Equation (7) should be replaced by the following:

$$\lambda_{\text{max}}(z^{-1} U_A + U_H) < 0$$

(7)

where

$$U_A = U_I \left[ \begin{array}{ccc} P_{11} & P_{12} & 0 \\ P_{21} & P_{22} & 0 \\ \end{array} \right] \left[ \begin{array}{c} z^2 I \\ 0 \\ 0 \end{array} \right] U_\perp$$

$$U_H = U_I \left[ \begin{array}{ccc} P_{11} & 0 & 0 \\ P_{21} & P_{22} & 0 \\ \end{array} \right] \left[ \begin{array}{c} 0 \\ 0 \\ z^2 I \end{array} \right] U_\perp$$

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Controllability and Observability of Linear Time-Varying Singular Systems

Chi-Jo Wang

Abstract—Controllability and observability of analytically solvable linear time-varying singular systems are considered. Our analysis includes the possible impulsive zero-input response. Based on the definitions of controllability and observability naturally extended from the ones adopted in the time-invariant setting, sufficient and necessary conditions for these two properties are derived. Furthermore, according to the specific definitions of these concepts used here, the authors show that the dual relationship between controllability and observability does not hold.

Index Terms—Controllability, duality, observability, singular systems.

I. INTRODUCTION

In the early 1980’s, linear time-varying singular systems

$$E(t) \ddot{x} = A(t) \dot{x} + B(t) u$$

$$y = C(t) x$$

began to attract the attention of researchers. Here $E$, $A$, $B$, $C$ are matrix functions of appropriate sizes; $E$ is assumed singular, and $E$ may have variable rank. Several papers focus on certain canonical forms and specify the systems transformable to these forms (see, e.g., [2] and [3]). References [1] and [11] look at systems with rectangular coefficients and examine how the closed-loop system can be made uniquely solvable via state and derivative feedback. Reference [9] develops a reduction procedure that transforms the solutions of linear time-varying singular systems into solutions of ordinary differential equations; [10] deals with systems with discontinuous inputs. The issue of observability is treated in [4]. Further in [5], controllability and observability concepts were examined. Based on an observability theory similar to the one presented in [4], projection operators are constructed in [12] to decompose the system into unobservable subspace and observable complement.

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One intrinsic feature of singular systems is the impulsive natural response due to “inconsistent initial conditions” as defined in [13]. Despite the fact that many papers on time-invariant singular systems have considered the impulsive behavior, very few works concerning time-varying singular systems attempted to do so. In order to generalize the existing theories to include linear time-varying singular systems, we have discussed state feedback impulse elimination in [15]. Further in this paper, we examine the issues of controllability and observability for analytically solvable linear time-varying singular systems, especially those in standard canonical form. Although similar topics have been investigated in [4] and [5], the possibility of impulsive solutions is excluded by the definition of observability therein. It is our intent to allow all initial conditions in this paper. For this end, we adopt the definitions used in [7] with modifications necessary to fit into the time-varying framework.

It has been shown in [2] that every analytically solvable singular state can be put into standard canonical form through appropriate coordinate transformation. Two independent subsystems, the slow subsystem and the fast subsystem, constitute the standard canonical form. In view of the fact that the slow subsystem is a regular state variable system whose properties, particularly observability and controllability, have been thoroughly studied, our analysis will center on the fast subsystem.

This paper is organized as follows. Problem formulation is given in Section II; Section III presents preliminaries essential to our exposition. We derive the criteria of observability and controllability in Sections IV and V, respectively. The results on observability we will obtain serve as a straightforward extension of the present time-invariant theory and indicate that observability at $t_0$ for the fast subsystem is determined only by the system parameters at $t_0$. However, controllability at $t_0$ may be contingent on system parameters at any point of $[t_0, \infty)$. As a result, controllability and observability are not dual concepts within this framework. Finally, we conclude this paper in Section VI.

