u \neq v \). Then clearly, \( v \neq z = x \). The paths \( xC'v + e + uPy \) and \( xC''v + e + uPy \) are paths joining \( x \) and \( y \) of different parity, and using Lemma 2.2 we obtain the contradiction \( \gamma_G(x, y) \leq (n - 1) - 1 = n - 2 \).

Finally, suppose that there is a loop of \( G \) which is not an edge of \( G^* \). Then \( r = 1 \) and
it follows that \(|C| = 1\) implying that there are at least two loops of \(G\). It is easy to see that if \(w\) is a vertex meeting a loop, then \(\gamma_G(w) \leq n - 1\). From this it now follows that \(\exp_G(2) \leq n - 1\). Since \(k \geq 2\), Lemma 2.1 implies that

\[
\exp_G(k) \leq \exp_G(2) + (k - 2) \leq n - 1 + (k - 2) < n - 1 + (k - 1),
\]

a contradiction because \(r = 1\). Therefore, \(G = G^*\) and \(G\) is isomorphic to \(G_{n,r}\).

We now suppose that \(k = 1\) and hence that \(r = 1\). Then \(\exp_G(1) = n - 1\). Arguing as above we obtain a spanning subgraph \(G^*\) of \(G\) consisting of a path \(P\) of length \(n - 1\) with a loop at one of its endvertices. If \(G\) contains an edge (loop or nonloop) which is not an edge of \(G^*\) and which is not a loop at the other endvertex of \(P\), then it is easy to verify that \(\exp_G(1) \leq n - 2\), a contradiction. Hence, \(G\) satisfies (ii) of the theorem. The theorem now follows. \(\square\)

5. The maximum \(k\)th generalized exponent for \(1 \leq k \leq r - 1\)

In this section we show that \(\exp_{G_{n,r}}(k)\) is the maximum \(k\)th generalized exponent in \(S_n(r)\) for each integer \(k\) with \(1 \leq k \leq r - 1\).

We first recall that a **unicyclic graph** is a connected graph with a unique cycle and that such a graph consists of a cycle \(C\) along with a tree rooted at each of its vertices. Some of these trees may be trivial; those trees which contain at least two vertices are called **branches**. Let \(a\) be a nonnegative integer. If \(C\) is a cycle in a graph \(G\) and \(x\) is a vertex of \(G\), then we denote by \(C(x; a)\) the set of all vertices \(z\) of \(C\) for which \(|xC'z| < a\) and by \(C'(x; a)\) the set of vertices \(z\) of \(C\) for which \(|xC'z| \geq a\).

Since \(G_{n,r}\) is a unicyclic graph in \(S_n(r)\) and since any graph in \(S_n(r)\) has a spanning unicyclic subgraph whose unique cycle has length \(r\), we first consider unicyclic graphs.

**Lemma 5.1.** Let \(G\) be a unicyclic graph in \(S_n(r)\) and let \(t\) be an integer with \(1 \leq 2t \leq r - 1\). Let \(C\) be the unique cycle of \(G\) and let \(B(x_1), \ldots, B(x_h)\) be the branches of \(G\) rooted, respectively, at \(x_1, \ldots, x_h\). Let \(b_i + 1\) be the number of vertices of \(B(x_i)\) and let

\[
a_i = b_i - \left( n - \frac{3r + 1}{2} + t \right) \quad (1 \leq i \leq h).
\]

Then

(i) for each vertex \(x\) in \(V(C) \setminus \bigcup_{i=1}^{h} C(x_i; a_i)\),

\[
\gamma_G(x) \leq \max \left\{ n - \frac{r + 1}{2} + t - 1, r - 1 \right\}, \quad \text{and} \quad (13)
\]

