An Iterative Learning Control Theory for a Class of Nonlinear Dynamic Systems*

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Key Words—Learning control; input update; prediction learning rule; current learning rule; convergence.

Abstract—An iterative learning control scheme is presented for a class of nonlinear dynamic systems which includes holonomic systems as its subset. The control scheme is composed of two types of control methodology: a linear feedback mechanism and a feedback learning strategy. At each iteration, the linear feedback provides stability of the system and keeps its state errors within uniform bounds. The iterative learning rule, on the other hand, tracks the entire span of a reference input over a sequence of iterations. The proposed learning control scheme takes into account the dominant system dynamics in its update algorithm in the form of scaled feedback errors. In contrast to many other learning control techniques, the proposed learning algorithm neither uses derivative terms of feedback errors nor assumes external input perturbations as a prerequisite. The convergence proof of the proposed learning scheme is given under minor conditions on the system parameters.

1. Introduction

THE VAST MAJORITIES of conventional control techniques have been devised for linear time-invariant systems which are assumed to be completely known and well understood. In most practical instances, however, the systems to be controlled are nonlinear and the basic physical processes in it are not completely known a priori. The situation becomes even more difficult when the unknown parameters of the system model change continuously or on a number of occasions. These types of parameter changes as well as model uncertainties are extremely difficult to manage even with the adaptive control techniques. The adaptive control techniques have been verified to work well for many dynamic systems with unknown but fixed parameters. However, their applications may not be easily extended to systems with unknown time-varying parameters.

As a way to overcome these difficulties, a number of iterative learning control techniques have been proposed, which improve tracking performance through a number of iterative operations (Arimoto et al. (1984); Bondi et al. (1988); Casalino and Bartolini (1984); Craig (1984); Kawamura et al. (1988); Kuc and Nam (1989); Messner et al. (1990); Miller et al. (1987); Miyamoto et al. (1988); Oh and Suh (1988)). In contrast to the adaptive control schemes which achieves asymptotic tracking in the time domain, the learning control schemes are capable of tracking the entire profile of the reference input because they are executed in an iterative manner. In other words, with the learning control schemes, the tracking errors in any phase of the time domain, either in the transient phase or in the steady state, can be made to be within the specified error bounds. Moreover, the iterative learning control schemes work equally well for unknown dynamic systems with or without parameter changes. Among the publications in the area of learning control, Asano et al. (1984) proposed a general learning method for a class of nonlinear systems whose input and output gain matrices are of linear time-invariant form. In their scheme, the time derivative of the system output error is used to modify the control input for the next iteration. Their learning algorithm converges if the learning gain matrix satisfies certain conditions and has been applied to a robot system via a simple nonlinear transformation. Bondi et al. (1980) developed a learning algorithm for robot systems which uses position, velocity and acceleration signals in updating the control input at each iteration. Their result is based on the high-gain feedback concept which sets up uniform upper bounds on the trajectory errors. This bound has been used to prove the convergence of their learning controller. Miller et al. (1987) applied the idea of CMAC learning (Albus (1981)) to learning rule and developed a general learning controller for robot manipulators. They also used a high-gain feedback control approach in providing stability of the closed-loop system at each iteration.

In this paper, we present a simple iterative learning control scheme which can be applied to a broad class of nonlinear systems—specifically to holonomic mechanical systems including robot systems. Its learning rule is based on feedback error signals from a linear PD controller, which provides several unique features. First, in contrast to many other learning schemes (Arimoto et al. (1984); Bondi et al. (1988); Miller et al. (1987)), the proposed learning controller does not use any derivative terms of system state/output errors, which implies that the measurement or estimation of the acceleration signal is not necessary when the learning control is applied to robot systems. Second, the learning controller does not require any external input perturbations. This feature enhances the robustness of the control scheme since the controller is free from any input perturbations which, when selected poorly, may jeopardize stable operation of the system. Third, the proposed learning controller essentially takes into account the dominant system dynamics in the sense that it uses feedback information in updating the feedforward learning input. Fourth, its convergence is derived under simple conditions which normally hold even when the system has some bounded uncertainties. Finally, bounded input disturbances have been accommodated in the learning controller and eventually eliminated at the final stage of learning.

This paper is organized as follows: Section 2 formulates the problem for a class of nonlinear systems. Section 3 derives the existence of a feedback law which keeps the tracking errors within the specified error bounds. Under stable closed loop systems, Section 4 develops the learning rules which drive the tracking error to zero.
2. Problem formulation

Consider a class of nonlinear dynamic systems which can be described as follows:

\[ \dot{x}(t) = f(x(t), t) + g(x(t), t)u(t), \quad (1) \]

where \( x(t) = [x_1(t), x_2(t)], \) \( x(t) \in \mathbb{R}^n, \) \( \dot{x}(t) \in \mathbb{R}^n, \)

\[ f(x(t), t) = \begin{bmatrix} x_2(t) \\ f_1(x(t), t) \end{bmatrix}, \quad g(x(t), t) = \begin{bmatrix} 0 \\ g_1(x(t), t) \end{bmatrix}. \]