II. PROBLEM FORMULATION

In this paper, we look at the class of analytically solvable singular systems as defined in [2]. A brief review of these systems can also be found in [15]. Consider linear time-varying singular systems

$$E(t) \ddot{z} = A(t) \dot{z} + G(t) u$$

(1)

where $E$, $A$, $G$ are, respectively, $m \times m$, $m \times m$, $m \times q$ matrix functions. Moreover, $E$ is singular; $E$, $A$, and $G$ are all analytic. According to [2, Th. 2], if (1) is analytically solvable, there exist analytic and invertible matrix functions $P(t)$ and $Q(t)$ such that the premultiplication of (1) by $Q(t)$ and the transformation $z = P(t) \bar{z}$ put (1) into standard canonical form

$$z_1 = A_1(t) \dot{z}_1 + G_1(t) u$$

(2)

$$N(t) \ddot{z}_2 = \ddot{z}_2 + G_2(t) u$$

(3)

where $N(t)$ is strictly upper triangular for all $t$. An algorithmic procedure to perform this decomposition has also been provided in [2]. Since properties of (2) are well known, only (3) remains to be investigated. Hence, we set the focal point of this paper on the controllability and observability of systems

$$N(t) \ddot{x} = x + B(t) u$$

(4)

$$y = C(t) x$$

(5)
where $N, B, C$ are $n \times n, n \times p, r \times n$ analytic matrix functions, respectively. Furthermore, $N(t)$ is strictly upper triangular for all $t$. State (4), along with output (5) will be referred to as the “fast subsystem” throughout this paper.

III. PRELIMINARIES

We first present a brief introduction to the theory of distributions. A more detailed exposition can be found in [8]. Let $C^i$ be the $i$ times continuously differentiable functions mapping $\mathbb{R}$ to $\mathbb{R}$, and $C_p^i$ the $i$ times piecewise continuously differentiable maps. $K$ is the space of $C^\infty$ functions $\phi : \mathbb{R} \to \mathbb{R}$, where $\phi$ has a bounded support, and let $K'$ be the dual space of $K$. A distribution $f$ is an element of $K'$, i.e., a continuous linear functional $\phi \rightarrow (f, \phi)$. $K_p^i$ denotes the space of piecewise continuous distributions as defined in [7]. Also from [7], $f|_{[t_0, \infty)}$ stands for the restriction of $f \in K_p^i$ on the interval $[t_0, \infty)$. $\delta(t)$ is defined by $\langle \delta(t), \phi \rangle = \phi(t_0)$. The multiplication of $f \in K'$ by $g \in C^\infty$, $gf \in K'$, is defined as the following: For $\phi \in K'$, $\langle gf, \phi \rangle = \langle f, \phi \rangle \cdot \phi(t_0)$. $\delta^{(i)}$ denotes the $i$th distributional derivative of $f \in K'$.

The issue of adopting distributions as viable solutions for linear time-invariant singular systems

$$\dot{x} = x + Bu$$

to account for inconsistent initial conditions was rigorously explored in [7]. It is straightforward to see that the results on this issue from [7] can be easily extended to (4) and (5). One result most relevant to our analysis is presented in Proposition 1 without proof.

**Proposition 1:** Let $u \in C^i_p$, then

$$N(t|x|_{y_0, \infty}) = y|_{y_0, \infty} + B(t)|_{y_0, \infty} + N(t)x(t_0)\delta(t_0)$$

and

$$y|_{y_0, \infty} = C(t)x(t_{\infty})$$

characterizes the response of the fast subsystem for $t \geq t_0$ due to initial condition $x(t_0)$. Since $u \in C^i_p$, according to Proposition 1 and [3, Th. 2.1], the unique solution for (6) is

$$- (I - N(t)D)^{-1}(B(t)u|_{y_0, \infty})$$

$$- (I - N(t)D)^{-1}(N(t_0)x(t_0)\delta(t_0)).$$

Next, we give Proposition 2 that will be essential in Section III. A proof for Proposition 2 can be easily implemented by induction. Details of such a proof can be found in [14, p. 61].

**Proposition 2:**

$$\sum_{k=0}^{n} (-1)^{i+k} \binom{i}{k} M^i(t)\delta_{y_0}(i)$$

where $M(t) \in C^\infty$, $i \in \mathbb{Z}^+$.

It is interesting to note that the product of a $C^\infty$ function and $\delta_{y_0}(i)$, $i \in \mathbb{Z}^+$, is a distribution with point support.

IV. OBSERVABILITY

**Definition 1:** The fast subsystem is observable at $t_0$ if knowledge of $u|_{y_0, \infty} \in C^\infty_p$, $y|_{y_0, \infty} \in K_p^i$, and $y(t_0)\delta(t_0)$ is sufficient to determine $x(t_0)\delta(t_0)$.