(ii) for each vertex \(z\) in \(V(C) \setminus \bigcup_{i=1}^{h} C(x_i; a_i + 1)\),

\[
\gamma_G(z) \leq \max \left\{ n - \frac{r + 1}{2} + t - 2, r - 1 \right\}. \quad (14)
\]
Proof. Let $x$ and $z$ be as prescribed in (i) and (ii) above, and let $y$ be a vertex of $G$. If $y$ is a vertex of $C$, then by Lemma 2.3

$$\gamma_G(x,y) \leq r - 1 \quad \text{and} \quad \gamma_G(z,y) \leq r - 1.$$ 

Now assume that $y$ is not a vertex of $C$ and let $y$ be a vertex of $B(x_j)$. Then $d(x_j,y) \leq b_x$. Since $x$ is not in $C(x_j; a_j)$, $|xC'x_j| \geq a_j$ and hence $|xC''x_j| \leq r - a_j$. Similarly, since $z$ is not in $C(x_j, a_j + 1)$, $|zC''x_j| \leq r - a_j - 1$. By Lemma 2.3 again, we have

$$\gamma_G(x,y) \leq |xC''x_j| + d(x_j,y) - 1 \leq r - a_j + b_j - 1 = n - \frac{r + 1}{2} + t - 1,$$

and similarly

$$\gamma_G(z,y) \leq r - a_j - 1 + b_j - 1 = n - \frac{r + 1}{2} + t - 2.$$

The lemma now follows. $\Box$

The following theorem along with Theorem 3.1 (see also the conclusion of its proof) shows that $G_{n,r}$ is also an extremal graph for each of the first $r - 1$ generalized exponents of graphs in $S_n(r)$.

Theorem 5.2. Let $r$ be an odd integer with $1 \leq r \leq n$, and let $t$ be an integer with $1 \leq 2t \leq r - 1$. Then

$$\exp_G(2t - 1) \leq \exp_G(2t) \leq \exp_{G_{n,r}}(2t) = \exp_{G_{n,r}}(2t - 1)$$

$$= \max \left\{ n - \frac{r + 1}{2} + t - 1, r - 1 \right\}$$  \hspace{1cm} (15)

for each graph $G$ in $S_n(r)$.

Proof. We have only to prove that $\exp_G(2t) \leq \max\{n - (r + 1)/2 + t - 1, r - 1\}$. If $r = n$, then $G$ is a cycle of length $n$ and is isomorphic to $G_{n,n}$, and hence the result holds by Theorem 3.1. Now assume that $r \leq n - 1$.

We first consider a unicyclic graph $G$ in $S_n(r)$, and we use the notation of Lemma 5.1. We then have $\sum_{i=1}^{h} b_i = n - r$ and

$$a_i \leq n - r - \left( n - \frac{3r + 1}{2} + t \right) = \frac{r + 1}{2} - t \leq \frac{r - 1}{2} \quad (1 \leq i \leq h).$$

We also have

$$|C(x_i; a_i)| = \begin{cases} 
0 & \text{if } a_i \leq 0, \\
2a_i - 1 & \text{if } a_i \geq 1.
\end{cases}$$  \hspace{1cm} (16)

We consider two cases.
Case 1: \( n - \frac{1}{2}(3r + 1) + t \geq 0 \). If \( a_i \leq 0 \) for each \( i \) with \( 1 \leq i \leq h \), then \( |\bigcup_{i=1}^h C(x_i; a_i)| = 0 \leq r - 2t \). Otherwise, we may choose the notation so that for some integer \( f \) with \( 1 \leq f \leq h \) we have \( a_1, \ldots, a_f \geq 1 \) and \( a_{f+1}, \ldots, a_h \leq 0 \). Then

\[
\left| \bigcup_{i=1}^h C(x_i; a_i) \right| = \left| \bigcup_{i=1}^f C(x_i; a_i) \right|
\leq \sum_{i=1}^f |C(x_i; a_i)|
= \sum_{i=1}^f (2a_i - 1)
= 2 \sum_{i=1}^f \left( b_i - \left( n - \frac{3r + 1}{2} + t \right) \right) - f
= 2 \left( \sum_{i=1}^f b_i - \left( n - \frac{3r + 1}{2} + t \right) \right) - 1 - (f - 1)
- 2(f - 1) \left( n - \frac{3r + 1}{2} + t \right)
\leq 2 \left( n - r - \left( n - \frac{3r + 1}{2} + t \right) \right) - 1
= r - 2t.
\]

In both instances we have

\[
\left| V(C) \setminus \bigcup_{i=1}^h C(x_i; a_i) \right| = r - \left| \bigcup_{i=1}^h C(x_i; a_i) \right| \geq 2t.
\]