\( f_1(x(t), t) \in \mathbb{R}^n \) and \( g_1(x(t), t) \in \mathbb{R}^{n \times n}. \) The constraints imposed on \( f_1(x(t), t) \) and \( g_1(x(t), t) \) are:

- \( f_1(x(t), t) \leq \kappa(t) |x_1(t) - x_1'(t)|, \)
- \( |g_1(x(t), t)| \leq a(t) |x_1(t) - x_1'(t)|, \)

where \( \kappa(t), a(t) \) are bounded positive functions. \( \cdot \) denotes the Euclidean norm and \( \| \cdot \| \) is an induced matrix norm defined as:

\[ \|A\| = \sup \{ |Av| / |v| \neq 0 \}. \]

(B) \( g_1(x(t), t) \) is assumed to be symmetric and positive definite for all \( t \in [0, t_f] \) and satisfies the inequality as follows:

\[ 0 < \lambda_1 \leq g_1(x(t), t) \leq \lambda_f, \]

where the inequality \( M \leq N \) is defined as \( \lambda_{\min}(M) \leq \lambda_{\min}(N). \)

Note that the described system covers a broad class of nonlinear systems which includes holonomic mechanical systems as its subset (Greenwood (1980)). For example, consider a general n-link robot dynamic system

\[ D(q(t)) \dot{q}(t) + B(q(t), \dot{q}(t)) + G(q(t)) = T(t), \]

where \( D(q(t)) \) is the \( n \times n \) inertia matrix, \( B(q(t), \dot{q}(t)) \) is the Coriolis plus centripetal force vector, \( G(q(t)) \) is the gravity vector and \( T(t) \) is the input torque/force vector with generalized joint variable \( q(t) \in \mathbb{R}^n. \) When the joint angle vector \( q(t) \) and the joint velocity vector \( \dot{q}(t) \) form the state vector

\[ \begin{bmatrix} q_1(t) \\ \dot{q}_1(t) \end{bmatrix} = \begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix}, \]

it yields a set of first order differential equations

\[ \dot{x}(t) = f(x(t)) + g(x(t))u(t), \]

where

\[ f(x(t)) = \begin{bmatrix} x_2(t) \\ -D^{-1}(x(t))(B(x(t)) + G(x(t))) \end{bmatrix}, \quad g(x(t)) = \begin{bmatrix} 0 \\ -D^{-1}(x(t)) \end{bmatrix}. \]

and

\[ u(t) = T(t). \]

Note that the inertia matrix \( D(q(t)) \) is positive definite and invertible. In addition, when the joint velocity is finite in a domain \( W, \) which is closed and bounded subset of \( \mathbb{R}^n, \) then, a piecewise continuous control input \( u(t) \in \mathbb{R}^n, \ (t \in [0, t_f]) \)

for the uncertain system (1) with which the system state \( x(t) \) follows \( x_d(t) \) with a given accuracy \( \epsilon, \)

\[ |x(t) - x_d(t)| \leq \epsilon, \quad \text{for all} \ t \in [0, t_f], \]

where \( j \) denotes the \( j \)-th iteration.

In the following, the operating conditions such as sampling frequency, initial configuration, etc. are assumed to be prespecified and the uncertain system (1) is assumed to be repetitive for all \( t \in [0, t_f]. \) The desired state/input trajectory and the actual state/input trajectory of the system at the \( j \)-th iteration are denoted by \( \{x_d(t), u_d(t)\} \) and \( \{x(t), u(t)\}, \) respectively. Additionally, the following notations will be used in the sequel:

\[ \|A\|_m = \max_{1 \leq i \leq n} |A_i|, \]

where \( v \in \mathbb{R}^n, \) \( A \in \mathbb{R}^{n \times n} \) and \( a(t) > 0, \) \( a(t) \) denotes is defined as.

3. Stabilization of the dynamic system

As stated in the Introduction, the tracking capability of the iterative learning process hinges upon the stability of the closed-loop system at each iteration. The aim of this section is, therefore, to design a feedback controller which, when applied to target system (1), forms a stable closed-loop system. The feedback controller used in this paper is based on the high-gain feedback concept (Gusev(1988)) which keeps the control structure simple and limits the tracking error within a uniform bound. The following theorem establishes the theoretical foundation on which the construction of the feedback control scheme is based.

**Theorem 1.** Suppose that \( f(x(t), t) \) and \( g(x(t), t) \) in system (1) satisfy conditions (A) and (B) for all \( t \in [0, t_f]. \) Then there exists a feedback control \( u(t) = K(x_d(t) - x(t)) \) for system (1), which keeps the state tracking error, \( x(t) - x_d(t) \), within a uniform bound as follows:

\[ |x(t) - x_d(t)| \leq \varepsilon(e > 0) \quad \text{for all} \ t \in [0, t_f]. \]

(2)

Specifically, let the feedback gain matrix \( K \) be

\[ K = [adb^{-1}l_{n \times n}; db^{-1}l_{n \times n}], \]

where \( a > 0, \ b \geq \lambda_1 \) and

\[ d = (\mu + a) \frac{b}{\lambda_1}, \]

in which \( \mu > a + \kappa_m/a + 2\kappa_m \). Here, \( L_{n \times n} \) is the \( n \times n \) identity matrix.