In [7], the definition of observability for

$$\dot{x} = x + Bu$$

$$y = Cx$$

did not specify a particular instant of interest, owing to the time-invariant nature. Definition 1 differs from that of [7] in its specific of initial time $t_0$. This feature (indicating the initial time) is most natural in the time-varying setting and typical in defining system properties for regular time-varying systems (see, e.g., [6, Definition 5-5]). Moreover, because $C$ is analytic, $y(t_0) = C(t_0)x(t_0)$.

We first need a lemma.

**Lemma 1:** Let $x(t_0) \in \mathbb{R}^n$. Then

$$(I - N(t)D)^{-1}(N(t_0)x(t_0)\delta(t_0))$$

can be expressed as

$$\sum_{i=0}^{n} \alpha_i \delta_{y_0}^{(i)}(x(t_0))$$

where

$$\alpha_0 = (I + P_0)N(t_0)$$

and

$$\alpha_i = (I + P_i)N(t_0)\alpha_{i-1}$$

for $i = 1, \cdots, n - 1$; $P_i$ is strictly upper triangular $n \times n$ matrices for $j = 0, \cdots, n - 1$.

**Proof:** Since $N(t)$ is strictly upper triangular for all $t$, then

$$(I - N(t)D)^{-1}(N(t_0)x(t_0)\delta(t_0))$$

equals

$$(I + N(t)D + (N(t)D)^2 + \cdots + (N(t)D)^{n-1})$$

$$(N(t_0)x(t_0)\delta(t_0))$$

According to Proposition 2, (8) is a distribution with point support; we can thus further express

$$\sum_{i=0}^{n} \alpha_i \delta_{y_0}^{(i)}(x(t_0))$$

where

$$\alpha_i = (I + n\hat{N}(t_0))^{-1}N(t_0)\delta_{y_0}^{(i)}$$

and

$$\alpha_i = (I + n\hat{N}(t_0))^{-1}N(t_0)\delta_{y_0}$$

We now derive the formula of $\alpha_i$ by induction. Note that the coefficients of $\delta_{y_0}^{(i-1)}$ on both sides of (9) are zero. Thus, we get

$$-N(t_0)\alpha_{n-2} + \alpha_{n-1} + n\hat{N}(t_0)\alpha_{n-2} = 0$$

or

$$\alpha_{n-1} = (I + n\hat{N}(t_0))^{-1}N(t_0)\alpha_{n-2}.$$
the claim is true for \( i = k - 1 \). Note that the coefficients for \( \delta_{k-1}^{(k-1)} \) on both sides of (9) are zero. By invoking Proposition 2, we have

\[
\sum_{\ell = k-1}^{n-1} (-1)^{\ell-k+1} \binom{k-1}{\ell-1} (t_0)^{k-\ell} \alpha_{\ell, k-2} \alpha_k = 0.
\]

Since the claim is true for \( k \leq i \leq n-1 \), the above can be rewritten as

\[
\left( I + \sum_{\ell = k-1}^{n-1} Q_\ell \right) \alpha_{k, k-2} = N(t_0) \alpha_{k, k-2}
\]

where \( Q_\ell \) are strictly upper triangular matrices. Again, because

\[
\left( I + \sum_{\ell = k-1}^{n-1} Q_\ell \right)
\]

is upper triangular with all diagonal elements being one, we have

\[
\alpha_{k, k-2} = \left( I + P_{k-1} \right) N(t_0) \alpha_{k, k-2}.
\]

Thus the claim is true when \( i = k - 1 \). Next, we show \( \alpha_0 = (I + P_0) N(t_0) \). Equating the coefficients of \( \delta_0 \) on both sides of (9) yields

\[
0 = \alpha_0 + \sum_{\ell = 0}^{n-1} (t_0)^{\ell+1} \sum_{i=0}^{\ell} \alpha_i \delta^{(\ell)}_0 \alpha_\ell = N(t_0).
\]

Since \( \alpha_i = (I + P_i) N(t_0) \alpha_{i+1} \) when \( i = 1, \ldots, n-1 \), repeating the process above gives us

\[
\alpha_0 = (I + P_0) N(t_0).
\]

For \( j = 0, \ldots, n-1 \), \( (I + P_j) N(t_0) \) are \( n \times n \) strictly upper triangular matrices. Since \( \alpha_{n-1} \) is the product of them, i.e., \( \alpha_{n-1} = (I + P_{n-1}) N(t_0) \cdots (I + P_1) N(t_0) (I + P_0) N(t_0) \),

\[
N(t_0) \alpha_{n-1} = \hat{N}(t_0) \alpha_{n-1} = 0.
\]

As a first step toward our observability theorem, we consider systems with time-invariant \( C \) matrix in the following lemma.