Applying Lemma 5.1, we have

\[
\exp_G(2t) \leq \max \left\{ \gamma_G(x) : x \in V(C) \setminus \bigcup_{i=1}^h C(x_i; a_i) \right\}
\leq \max \left\{ n - \frac{r + 1}{2} + t - 1, r - 1 \right\}.
\]

Case 2: \( n - \frac{1}{2}(3r + 1) + t < 0 \). Since \( r \leq n - 1 \), there exists an integer \( t_0 \) with \( t < t_0 \leq (r - 1)/2 \) such that \( n - (3r + 1)/2 + t_0 = 0 \). Applying Case 1 to \( t_0 \) we obtain

\[
\exp_G(2t) \leq \exp_G(2t_0)
\leq \max \left\{ n - \frac{r + 1}{2} + t_0 - 1, r - 1 \right\}.
\]
The theorem therefore holds for unicyclic graphs in $S_n(r)$. If $G$ is not unicyclic, then there is a spanning, unicyclic subgraph $G_0$ of $G$ which belongs to $S_n(r)$, and we have

$$\exp_G(2t) \leq \exp_{G_0}(2t) \leq \max \left\{ n - \frac{r + 1}{2} + t - 1, r - 1 \right\}. \quad \square$$

Combining Theorems 3.1, 4.1 and 5.2, we obtain the following theorem which evaluates the maximum generalized exponents for the class $S_n(r)$.

**Theorem 5.3.** Let $r$ be an odd integer with $1 < r < n$. Then

$$\max \{ \exp_G(k) : G \in S_n(r) \} = \exp_{G_{\ast}}(k)$$

$$= \begin{cases} \left( n - 1 \right) + \left( k - r \right), & r \leq k \leq n \\ \max \left\{ n - \frac{r + 1}{2} + \left\lfloor \frac{k + 1}{2} \right\rfloor - 1, r - 1 \right\}, & 1 \leq k \leq r - 1. \end{cases}$$

(17)

Let $e^*(n,k,r)$ denote the maximum value of $\exp_G(k)$ for $G$ in $S_n^*(r)$. The following corollary is a simple consequence of the fact that a graph in $S_n^*(r)$ has a spanning, unicyclic subgraph which is contained in $S_n(r)$.

**Corollary 5.4.** Let $r$ be an odd integer with $1 < r < n$. Then

$$e^*(n,r,k) = \exp_{G_{\ast}}(k)$$

$$= \begin{cases} \left( n - 1 \right) + \left( k - r \right) & \text{if } r \leq k \leq n, \\ \max \left\{ n - \frac{r + 1}{2} + \left\lfloor \frac{k + 1}{2} \right\rfloor - 1, r - 1 \right\} & \text{if } 1 \leq k \leq r - 1. \end{cases}$$

(18)

6. Characterization of the extremal graphs 1

In this and the subsequent two sections we characterize the graphs $G$ in $S_n(r)$ for which the $k$th generalized exponent is maximum for those integers $k$ satisfying $1 \leq k \leq r - 1$. The characterization depends on the relationship between $n,k$ and $r$. In each case we take $t = \lfloor(k + 1)/2 \rfloor$, so that $k = 2t - 1$ or $2t$. Since $1 \leq k \leq r - 1$ and $r$ is odd, we have $1 \leq t \leq (r - 1)/2$. The next theorem shows that if $n - (3r + 1)/2 + t \leq 0$, then every graph in $S_n(r)$ has the same $k$th generalized exponent. In particular, in this case the extremal graph is not unique in general.
Theorem 6.1. Let $r$ be an odd integer with $1 \leq r \leq n$, and let $t$ be an integer with $1 \leq t \leq (r - 1)/2$. If

$$n - (3r + 1)/2 + t \leq 0,$$

then for each graph $G$ in $S_n(r)$ we have

$$\exp_G(2t - 1) - \exp_G(2t) = r - 1.$$  \hspace{1cm} (20)

Proof. By Theorem 5.2, we have

$$\exp_G(2t - 1) \leq \exp_G(2t) \leq \max \left\{ n - \frac{r + 1}{2} + t - 1, r - 1 \right\} = r - 1. \hspace{1cm} (21)$$