Then, the state tracking error is bounded as:

\[ |x(t) - x_d(t)| \leq \sqrt{1 + 4a \frac{\lambda_1}{a(v - \nu)}} \leq \varepsilon, \]

(4)

where \( \nu = a + \kappa_m/a + 2\kappa_m \) and \( u_d(t) \) denotes the desired control input for the desired state trajectory \( x_d(t). \)

The proof of Theorem 1 is similar to Gusev’s (1988) and is given in Appendix A.

Note that from (4) that as long as the desired input \( u_d(t) \) is bounded, the state error is bounded, which implies that the closed-loop system is BIBO (or BIBS) stable with respect to the pair \( \{u_d(t), x_d(t) - x(t)\}. \) If we set \( b = \lambda_1 \) in (3), then it follows that

\[ d \geq (\mu + a) \frac{\lambda_2}{\lambda_1} + 1, \]

where \( \lambda_2/\lambda_1 \) and \( \lambda_2 - \lambda_1 \) denote the input gain band-ratio and bandwidth. The inequality (5) implies that the lower bound of \( d \) is linearly related to \( \lambda_2/\lambda_1, \) or to \( \lambda_2 - \lambda_1. \) That is, the
larger the band-ratio (or bandwidth), the larger the feedback gain is required to keep the tracking error within the prespecified error bound.

4. An iterative learning control scheme

As stated in Section 3, the high-gain feedback technique offers a simple and useful approach to the nonlinear stability problem. However, the tracking error \( \varepsilon'(t) = x_d(t) - x'(t) \) in (4) cannot be made arbitrarily small unless the feedback gain \( d \) is set infinitely large (see (3), (4)). In practical applications, the feedback gain \( d \) cannot be made arbitrarily large to avoid actuator saturation, unmodelled dynamics, noise vulnerability, etc. Our aim is then to introduce a learning controller in the feedforward loop, which provides tracking of the entire profile of \( x_d(t) \) over a sequence of iterative operations. At the initial stage of learning an appropriate feedback gain is selected that ensures system stability with an initial error bound of \( \varepsilon \) much greater than the prespecified target tolerance \( \varepsilon \). Then, the feedforward learning controller reduces the system state error, \( \varepsilon'(t) \), every iteration and finally makes it smaller than \( \varepsilon \). The following theorem formalizes the learning operation by means of feedforward input correction strategy.

Theorem 2. Let the control input \( u'(t) \) at the \( j \)th iteration be a linear combination of feedback and feedforward control input as follows:

\[
u'(t) = u_d(t) + u_f(t),
\]

where \( u_d(t) \in \mathbb{R}^n \) is the feedback error input, \( K(x_d(t) - x'(t)) \) and \( u_f(t) \in \mathbb{R}^n \) is the feedforward learning input which will be generated from a learning controller. Then the state tracking error of the nonlinear system (1) combined with the above controller (6) is bounded as follows:

\[
|\varepsilon'(t)| \leq \frac{1}{4d^2} \frac{\lambda_2|\mu_d'(t) - \mu_d(t)|}{a} \leq \varepsilon,
\]

where \( \mu > v = a + (2 + 1/a)\kappa_m + a_m|u_d(t)| \). As in (3), \( a > 0, b \geq \lambda_2 \) and \( d \geq (\mu + a)b/\lambda_2 \).

The proof is given in Appendix B.

In view of Theorem 2, the tracking error \( \varepsilon'(t) = x_d(t) - x'(t) \) is bounded by a term which depends linearly on \( |\mu_d'(t) - \mu_d(t)| \) in (7) and hence it can be made arbitrarily small as long as a sequence \( \{\mu_d'(t)\} \) converges to \( \mu_d(t) \) for all \( t \in [0, t_f] \). Then, the original tracking problem reduces to the problem of finding a feedforward input sequence \( \{u_f(t)\} \) which converges to \( \mu_d(t) \) uniformly for all \( t \in [0, t_f] \).

To this end, we propose to generate a sequence of feedforward input \( \{u_f(t)\} \) which minimizes the index functional

\[
J_f = \frac{1}{2} \sum_{i=1}^{n} |u_d(t) - u_d(t)|^2,
\]

for \( t \in [0, t_f] \). Applying the gradient descent method to this problem, we obtain an update equation for \( u_f(t) \) at the \( j \)th-iteration:

\[
u_f^{j+1}(t) = u_f^j(t) - \beta \frac{\delta J_f}{\delta u_f} = u_f^j(t) + \beta (u_d(t) - u_d(t)), \tag{8}
\]

where \( \beta \) is a training factor, which must be \( 0 < \beta < 2 \) for (8) to converge (see Appendix C). As long as the desired control input \( u_d(t) \) is known, the above learning rule will drive the state error to zero. The learning rule (8) can be interpreted as a nonlinear compensating controller in the feedforward path. However, since \( u_d(t) \) is not known a priori due to unknown system dynamics, we can not use (8) directly. Hence, we propose to replace the unknown error term \( u_d(t) \) in (8) with the known feedback error term \( u_f(t) \) as follows:

\[
u_f^{j+1}(t) = u_f^j(t) + \beta u_f^j(t), \quad j = 1, 2, 3, \ldots \tag{9}
\]

where \( u_f(0) = 0 \) for all \( j \) and \( u_f(t) = 0 \) for all \( t \in [0, t_f] \).