**Lemma 2:** \( \bar{\alpha} \), \( \bar{\alpha} \)

\[
N(t) \bar{\alpha} = x + B(t) u
\]

is observable at \( t_0 \) if and only if

\[
\text{Im} \left( C^T (t_0) \right) + \text{Im} \left( N \left( t_0 \right) \right) = \mathbb{R}^n.
\]

**Proof:** Without loss of generality, let \( u |_{t_0 \to \infty} = 0 \). Consequently

\[
y |_{t_0 \to \infty} = -C \left( \sum_{i=0}^{n-1} \alpha_i \delta_0^{(i)} \right) x(t_0)
\]

where \( \alpha_i, i = 0, \ldots, n-1 \) are as described in Lemma 1.

We first show the necessity. If

\[
\text{Im} \left( C^T (t_0) \right) + \text{Im} \left( N \left( t_0 \right) \right) \neq \mathbb{R}^n,
\]

there exists nonzero \( x_0 \in \mathbb{R}^n \) such that \( Cx_0 \) and \( N(t_0)x_0 \) are both zero. Obviously, \( \alpha_0, x_0 = 0 \) for all \( i \). Thus

\[
N(t) \bar{\alpha} = x + B(t) u
\]

is not observable at \( t_0 \).

Next we show the sufficiency. Assume

\[
\text{Im} \left( C^T (t_0) \right) + \text{Im} \left( N \left( t_0 \right) \right) = \mathbb{R}^n.
\]

Yet

\[
\begin{align*}
N(t) \bar{\alpha} &= x + B(t) u \\
y &= Cx
\end{align*}
\]

is not observable at \( t_0 \). As a result, nonzero \( x_0 \in \mathbb{R}^n \) exists such that

\[
C x_0 = 0 \quad \text{and} \quad C \left( \sum_{i=0}^{n-1} \alpha_i \delta_0^{(i)} \right) x_0 = 0
\]

which implies \( C \alpha \) is zero for \( 0 \leq i \leq n-1 \). Notice that \( Cx_0 = 0 \) implies \( N(t_0)x_0 \neq 0 \); consequently, \( \alpha_0 x_0 = (I + P_0) N(t_0) x_0 \) is not zero. Similarly, because

\[
\begin{align*}
C \alpha_0 x_0 &= C(I + P_0) N(t_0) x_0 \\
\alpha_i x_0 &= (I + P_i) N(t_0) x_0 \neq 0
\end{align*}
\]

Repeating this argument yields \( \alpha_{n-1} x_0 \neq 0 \) and \( C \alpha_{n-1} x_0 = 0 \), which implies

\[
N(t_0) \alpha_{n-1} x_0 \neq 0.
\]

This is clearly a contradiction, since \( N(t_0) \) is strictly upper triangular.

We now show the main theorem of this section.

**Theorem 1:** \( \bar{\alpha} \), \( \bar{\alpha} \)

\[
N(t) \bar{\alpha} = x + B(t) u
\]

is observable at \( t_0 \) if and only if

\[
\text{Im} \left( C^T (t_0) \right) + \text{Im} \left( N \left( t_0 \right) \right) = \mathbb{R}^n.
\]

**Proof:** Assume \( \text{Im} \left( C^T (t_0) \right) + \text{Im} \left( N \left( t_0 \right) \right) \neq \mathbb{R}^n \). There exists a nonzero \( x_0 \in \mathbb{R}^n \) such that

\[
C(t) x_0 = 0 \quad \text{and} \quad N(t_0) x_0 = 0.
\]

As a result

\[
(I - N(t) D)^{-1} \left( N(t_0) x_0 \right) \delta_0 = 0.
\]