On the other hand, for each vertex $x$ of $G$,

$$\gamma_G(x) \geq \gamma_G(x,x) \geq r - 1,$$

since there can be no walk from $x$ to $x$ of odd length $r - 2$. Hence,

$$\exp_G(2t) \geq \exp_G(2t - 1) \geq \exp_G(1) \geq r - 1. \hspace{1cm} (22)$$

The theorem now follows from (21) and (22). \hspace{1cm} \Box

7. Characterization of the extremal graphs II

We continue with the notation of the previous section but now we investigate the case in which $n - (3r + 1)/2 + t \geq 2$ where again $t = \lceil (k + 1)/2 \rceil$. We first give some necessary conditions for a unicyclic graph $G$ in $S_n(r)$ to be extremal for $\exp_G(2t - 1)$ and $\exp_G(2t)$ in case that

$$n - \frac{3r + 1}{2} + t \geq 1. \hspace{1cm} (23)$$

Note that (23) implies that

$$\max \left\{ n - \frac{r + 1}{2} + t - 1, r - 1 \right\} = n - \frac{r + 1}{2} + t - 1.$$ 

Lemma 7.1. Let $r$ be an odd integer with $1 \leq r \leq n$, and let $t$ be an integer with $1 \leq 2t \leq r - 1$. Assume that (23) holds. Let $G$ be a unicyclic graph in $S_n(r)$ as given in Lemma 5.1. Assume that $a_i \geq 0$ for $1 \leq i \leq p$ and that $a_j < 0$ for $p + 1 \leq j \leq p + q$ where $h = p + q$. Then the following hold:

(i) If $\exp_G(2t) = n - (r + 1)/2 + t - 1$, then

(a) $p = 1$ and $q = 0$ when $n - (3r + 1)/2 + t \geq 2$, and

(b) $p = 1$ and $q = 0$, or $p = 2$ and $q = 0$, when $n - (3r + 1)/2 + t = 1$;

(ii) If $\exp_G(2t - 1) = n - (r + 1)/2 + t - 1$, then $p = 1$ and $q = 0$;
(iii) If \( p = 1 \) and \( q = 0 \) but \( G \) is not isomorphic to \( G_{n,r} \), then
\[
\exp_G(2t - 1) \leq \exp_G(2t) < n - \frac{r + 1}{2} + t - 1.
\]

**Proof.** We first assume that \( \exp_G(2t) = n - (r + 1)/2 + t - 1 \) and prove (i). Since
\[
n - \frac{r + 1}{2} + t - 1 > n - \frac{r + 1}{2} + t - 2 = \max \left\{ n - \frac{r + 1}{2} + t - 2, r - 1 \right\},
\]
Lemma 5.1 implies that
\[
\left| V(C) \setminus \bigcup_{i=1}^{h} C(x_i; a_i + 1) \right| < 2t. \tag{24}
\]
As in the proof of Theorem 5.2 we have \( \sum_{i=1}^{h} b_i = n - r, a_i \leq (r - 1)/2 (1 \leq i \leq h) \) and
\[
|C(x_i; a_i + 1)| = \begin{cases} 
0 & \text{if } p + 1 \leq i \leq p + q, \\
2a_i + 1 & \text{if } 1 \leq i \leq p.
\end{cases}
\]
If \( p = 0 \), then from (24) we obtain the contradiction \( r = |V(C)| < 2t \). Hence \( p \geq 1 \) and
\[
r - 2t < \left| \bigcup_{i=1}^{h} C(x_i; a_i + 1) \right|
\leq \sum_{i=1}^{p} |C(x_i; a_i + 1)|
= \sum_{i=1}^{p} (2a_i + 1)
= 2 \sum_{i=1}^{p} \left( b_i - \left( n - \frac{3r + 1}{2} + t \right) \right) + p
= 2 \left( \sum_{i=1}^{h} b_i - \left( n - \frac{3r + 1}{2} + t \right) \right) - 1
- \left( 2(p - 1) \left( n - \frac{3r + 1}{2} + t \right) - (p + 1) + 2 \sum_{i=p+1}^{p+q} b_i \right)
= r - 2t - \left( 2(p - 1) \left( n - \frac{3r + 1}{2} + t \right) - (p + 1) + 2 \sum_{i=p+1}^{p+q} b_i \right).
\]