The learning rule (9) estimates the nonlinear function \( u_d(t) \) in the form of \( u_f(t) \) and tries to provide nonlinear compensation to system (1). Moreover, after complete learning with zero tracking error, an approximate inverse dynamics model will be formed for \( x_d(t) \) in the feedforward controller (see Fig. 1). The training factor of positive constant, \( \beta \), is usually set less than one for sensitivity considerations. The learning rule (9) is referred to as the prediction learning rule (PLR), because the \( x_d(t) - x'(t) \) is used to update \( u_f^{j+1}(t) \).

The next theorem establishes the convergence proof of the suggested learning control scheme.

Theorem 3. Assume that \( u_d(t) \) and \( g^{-1}_f(x'(t), t) \) are bounded as follows:

\[
|u_d(t)|_{\infty} = u_0 < \infty,
\]

\[
\|g^{-1}_f(x'(t), t)|_{\infty} = r_0 < \infty.
\]

Furthermore, assume that the feedback gains \( a, b \) and \( d \) given in Theorem 2 satisfies the following inequalities:

\[
l_1 = (1 - \beta)db^{-1} - 2r_0 > 0,
\]

\[
l_2 = (2 - \beta)db^{-1} - \frac{\left(r_0 + 2b^{-1}\lambda_1\right)}{\lambda_2} > 0,
\]

\[
l_3 = 1 - \frac{\left(r_0 + \frac{1}{a}\lambda_1\right)^2 + 4\lambda_2}{p} > 0,
\]
where $p$ and $q$ are defined as $p = \min \{a_l, l_2\}$ and $q = (n_m + \omega_0\mu_0)/\lambda_1$, respectively. Then, with the iterative learning algorithm (6) and (9), the system (1) converges as follows:

(i) $V^{+1}(t) \leq V(t)$,

(ii) $\lim_{t \to \infty} x'(t) = x_d(t)$,

(iii) $\lim_{t \to \infty} u'(t) = u_d(t)$,

where the index functional $V(t)$ is defined as

$$V(t) = \int_{t_0}^{t} b \tilde{\varphi}(t)^{T} \tilde{\varphi}(t) dt,$$

for all $t \in [0, t_J]$ in which $\tilde{\varphi}(t) = u_d(t) - u'(t)$.

The proof is given in Appendix D.

When the previous state error $x_d(t) - x'(t)$ is replaced with the current state error $x_d(t) - x^{'i}(t)$ in (9) an improved learning rule is achieved as follows:

$$u_{d}^{i} = u_d(t) + \beta u^{i}_{d}, \quad i = 1, 2, 3, \ldots$$

Theorem 4. Assume that $u_d(t)$ and $g_{d}^{i}(x'(t), t)$ are bounded as in Theorem 3. Assume also that the feedback gains $a$, $b$ and $d$ satisfy the following inequalities:

$$l_1 = (2 + \beta)db - 2n_0 > 0,$$

$$l_2 = (2 + \beta)db - \left(2 + \frac{2}{\lambda_1}\right) > 0,$$

$$l_3 = 1 - \left(\frac{1}{1 + \alpha} + 1\right) > 0,$$

where $p$ and $q$ are defined in Theorem 3. Then, with the set of iterative learning algorithm (6) and (10), the system (1) converges as follows:

(i) $V^{+1}(t) \leq V(t)$,

(ii) $\lim_{t \to \infty} x'(t) = x_d(t)$,

(iii) $\lim_{t \to \infty} u'(t) = u_d(t)$,

where the index functional $V(t)$ is defined in Theorem 3.

The proof is given in Appendix E.

Remark. Comparing Theorem 3 and Theorem 4, we notice that the feedback gain $d$ used for CLR may be made significantly less than that for PLR. This fact is expected because the feedback error term is used to update the feedforward learning input. Specifically, when we use the current learning rule, the schematic diagram looks similar to the "feedback error learning method" in Miyamoto et al. (1988) but the difference lies in the learning operation in the feedforward controller. In the model given in Miyamoto et al. (1988), the update mechanism is essentially a parameter adaptive scheme which is based on a single layer perceptron, while the current learning rule in our scheme corrects the feedforward input directly by adding a fraction of feedback error. In implementing the feedforward controller, an associative memory technique has been used to reduce the required memory size (Kue et al. 1989)). The distinctive feature of the learning control scheme is that the precise description of the system dynamics is not necessary, which makes the controller flexible under varying operation conditions. Moreover, only mild conditions (such as the boundedness of $u_d(t)$ and $g_{d}^{i}(x'(t), t)$, etc.) are required to establish the convergence proof of the controlled system. In fact, the boundedness of $g_{d}^{i}(x'(t), t)$ implies that the rate of change of the kinetic energy matrix is bounded for holonomic mechanical systems. It is also important to point out that the learning controller is robust to bounded input disturbances, since it learns and rejects the bounded disturbances over a sequence of iterations. Let us assume that the disturbance vector function $v(t) \in R^n$ is unknown, except that it is bounded as