Hence

\[
\text{Im} \left( C^T (t_0) \right) + \text{Im} \left( N \left( t_0 \right) \right) = \mathbb{R}^n.
\]
implies
\[
\text{Im}\left([C^T(t_0)\alpha_0 C^T(t_0)\alpha_1 C^T(t_0)\cdots C^T(t_0)]\right) = \mathbb{R}^n.
\]
To show the sufficiency, it is enough to show
\[
\text{Im}\left([C^T(t_0)\beta_0 C^T(t_0)\beta_1 C^T(t_0)\cdots C^T(t_0)]\right) = \mathbb{R}^n.
\]
Since
\[
\text{Im}\left([C^T(t_0)\alpha_0 C^T(t_0)\alpha_1 C^T(t_0)\cdots C^T(t_0)]\right) = \mathbb{R}^n
\]
any \(w \in \mathbb{R}^n\) can be expressed as
\[
C^T(t_0)\alpha + \sum_{i=0}^{n-1} \alpha_i C^T(t_0)\alpha_i = \mathbb{R}^n
\]
where \(\alpha, \alpha_i,\) and \(b_i\) are appropriately chosen real vectors. From Lemma 1 and the fact that \(C^T(t_0)\), \(C^T(t_0)\), \(\cdots, C^T(t_0)\) span \(\mathbb{R}^n\),
\[
\sum_{i=1}^{n} \alpha_i b_i
\]
can be rewritten as
\[
\alpha_i^T \left[C^T(t_0)\gamma + \sum_{i=0}^{n-1} \alpha_i^T \gamma_i \right]
\]
where \(\gamma\) and \(\gamma_i\) are some appropriately chosen real vectors. Equation (12) can be further put as \(\beta_i^T \gamma + \alpha_i^T \rho\) where \(\rho \in \mathbb{R}^n\). This is so because
\[
\alpha_i^T \alpha_0^T = \alpha_i^T N^T(t_0)(I + \rho_0)^T
\]
\[
= \alpha_i^T \left[(I + P^T_2)^T \right]^{-1} (I + \rho_0)^T.
\]
Since \(\rho\) can be further expressed as
\[
C^T(t_0)\epsilon + \sum_{i=0}^{n-1} \alpha_i^T \epsilon_i
\]
for some \(\epsilon \in \mathbb{R}^r\) and \(\epsilon_i \in \mathbb{R}^n\), we can continue this process until \(\sum_{i=1}^{n-1} \beta_i b_i\) in (11) is expressed solely as a linear combination of \(\beta_i\), \(i = 0, \cdots, n-1\). In the view of fact that \(w\) is chosen arbitrarily, the proof is thus completed.

Since condition (10) is equivalent to \(\ker(N(t_0)) \cap \ker(C(t_0)) = 0\), Theorem 1 is a natural extension of [7, Th. 6].

V. CONTROLLABILITY

Definition 2: The fast subsystem is controllable at \(t_0\) if for every \(w \in \mathbb{R}^n\), there exist \(u \in C^{n-1}, \tau > t_0\) such that \(x(\tau) = w\).

Again, the only difference between Definition 2 and the definition of controllability adopted in [7] is in the indication of initial time \(t_0\). Let \((I - N(t)D)^{-1}(B(t)u)\) be expressed as
\[
\sum_{i=0}^{n-1} F_i(t)u^{(i)}.
\]
It is straightforward to see that
\[
F_0(t) = (I - N(t)D)^{-1} B(t)
\]
and
\[
F_i(t) = (I - N(t)D)^{-1} (N(t)F_{i-1}(t))
\]
for \(i = 1, \cdots, n-1\).

We now present our theorem for controllability.

Theorem 2:
\[
N(t)\dot{x} = x + B(t)u
\]
is controllable at \(t_0\) if and only if there exists \(\tau > t_0\) such that
\[
\text{Im}(N(\tau)) + \text{Im}(B(\tau)) = \mathbb{R}^n.
\]

Proof: We first show that columns of \([N(\tau) : B(\tau)]\) span \(\mathbb{R}^n\) if and only if columns of
\[
\left[F_0(\tau), F_1(\tau), \cdots, F_{n-1}(\tau)\right]
\]
span \(\mathbb{R}^n\). If there exists nonzero \(d \in \mathbb{R}^n\) such that \(d^T N(\tau) = 0\) and \(d^T B(\tau) = 0\), it is clear that \(d^T F_i(\tau) = 0\), for \(i = 0, \cdots, n-1\).

Conversely, if columns of \([N(\tau) : B(\tau)]\) span \(\mathbb{R}^n\), columns of
\[
\left[B(\tau) : N(\tau)B(\tau), \cdots, N^{n-1}(\tau)B(\tau)\right]
\]
also span \(\mathbb{R}^n\), because \(N\) is strictly upper triangular. Following an analysis similar to the one presented in Lemma 2 yields that the columns of \([F_0(\tau) : F_1(\tau), \cdots, F_{n-1}(\tau)]\) span \(\mathbb{R}^n\).