If \( n - (3r + 1)/2 + t \geq 2 \), then since \( p \geq 1 \) and \( b_i \geq 1 \) we now conclude that \( 0 > 4(p - 1) - (p + 1) + 2q = 3p + 2q - 5 \) and so \( p = 1 \) and \( q = 0 \). If \( n - (3r + 1)/2 + t = 1 \),
we conclude that $0 > p + 2q - 3$ and so $p = 1$ and $q = 0$, or $p = 2$ and $q = 0$. Hence (i) holds.

We now assume that $\exp_G(2t - 1) = n - (r + 1)/2 + t - 1$ and prove (ii). In place of (24) we have $|V(C) \setminus C(x_1; a_1 + 1)| < 2t - 1$, and we obtain

$$r - 2t + 1 < r - 2t - \left(2(p - 1) \left(n - \frac{3r + 1}{2} + t\right) - (p + 1) + 2 \sum_{i=p+1}^{p+q} b_i\right).$$

This implies that

$$-1 > 2(p - 1) - (p + 1) + 2q = p + 2q - 3$$

and hence that $p = 1$ and $q = 0$. Hence (ii) holds.

We now assume that $p = 1$, $q = 0$ and $G$ is not isomorphic to $G_{n,r}$. We then have $h = 1$ and the unique branch $B_1 = B(x_1)$ of $G$ is not a path with $x_1$ as an endvertex. Thus, for any vertex $y$ of $B_1$ we have

$$d_1(x_1, y) \leq |V(B_1)| - 2 = b_1 - 1,$$

where $d_1(\cdot, \cdot)$ is the distance function of $B_1$. Let $x$ be any vertex in $V(C) \setminus C(x_1; a_1)$. By Lemma 2.3, if $y$ is a vertex of $C$, we have $\gamma_G(x, y) \leq r - 1$, and if $y$ is a vertex of $B_1$,

$$\gamma_G(x, y) \leq |x \cap C'| + d_1(x_1, y) - 1$$

$$\leq r - a_1 + b_1 - 2$$

$$= r - 2 + \left(n - \frac{3r + 1}{2} + t\right)$$

$$= n - \frac{r + 1}{2} + t - 2.$$

This implies that

$$\gamma_G(x) \leq \max \left\{ n - \frac{r + 1}{2} + t - 2, r - 1 \right\}.$$

Now $b_1 = n - r$ implies that

$$|C(x_1; a_1)| = 2a_1 - 1 = 2b_1 - 2 \left(n - \frac{3r + 1}{2} + t\right) - 1 = r - 2t,$$

and hence $|V(C) \setminus C(x_1; a_1)| = 2t$. Therefore,

$$\exp_G(2t) \leq \max \left\{ \gamma_G(x) : x \in V(C) \setminus C(x_1; a_1) \right\}$$

$$\leq \max \left\{ n - \frac{r + 1}{2} + t - 2, r - 1 \right\}$$

$$= n - \frac{r + 1}{2} + t - 2$$

$$< n - \frac{r + 1}{2} + t - 1,$$

and (iii) also holds. □
The proof of the next lemma is quite technical and we omit it.

**Lemma 7.2.** Let \( r \) be an odd integer with \( 1 \leq r \leq n \), and let \( t \) be an integer with \( 1 \leq 2t \leq r - 1 \). Assume that (23) holds. Let \( G \) be a graph in \( S_n(r) \) such that \( G \) contains a proper spanning subgraph \( H \) where \( H \) is isomorphic to \( G_{n,r} \). Then either

(i) \( \exp_G(2t) \leq n - (r + 1)/2 + t - 2 \), or

(ii) \( r = n - 2 \) (and so \( t = (r - 1)/2 \) and \( n = (3r + 1)/2 + t - 1 \), since \( n - (3r + 1)/2 + t \geq 1 = n - (3r + 1)/2 + (r - 1)/2 \) and \( G \) is isomorphic to the graph \( G_{n,n-2}^* \) contained in \( S_n(r) \) consisting of a cycle of length \( n - 2 \) and a cycle of length 4 with a common edge.