$$|v(t)|_{m} < v_0,$$

and $v(t)$ is uncorrelated with the state of the control system. Then the resulting system is described as

$$\dot{x}'(t) = f(x'(t), t) + g(x'(t), t)(u_d(t) + v(t)).$$

Applying the same learning control input (6) we have developed, we obtain

$$|x'(t)|_{m} < \sqrt{1 + \frac{4a}{ac} bu^{*}} (see Appendix F).$$

where $u_{d}(t) = u_d(t) - v(t)$. In view of Theorem 3 and 4, with $u_d$ being replaced with $u_d = u_{d} + v_0$, it is trivial to show that the disturbed feedback error input converges to zero, $|x_d(t) - u_d(t)| \rightarrow 0$, for all $t \in [0, t_J]$ as $j \rightarrow \infty$.

Because the admissible state space depends on the physical characteristics of the controlled system, the physical constraints such as actuator limits must be considered in computing the desired state trajectories $x_d(t)$ as well as the feedback gain matrices. If an upper bound of $u_d(t)$ is not known a priori, it is impossible to choose an appropriate feedback gain $d$. In this case, the physical limit of the actuator input $u^{*}$ can be used instead of $u_d(t)$. The actuator limit $u^{*}$ imposes a conservative upper bound on the control input $u_d(t)$, because $u_d(t)$ must satisfy $|u_d(t)|_{m} \leq u^{*}$ to be a feasible input. When $u^{*}$ is used for a bound of feedback control input, an upper bound of $d$ can be set as $d < \sqrt{1 + \frac{4a}{ac} bu^{*}}$ (see Appendix F).

5. Conclusion

An iterative learning control scheme is presented in this paper, which is capable of tracking the entire profile of the desired state over a sequence of iterations. It is based on two control mechanisms: the linear feedback controller which ensures stability of the closed-loop system and the feedforward learning controller which is in charge of tracking. The features of the learning control scheme are as follows. First, in contrast to many other learning schemes, the proposed learning controller does not use any derivative terms of system state/output errors, which implies that the measurement or estimation of the acceleration signal is not necessary when the learning control is applied to robot systems (e.g. Kuc et al. 1991). Second, the learning control mechanism does not require any external input perturbations. It enhances the robustness of the control scheme since the controller is free from any input perturbations which, when selected poorly, may jeopardize stable operation of the system. Third, the learning structure essentially takes into account the dominant system dynamics in the sense that it uses feedback information in updating the feedforward learning input. Fourth, its convergence under simple conditions which normally hold even when the system has some bounded uncertainties. On the contrary, under the assumptions imposed on Arimoto's (Arimoto 1984) the learning rule may not work if the system contains uncertainties in the system model and in the system parameters. Finally, bounded input disturbances have been accommodated in the learning controller and eventually eliminated at the final stage of learning.
References


Appendix A

Proof of Theorem 1. With \( u(t) = K(x_d(t) - x(t)) \), the matrix differential equation for the error \( x(t) = x_d(t) - x(t) \) is described as follows:

\[
\dot{x}(t) = \begin{bmatrix} 0 & L \end{bmatrix} \begin{bmatrix} x(t) \\ x_d(t) - x(t) \end{bmatrix} + \begin{bmatrix} 0 \\ u(t) \end{bmatrix}
\]

where

\[
p(t) = b - g(x(t), t),
\]

\[
q(t) = f_1(x_d(t), t) - f_1(x(t), t) + g_1(x_d(t), t) u(t).
\]

Since \( a > 0 \) and \( b > b_2 \), it follows that \( 0 < b^{-1} b < p(t) \leq b^{-1} b_2 \). If we set \( w_0(t) = \dot{x}_d(t) \) and \( w_1 = \dot{x}_d(t) + x(t) \), then

\[
\dot{w}_0(t) = \dot{x}_0(t) = -a w_0(t) + w_0(t),
\]

and

\[
\dot{w}_1(t) = \dot{x}_1(t) + \dot{x}_2(t) = a \dot{x}_1(t) + a \dot{x}_2(t) - a \dot{x}_1(t) + q(t)
\]

Since the feedback gain \( d \) satisfies

\[
d \geq (\mu + a) b / \lambda_1 \quad \text{for a given} \quad \mu > a + 2 \kappa_m .
\]

the term \( (\mu + a) b / \lambda_1 \) is negative semi-definite.