Let arbitrarily chosen \(w \in \mathbb{R}^n\) be put as \(\sum_{i=0}^{n-1} F_i(\tau)u_i\). Choose \(u(t) = \sum_{i=0}^{n-1} (t - \tau)u_i / i!\) when \(t \geq t_0\) and arbitrary elsewhere. Then \(x(\tau) = w\). Conversely, if
\[
\text{Im}(N(\tau)) + \text{Im}(B(\tau)) \neq \mathbb{R}^n
\]
there exists a \(w \in \mathbb{R}^n\) that is not in
\[
\text{Im}\left([F_0(\tau), F_1(\tau), \cdots, F_{n-1}(\tau)]\right)
\]
Since \(x(\tau) = \sum_{i=0}^{n-1} F_i(\tau)u^{(i)}(\tau)\), obviously, \(w\) is not reachable.

It is interesting to observe that no dual relation exists between conditions (13) and (10). In fact, because \(N, B,\) and \(C\) are all analytic, the observability of
\[
N(t)\dot{x} = x + B(t)u
\]
at \(t_0\) implies the controllability of \(N^T(t)\dot{x} = x + C^T(t)u\) at \(t_0\); however, the converse is not true in general.

VI. CONCLUSIONS

We have characterized controllability and observability of linear time-varying singular systems in standard canonical form. Definitions of these two concepts are most naturally extended from the ones used in the time-invariant setting. The condition for observability at \(t_0\) depends solely on system parameters at \(t_0\) and is a natural extension of the existing time-invariant theory. Controllability at \(t_0\), however, may depend on system parameters at any point over the interval \((t_0, \infty)\). Consequently, the criterion for controllability is no longer the algebraic dual to that for observability in this time-varying framework. Regarding future research topics, there are two
interesting issues that can be further explored: deriving conditions of controllability and observability in terms of original systems parameters and developing numerical algorithms for verifying controllability and observability of the original system.

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Evaluation of Convergence Rate in the Central Limit Theorem for the Kalman Filter

Fazil A. Aliev and Levent Ozbek

Abstract—State-space models are used for modeling of many physical and economic processes. An asymptotic distribution theory for the state estimate from a Kalman filter in the absence of the usual Gaussian assumption is presented in [1]. They proved the central limit theorem for state estimators when the random terms in the model have arbitrary distribution. In this study, some convergence rates in the central limit theorem are given. These convergence rates are used for the development of a nonparametric test of the validity of the model.

Index Terms—Central limit theorem, convergence rate, Kalman filter, model validity tests, nonparametric tests.

I. INTRODUCTION

The linear state-space model with random noise is widely used for the modeling of dynamic systems. This paper presents an asymptotic distribution theory for the popular Kalman filter state estimator in the absence of the usual Gaussian assumptions on the random terms in the underlying model. We establish some rates for convergence results in the central limit theorem for the Kalman filter state estimator. Our results provide further insight into the asymptotic properties of the Kalman filter and serve as a basis for the development of nonparametric hypothesis testing procedures useful in the important problem of model validation. The state-space model considered here has the standard discrete-time form for $k = 1, 2, \cdots, n$.

$$x_k = \Phi_{k-1}x_{k-1} + w_{k-1}$$
$$z_k = H_kx_k + v_k.$$ (1)

Here $x_k \in \mathbb{R}^p$ is an unobserved vector of state variables and $z_k \in \mathbb{R}^m$ represents the system observations. $\Phi_{k-1}$, $H_k$ are matrices of appropriate dimensions. The terms $w_{k-1}$ and $v_k$ correspond to random disturbances in the dynamics and measurement errors, respectively. Also, the initial state $x_0$ is allowed to be random. The random terms $x_0$, $\{w_{k-1}\}$ and $\{v_k\}$ of the state space model (1) satisfy the following standard Kalman filter assumptions for all $k = 1, 2, \cdots, n$.

A1) $Ew_{k-1} = 0$ and $Ev_k = 0$.

A2) $E[(x_0 - E_x)(x_0 - E_x)^\top] \equiv P_0 \leq \beta I$

$E(w_{k-1}w_{k-1}^\top) \equiv Q_{k-1} \leq \beta I$

$\gamma I \leq E(v_kv_k^\top) \equiv R_k \leq \beta I$

for some $0 < \gamma \leq \beta < \infty$.

A3) $\{w_{k-1}\}$ and $\{v_k\}$ are each independent sequences, $w_{j-1}$ and $v_k$ are independent for all $j \neq k$, and $E(w_{k-1}v_k^\top) = 0$.

A4) $x_0$ is independent of $w_{k-1}$ and $v_k$.

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