We now state and prove the main theorem of this section.

**Theorem 7.3.** Let \( r \) be an odd integer with \( 1 \leq r \leq n \), and let \( t \) be an integer with \( 1 \leq t \leq (r - 1)/2 \). Assume that

\[
n - (3r + 1)/2 + t \geq 2.
\]

Let \( G \) be a graph in \( S_n(r) \). Then \( \exp_G(2t) = \exp_{G_{n,r}}(2t) \) if and only if \( G \) is isomorphic to \( G_{n,r} \), and \( \exp_G(2t - 1) = \exp_{G_{n,r}}(2t - 1) \) if and only if \( G \) is isomorphic to \( G_{n,r} \).

**Proof.** First suppose that \( \exp_G(2t) = \exp_{G_{n,r}}(2t) \). Let \( H \) be a spanning unicyclic subgraph of \( G \) in \( S_n(r) \). Since \( \exp_G(2t) \leq \exp_H(2t) \), \( \exp_H(2t) = \exp_{G_{n,r}}(2t) \). It follows from Lemma 7.1 that \( H \) is isomorphic to \( G_{n,r} \), and thus \( G \) has a spanning subgraph isomorphic to \( G_{n,r} \). Now Lemma 7.2 implies that \( G = H \). If \( \exp_G(2t - 1) = \exp_{G_{n,r}}(2t - 1) \) then by Theorem 5.2 we also have \( \exp_G(2t) = \exp_{G_{n,r}}(2t) \) and the theorem follows.

\[\square\]

8. Characterization of the extremal graphs III

We continue with the notation of the previous two sections but now we consider the case in which \( n = (3r + 1)/2 + t = 1 \) where \( t = \left\lceil (k + 1)/2 \right\rceil \). In the next lemma we evaluate some of the generalized exponents of the graph \( G_{n,n-2}^* \) defined in (ii) of Lemma 7.2.

**Lemma 8.1.** Let \( n \) be an odd integer and let \( r = n - 2 \). Let \( t = (n - 3)/2 \). Let \( G_{n,n-2}^* \) be the graph in \( S_n(n-2) \) consisting of an \( (n-2) \)-cycle and a 4-cycle with a common edge. Then

\[
\exp_{G_{n,n-2}^*}(2t) - \exp_{G_{n,n-2}^*}(n - 3) = n - 2 = n - \frac{r + 1}{2} + t - 1
\]

and

\[
\exp_{G_{n,n-2}^*}(2t - 1) = \exp_{G_{n,n-2}^*}(n - 4) \leq n - 3 = n - \frac{r + 1}{2} + t - 2.
\]
Proof. Let the 4-cycle of $G_{n,n-2}^*$ be $x_0, x_1, x_2, u, x_0$ where $\{u, x_0\}$ is an edge of the $(n-2)$-cycle of $G_{n,n-2}^*$. It is easy to check that for each vertex $z$ of $G_{n,n-2}^*$ different from any of $x_0, x_1, x_2, u$, we have $\gamma_{G_{n,n-2}^*}(z) \leq r-1 = n-3$. Hence $\exp_{G_{n,n-2}^*}(n-4) \leq n-3$, proving (27).

Also since there is no walk from $u$ to $x_2$ of even length $r-1$

$$\gamma_{G_{n,n-2}^*}(x_0) = \gamma_{G_{n,n-2}^*}(u) \geq \gamma_{G_{n,n-2}^*}(u, x_2) \geq r,$$

and since there is no walk from $x_2$ to $u$ of length $r-1$,

$$\gamma_{G_{n,n-2}^*}(x_1) = \gamma_{G_{n,n-2}^*}(x_2) \geq \gamma_{G_{n,n-2}^*}(x_2, u) \geq r.$$

Therefore $\exp_{G_{n,n-2}^*}(n-3) \geq r = n-2$. But by Theorem 5.2 we have

$$\exp_{G_{n,n-2}^*}(n-3) = \exp_{G_{n,n-2}^*}(2t) \leq \max \left\{ n - \frac{r+1}{2} + t - 1, r - 1 \right\} = n-2,$$

and hence (26) holds. □

We now show under the assumptions of this section that $G_{n,r}$ is the only extremal graph for the $k$th generalized exponent when $k$ is odd.