Now, from the fact that

\[
\frac{d[v(t)]}{dt} = \frac{\partial v(t)}{\partial t} + \frac{\partial v(t)}{\partial x(t)} \frac{d[x(t)]}{dt}
\]

it becomes

\[
\frac{d[w(t)]}{dt} = \frac{\partial w(t)}{\partial t} + \frac{\partial w(t)}{\partial x(t)} \frac{d[x(t)]}{dt}
\]

and

\[
\frac{d[w(t)]}{dt} \leq -\mu |w(t)| + |q(t)|
\]

(because \( (\mu + a) / d \) is negative semi-definite)

\[
\leq -\mu |w_0(t)| + |w_1(t)| + d |w_2(t)|
\]

\[
+ \kappa_m |x_d(t)| + \lambda_2 |u(t)|
\]

(therefore \( f(x(t), t) \) is Lipschitz continuous)

\[
\leq -\mu |w_0(t)| + (a^2 + \kappa_m + \alpha \kappa_m) |w_1(t)|
\]

\[
+ \lambda_2 |u(t)|
\]

Now, if we define \( y(t) = \max (a^{-1} |w_0(t)|, |w_1(t)|) \) for \( a > 0 \), then

\[
|w_0(t)| \leq a y(t), \quad |w_1(t)| \leq a y(t) \quad \text{and} \quad y(t) \text{is piecewise continuous}.
\]

Moreover,

\[
\frac{dy(t)}{dt} = \frac{a^{-1} |w(t)|}{dt} \frac{d[w(t)]}{dt}
\]

\[
\leq \frac{a^{-1} |w_0(t)|}{dt} + \frac{d[w_1(t)]}{dt}
\]

which implies from the above that

\[
\frac{dy(t)}{dt} \leq \begin{cases} 
-\mu y(t) + a y(t) & \text{if } a^{-1} |w_0(t)| \geq |w_1(t)| \\
0 & \text{if } a^{-1} |w_0(t)| < |w_1(t)|.
\end{cases}
\]

Hence, it follows that

\[
y(t) \leq e^{-\mu t} + \frac{1 - e^{-\mu t} v}{a - v}
\]

\[
\leq \frac{\lambda_2 |u(t)|}{a - v}
\]

where \( \mu > v \).

If we set \( y(0) = \frac{\lambda_2 |u(0)|}{a - v} \), then

\[
y(t) \leq \frac{\lambda_2 |u(t)|}{a - v}
\]

\[
= e^{-\mu t} + \frac{1 - e^{-\mu t} v}{a - v}
\]

\[
\leq \frac{\lambda_2 |u(t)|}{a - v}
\]

for all \( t \in [0, t_f] \).

On the other hand, if \( |\xi(t)| \leq \lambda_2 |u(t)|/a \), then \( \gamma(t) = \max (a^{-1} |w_0(t)|, |w_1(t)|) \),

\[
\gamma(t) = \gamma(0) e^{-\mu t} + \frac{1 - e^{-\mu t} v}{a - v}
\]

\[
\leq \frac{\lambda_2 |u(t)|}{a - v}
\]

\[
\leq \frac{\lambda_2 |u(t)|}{a - v}
\]

which implies from (A1) that

\[
\gamma(t) \leq \frac{\lambda_2 |u(t)|}{a - v}
\]

Hence,

\[
|\xi(t)| \leq \sqrt{\|\xi(t)\|^2 + \gamma(t)}
\]

\[
= \sqrt{\|\xi(t)\|^2 + \gamma(t)}
\]

\[
\leq \sqrt{\|w_1(t)\|^2 + \gamma(t)}
\]

\[
\leq \sqrt{\|w_0(t)\|^2 + \gamma(t)}
\]

\[
\leq \sqrt{\|\gamma(t)\|^2 + \gamma(t)}
\]

\[
= \sqrt{1 + a^2 \gamma(t)}
\]

\[
\leq \sqrt{1 + 4a^2 \lambda_2 |u(t)|}
\]
Therefore, for fixed $a$, we can select $\mu$ such that
\[ \sqrt{1 + 4a^2} \frac{\lambda_2[u(t)]_m}{\mu - \nu} \leq \varepsilon \quad \text{for all} \quad t \in [0, T]. \]
Q.E.D.

Appendix B

Proof of Theorem 2. As in (6), the control input to be used at the $j$th iteration is
\[ u'(t) = u(t) + u_2(t) = K(x(t) - x'(t)) + u_2(t). \]
Substituting this input into the system equation (1), we have
\[ \dot{x}(t) = \begin{bmatrix} 0 & I \end{bmatrix} x(t) + \begin{bmatrix} 0 \end{bmatrix}, \quad (B1) \]
where
\[ p(t) = b \dot{g}_i(x(t), t), \quad q(t) = (f_i(x(t), t) - f_i(x'(t), t)) + g_i(x(t), t)u(t) - g_i(x'(t), t)u'_2(t). \]
Now, as in Appendix A, let's define $\alpha(t) = \dot{x}_2(t)$ and $\alpha_2(t) = \dot{x}_2(t) + \dot{x}_3(t)$ for all $j$. Then, following the same reasoning as in Appendix A, it follows that
\[ \frac{d\alpha(t)}{dt} = -a_1\alpha(t) + \alpha_2(t) \]
and integration by parts,
\[ \frac{d\alpha(t)}{dt} = -a_1\alpha(t) + \alpha_2(t) \]
which implies that
\[ \gamma(t) = \gamma(0)e^{-(\alpha_1 + \nu)t} + \frac{1 - e^{-(\alpha_1 + \nu)t}}{\alpha_1} \lambda_2[u(t)]_m \leq \varepsilon. \]
Hence, it follows that
\[ \gamma(t) \leq \gamma(0)e^{-\nu T} + \frac{1 - e^{-\nu T}}{\alpha_1} \lambda_2[u(T)]_m, \]
where $\nu = a + \frac{2}{1 + a}(\lambda_2[u(T)]_m)$. Then, as in Appendix A, $|\dot{x}'(t)| \leq \lambda_2[u(T)]_m/(1 + a)(\mu - \nu)$ implies that
\[ |\dot{x}'(t)| \leq \sqrt{1 + 4a^2} \frac{\lambda_2[u(T)]_m}{\mu - \nu}. \]
Therefore, we can select $\mu$ such that
\[ \sqrt{1 + 4a^2} \frac{\lambda_2[u(T)]_m}{\mu - \nu} \leq \varepsilon. \]
Q.E.D.

Appendix C

Determining bounds on the training factor $\beta$. Subtracting both sides of equation (8) from $u_2(t)$, we obtain
\[ (u(t) - u_2(t)) = (u(t) - u_2(t)) - \beta(u(t) - u_2(t)) = (1 - \beta)(u(t) - u_2(t)). \]
Hence, in order for the update rule (8) to converge, $\beta$ must be such that $0 < \beta < 2$.

Appendix D

Proof of Theorem 3. Let
\[ z(t) = \alpha_1(t) + \alpha_2(t), \quad \beta(t) = u(t) - u_2(t), \]
and
\[ \Delta \beta(t) = \beta(t) - \beta(t), \]
Then, the learning rule (9) is described as
\[ \Delta \beta(t) = u(t) - u_2(t) - \beta(t) = \beta(t). \]
Next, note from (B1) that
\[ \dot{x}(t) = \alpha_1(t) + \alpha_2(t) = \alpha_2(t) + \alpha_2(t) = \alpha_2(t) + \alpha_2(t) + \alpha_2(t) + \alpha_2(t), \]
where
\[ \alpha_2(t) = f_i(x(t), t) - f_i(x'(t), t) \]
and $\alpha_2(t) = g_i(x(t), t)u(t) - g_i(x'(t), t)u'(2(t))$. 
Now, when $j = 1$, $u_2(t) = 0$ and
\[ V(t) = \int_0^t b \beta d\beta \leq \varepsilon \quad \text{for all} \quad t \in [0, T]. \]
If we define $\Delta V(t) = V(t+1) - V(t)$, then
\[ \Delta V(t) = \int_0^t b \beta d\beta - \varepsilon \quad \text{for all} \quad t \in [0, T]. \]
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where
\[ \alpha_2(t) = f_i(x(t), t) - f_i(x'(t), t), \quad \beta(t) = u(t) - u_2(t), \]
and $\alpha_2(t) = g_i(x(t), t)u(t) - g_i(x'(t), t)u'(2(t))$. 
Now, when $j = 1$, $u_2(t) = 0$ and
\[ V(t) = \int_0^t b \beta d\beta \leq \varepsilon \quad \text{for all} \quad t \in [0, T]. \]
If we define $\Delta V(t) = V(t+1) - V(t)$, then
\[ \Delta V(t) = \int_0^t b \beta d\beta - \varepsilon \quad \text{for all} \quad t \in [0, T]. \]
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Then, the learning rule (9) is described as
\[ \Delta \beta(t) = u(t) - u_2(t) - \beta(t) = \beta(t). \]
Next, note from (B1) that
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where
\[ \alpha_2(t) = f_i(x(t), t) - f_i(x'(t), t), \quad \beta(t) = u(t) - u_2(t), \]
and $\alpha_2(t) = g_i(x(t), t)u(t) - g_i(x'(t), t)u'(2(t))$. 
Now, when $j = 1$, $u_2(t) = 0$ and
\[ V(t) = \int_0^t b \beta d\beta \leq \varepsilon \quad \text{for all} \quad t \in [0, T]. \]
If we define $\Delta V(t) = V(t+1) - V(t)$, then
\[ \Delta V(t) = \int_0^t b \beta d\beta - \varepsilon \quad \text{for all} \quad t \in [0, T]. \]
Hence, in order for the update rule (8) to converge, $\beta$ must be such that $0 < \beta < 2$. 