Theorem 8.2. Let $r$ be an odd integer with $1 \leq r \leq n$, and let $t$ be an integer with $1 \leq t \leq (r-1)/2$. Assume that

$$n - 3r + 1 \geq t = 1. \quad (28)$$

Let $G$ be a graph in $G_{n,r}$. Then $\exp_G(2t-1) = \exp_{G_{n,r}}(2t-1)$ if and only if $G$ is isomorphic to $G_{n,r}$.

Proof. Suppose that $\exp_G(2t-1) = \exp_{G_{n,r}}(2t-1)$, which by Theorem 3.1 equals $n - \frac{1}{2}(r+1) + t - 1$. Let $G_0$ be a unicyclic spanning graph of $G$ which is contained in $S_n(r)$. Then $\exp_{G_0}(2t-1)$ also equals $n - \frac{1}{2}(r+1) + t - 1$. By (ii) of Lemma 7.1 applied to $G_0$ and using the notation in the lemma, we must have $p = 1$ and $q = 0$, and by (iii) of that lemma, $G_0$ is isomorphic to $G_{n,r}$. Since (i) of Lemma 7.2 does not hold, it follows from the lemma that either $G = G_0$ or $r = n - 2$ and $G$ is isomorphic to $G_{n,n-2}^*$. Since the latter possibility contradicts (27) of Lemma 8.1, we conclude that $G = G_0$ and hence $G$ is isomorphic to $G_{n,r}$. The theorem now follows. □

In order to obtain a characterization of the extremal graphs when $k$ is even we prove the following lemma.

Lemma 8.3. Let $r$ be an odd integer with $1 \leq r \leq n$, and let $t$ be an integer with $1 \leq t \leq (r-1)/2$. Assume that (28) holds. Let $g$ and $f$ be positive integers with

$$\frac{n - r}{2} \leq g \leq n - r - 1 \leq f \leq \frac{r - 1}{2}. \quad (29)$$
Let $G = G(n, r, g, f)$ be the graph in $S_{n}(r)$ pictured in Fig. 2 consisting of an $r$-cycle $C$, a path $P_{1}$ of length $g$ joining a vertex $y_{1}$ to a vertex $x_{1}$ of $C$ and a path $P_{2}$ of length $h = n - r - g \geq 1$ joining a vertex $y_{2}$ to a vertex $x_{2}$ of $C$, where it is assumed that $|x_{1}C'x_{2}| = f$. Then

$$\exp_{G}(2t) = \max \left\{ n - \frac{r + 1}{2} + t - 1, r - 1 \right\} = r.$$

**Proof.** Since (28) holds, we have $n - (r + 1)/2 + t - 1 = r$. Also $n - (3r + 1)/2 + (r - 1)/2 \geq n - (3r + 1)/2 + t = 1$ implies that $n \geq r + 2$ and $(n - r)/2 \leq n - r - 1$. Also $n - (3r + 1)/2 - 1 - t \leq 0$ implies that $n - r - 1 \leq (r - 1)/2$. We conclude that there do exist positive integers $g$ and $f$ satisfying (29). Also $g \geq (n - r)/2$ implies that $h = n - r - g \leq g$.

By Theorem 5.2, it suffices to prove that $\exp_{G}(2t) \geq r$. First we show that $\gamma_{G}(x) \geq r$ for each vertex $x$ of $V(P_{1}) \cup V(P_{2}) \cup C(x_{1}; g) \cup C(x_{2}; h)$. We consider three cases.

**Case 1:** $x$ is not a vertex of $C$. Since $r$ is odd, there is no walk of length $r$ from $x$ to $y_{1}$ and hence $\gamma_{G}(x) > r$.

**Case 2:** $x$ is a vertex of $C(x_{1}; g)$ so that $|x_{1}C'x| \leq g - 1$. Suppose that $W$ is a walk from $x$ to $y_{1}$ with $|W| = r - |xC'x_{1}| + g - 2$. Then

$$|xC''x_{1} + P_{1}| = r - |xC'x_{1}| + g > |W|$$

and

$$|xC'x_{1} + P_{1}| + r = r + |xC'x_{1}| + g > |W|,$$

and it follows that $W$ must be obtained from $xC'x_{1} + P_{1}$ by attaching closed walks of length 2 to some of its vertices. But then

$$r - |xC'x_{1}| + g - 2 = |W| \equiv |xC'x_{1}| + g \pmod{2},$$

a contradiction since $r$ is odd. Thus, there is no walk from $x$ to $y_{1}$ of length $r - |xC'x_{1}| + g - 2$, and hence

$$\gamma_{G}(x) \geq \gamma_{G}(x, y_{1}) \geq r - |xC'x_{1}| + g - 1 \geq r,$$

since $|xC'x_{1}| < g - 1$. 

![Fig. 2. The graph $G(n, r, g, f)$](image_url)
Case 3: $x$ is a vertex of $C(x_2; h)$. Using an argument similar to that in Case 2 we conclude that

$$
\gamma_G(x) \geq \gamma_G(x, y_2) \geq r - |xC'x_2| + h - 1 \geq r.
$$

In summary, we now know that $\gamma_G(x) \geq r$ for each vertex $x$ of $V(P_1) \cup V(P_2) \cup C(x_1; g) \cup C(x_2; h)$.

Now $|x_1 C'x_2| = f \geq n - r - 1 = g + h - 1 > (g - 1) + (h - 1)$ implies that $C(x_1; g) \cap C(x_2; h) = \emptyset$, and using $n - (3r + 1)/2 + t = 1$ we obtain

$$
|V(P_1) \cup V(P_2) \cup C(x_1; g) \cup C(x_2; h)|
= |V(P_1)| + |V(P_2)| + |C(x_1; g)| + |C(x_2; h)|
= (g + 1) + (h + 1) + (2g - 1) + (2h - 1) - 2
= 3g + h - 2
= 3n - 3r - 2
= n - (2t - 1).
$$

It now follows that $\exp_G(2t) \geq r$. \(\square\)

Under the assumptions of this section, the extremal graphs for the $k$th generalized exponent when $k$ is even can now be characterized.

**Theorem 8.4.** Let $r$ be an odd integer with $1 \leq r \leq n$, and let $t$ be an integer with $1 \leq t \leq (r - 1)/2$. Assume that

$$
n - \frac{3r + 1}{2} + t = 1. \tag{30}
$$

Let $G$ be a graph in $S_n(r)$. Then $\exp_G(2t) = \exp_{G_{n,r}}(2t)$ if and only if one of the following holds:

(i) $G$ is isomorphic to $G_{n,r}$,

(ii) $n$ is odd, $r = n - 2$, $2t = r - 1$, and $G$ is isomorphic to $G_{n,n-2}^*$, and

(iii) there exist integers $g$ and $f$ with $(n - r)/2 \leq g \leq n - r - 1 \leq f \leq (r - 1)/2$ such that $G$ is isomorphic to $G(n, r, g, f)$.

**Proof.** By Theorem 3.1 and Lemmas 8.1 and 8.3, it suffices to show that a graph $G$ in $S_n(r)$ for which $\exp_G(2t) = \exp_{G_{n,r}}(2t)$ satisfies either (i), (ii) or (iii). Let $G_0$ be a spanning, unicyclic subgraph of $G$ in $S_n(r)$. Then $\exp_{G_0}(2t) = \exp_{G_{n,r}}(2t)$ and applying Lemma 7.1 to $G_0$ (and using its notation) we have $p = 1$ and $q = 0$, or $p = 2$ and $q = 0$.

First assume that $p = 1$ and $q = 0$. Then it follows from (iii) of Lemma 7.1 that $G_0$ is isomorphic to $G_{n,r}$. By Lemma 7.2 either $G_0$ equals $G$ and $G$ is isomorphic to $G_{n,r}$, or $r = n - 2$ and $t = (r - 1)/2$, and $G$ is isomorphic to $G_{n,n-2}^*$. 
The proof in the case $p = 2$ and $q = 0$ is accomplished by first showing that (a) $G_0$ is isomorphic to one of the graphs $G(n, r, g, f)$ and then that (b) $G_0$ equals $G$. We omit the technical details. □

References