Appendix D

Proof of Theorem 3. Let
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and
\[ \Delta \beta(t) = \beta(t) - \beta(t) = \beta(t). \]
Then, the learning rule (9) is described as
\[ \Delta \beta(t) = u(t) - u_2(t) - \beta(t) = \beta(t). \]
Next, note from (B1) that
\[ \dot{x}(t) = \alpha_1(t) + \alpha_2(t) = \alpha_2(t) + \alpha_2(t) = \alpha_2(t) + \alpha_2(t) + \alpha_2(t) + \alpha_2(t), \]
where
\[ \alpha_2(t) = f_i(x(t), t) - f_i(x'(t), t), \quad \beta(t) = u(t) - u_2(t), \]
and $\alpha_2(t) = g_i(x(t), t)u(t) - g_i(x'(t), t)u'(2(t))$. 
Now, when $j = 1$, $u_2(t) = 0$ and
\[ V(t) = \int_0^t b \beta d\beta \leq \varepsilon \quad \text{for all} \quad t \in [0, T]. \]
If we define $\Delta V(t) = V(t+1) - V(t)$, then
\[ \Delta V(t) = \int_0^t b \beta d\beta - \varepsilon \quad \text{for all} \quad t \in [0, T]. \]
Hence, the sequence \( \{W(t)\} \) is monotonically decreasing, which confirms (i). Because \( V(t) \) is bounded, the monotonically decreasing nonnegative sequence \( \{V(t)\} \) converges to a nonnegative fixed value, and consequently \( \Delta V(t) \to 0 \) as \( j \to \infty \). This implies that \( z'(t), \tilde{z}'(t) \) and hence \( \tilde{z}'(t) \) converge to zero for all \( t \in [0, \tau] \), because

\[
\Delta V(t) \leq -z(t)g \tilde{z}(t) + z(t)g \tilde{z}(t) - a \tilde{z}(t)(2 - \beta)db^{-l}I - ag \tilde{z}(t)(x(t), t) \tilde{z}(t)
\]

\[
\leq 0.
\]

This implies (ii). Finally \( \tilde{z}'(t) \to 0 \) because \( \tilde{z}'(t) \to 0 \) for all \( t \in [0, \tau] \). Then, from the error equation (C2), we have (iii).

Q.E.D.

Appendix E

Proof of Theorem 4. If we define \( z(t), \tilde{z}(t), \Delta \tilde{z}(t) \) as in the proof of Theorem 3, then, from the CLR (10), we have

\[
\Delta \tilde{z}(t) = -\beta u_0 \tilde{z}(t) = -\beta db^{-l}z^{-1}(t).
\]

Then, the error system at the \((j + 1)\)th iteration is

\[
z^{-1}(t) = -db^{-l}g_1(x^{-1}(t), t)z^{-1}(t) + \tilde{f}^{-1}(t) + \tilde{g}^{-1}(t)u_0(t)
\]

\[
+ z^{-1}(t) + g_1(x^{-1}(t), t) \tilde{u}(t)
\]

\[
= -(1 + \beta)db^{-l}g_1(x^{-1}(t), t)z^{-1}(t) + \tilde{f}^{-1}(t) + \tilde{g}^{-1}(t)u_0(t) + z^{-1}(t) + g_1(x^{-1}(t), t) \tilde{u}(t),
\]

where \( \tilde{f}^{-1}(t) = f(x(t), t) - f(x^{-1}(t), t) \) and \( \tilde{g}^{-1}(t) = g_1(x(t), t) - g_1(x^{-1}(t), t) \). When \( j = 1 \), \( V(t) \) is bounded as in Appendix D, because \( u_0(t) \) is bounded. If we define \( \Delta V^{-1}(t) = V^{-1}(t) - V(t) \), then, as in Appendix D, we have

\[
\Delta V^{-1}(t) = \int_0^b \Delta \tilde{w}^\top(t) \Delta \tilde{w}(t) + 2 \Delta \tilde{w}^\top(t) \tilde{w}(t) dt
\]

\[
= \int_0^b ((\beta db^{-l})z^{-1}(t) + \tilde{f}^{-1}(t) + \tilde{g}^{-1}(t)u_0(t) + z^{-1}(t) + g_1(x^{-1}(t), t) \tilde{u}(t)) dt
\]

\[
= z^{-1}(t)g_1(x^{-1}(t), t)z^{-1}(t)
\]

\[
- a \tilde{z}^{-1}(t)(x(t), t) \tilde{u}(t).
\]

The rest of the proof follows the same argument as in the proof of Theorem 3.

Q.E.D.

Appendix F

Determining the upper bound of feedback gain \( d \). The feedback error input, \( u(t) \), is given as follows:

\[
u(t) = Kf(t) = adb^{-l}f(t) + db^{-l}f(t) = db^{-l}u(t) = db^{-l}u(t) + \tilde{u}(t).
\]

Hence, from the proof of Theorem 2 and (7), it follows that

\[
\|u(t)\| = db^{-l}\|u(t)\| \leq \|d\| \|u(t)\| \leq \|u(t)\| + \|\tilde{u}(t)\|.
\]

\[
\leq db^{-l} \frac{3\|u(t)\|}{\|u(t)\|} (\mu - \nu)
\]

\[
\leq db^{-l} \frac{\alpha e}{\sqrt{1 + 4\alpha^2}} (\text{from (7)}).
\]

Now, in order for \( u(t) \) to become a feasible input, it is sufficient if its upper bound is bounded as

\[
\|u(t)\| \leq db^{-l} \frac{\alpha e}{\sqrt{1 + 4\alpha^2}} \leq u^*.
\]

which in turn implies that

\[
d \leq \sqrt{1 + 4\alpha^2} \frac{e}{\alpha}.
\]