Planck Mass Plasma Vacuum Conjecture

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As an alternative to string field theories in R10 (or M theory in R11) with a large group and a very large number of possible vacuum states, we propose SU2 as the fundamental group, assuming that nature works like a computer with a binary number system. With SU2 isomorphic to SO3, the rotation group in R3, explains why R3 is the natural space. Planck's conjecture that the fundamental equations of physics should contain as free parameters only the Planck length, mass and time, requires to replace differentials by rotation – invariant finite difference operators in R3. With SU2 as the fundamental group, there should be negative besides positive Planck masses, and the freedom in the sign of the Planck force permits to construct in a unique way a stable Planck mass plasma composed of equal numbers of positive and negative Planck mass particles, with each Planck length volume in the average occupied by one Planck mass particle, with Planck mass particles of equal sign repelling and those of opposite sign attracting each other by the Planck force over a Planck length. From the thusly constructed Planck mass plasma one can derive quantum mechanics and Lorentz invariance, the latter for small energies compared to the Planck energy. In its lowest state the Planck mass plasma has dilaton and quantized vortex states, with Maxwell's and Einstein's field equations derived from the antisymmetric and symmetric modes of a vortex sponge. In addition, the Planck mass plasma has excitonic quasiparticle states obeying Dirac's equation with a maximum of four such states, and a mass formula of the lowest state in terms of the Planck mass, permitting to compute the value of the finestructure constant at the Planck length, in surprisingly good agreement with the empirical value.

Key words: Planck Scale Physics; Analog Models of General Relativity and Elementary Particles Physics.

1. Introduction: Fundamental Group and Non-Archimedean Analysis

String (resp. M) theory unification attempts are characterized by going to ever larger groups and higher dimensions. Very different, but at the present time much less pursued attempts are guided by analogies between condensed matter and elementary particle physics, and analogies between condensed matter physics and general relativity [1 – 9]. These analogies suggest that the fundamental group of nature is small, with higher groups and Lorentz invariance derived from the dynamics at a more fundamental level. A likewise dynamic reduction of higher symmetries to an underlying simple symmetry is known from crystal physics, where the large number of symmetries actually observed are reduced to the spherical symmetry of the Coulomb field.

We make here the proposition that the fundamental group is SU2, and that by Planck's conjecture the fundamental equations of physics contain as free parameters only the Planck length r_p, the Planck mass m_p and Planck time t_p (G Newton's constant, h Planck's constant, c the velocity of light):

$$r_p = \sqrt{\frac{\hbar G}{c^3}} \approx 10^{-33}\ \text{cm},$$

$$m_p = \sqrt{\frac{\hbar c}{G}} \approx 10^{-5}\ \text{g},$$

$$t_p = \sqrt{\frac{\hbar G}{c^5}} \approx 10^{-44}\ s.$$

The assumption that SU2 is the fundamental group means nature works like a computer with a binary number system. As noted by von Weizsäcker [10] with SU2 isomorphic to SO3, the rotation group in R3, then immediately explains why natural space is three-dimensional. Planck's conjecture requires that differentials must be replaced by rotation invariant finite difference operators, which demands a finitistic non-Archimedean formulation of the fundamental laws[1]. And because of the two-valuedness of SU2 there must be negative besides positive Planck masses[2]. Planck's conjecture further implies that the force between the Planck masses must be equal the Planck force $F_p = m_p c^2/r_p = c^4/G$ acting over a Planck length. The remaining freedom in the sign of the Planck force makes it possible to construct in a unique way a stable "plasma" made up of an equal number of positive and negative Planck masses, with each volume in space occupied in the average by one Planck mass.

For the finitistic non-Archimedean analysis we proceed as follows: the finite difference quotient in one dimension is

$$\frac{\Delta \text{y}}{\Delta \text{x}} = \frac{f(x+l_0/2) - f(x-l_0/2)}{l_0}, \tag{1.1}$$

where $l_0 > 0$ is the finite difference. We can write [11]

$$f(x+h) = e^{hd/dx} f(x) \tag{1.2}$$

$$= f(x) + h\frac{\mathrm{d}f(x)}{\mathrm{d}x} + \frac{h^2}{2!}\frac{h^2\mathrm{d}^2 f(x)}{\mathrm{d}x^2} + \cdots$$

and thus for (1.1)

$$\frac{\Delta y}{\Delta x} = \frac{\sinh[(l_0/2)\mathrm{d}/\mathrm{d}x]}{l_0/2} f(x). \tag{1.3}$$

We also introduce the average

$$\bar{y} = \frac{(f(x+l_0/2) + f(x-l_0/2))}{2} \tag{1.4}$$

$$= \cosh[(l_0/2)\mathrm{d}/\mathrm{d}x] f(x).$$

In the limit $l_0 \to 0$, $\Delta y/\Delta x = \mathrm{d}y/\mathrm{d}x$, and $\bar{y} = y$. With $\mathrm{d}/\mathrm{d}x = \partial$ we introduce the operators

$$\Delta_0 = \cosh[(l_0/2)\partial],\ \ \Delta_1 = (2/l_0)\sinh[(l_0/2)\partial] \tag{1.5}$$

whereby

$$\frac{\Delta y}{\Delta x} = \Delta_1 f(x),\ \ \bar{y} = \Delta_0 f(x) \tag{1.6}$$

and

$$\Delta_1 = \left(\frac{2}{l_0}\right)^2 \frac{\mathrm{d}\Delta_0}{\mathrm{d}\partial}. \tag{1.7}$$

The operators Δ_0 and Δ_1 are solutions of

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\partial^2} - \left(\frac{l_0}{2}\right)^2\right] \Delta(\partial) = 0. \tag{1.8}$$

The generalization to an arbitrary number of dimensions is straightforward. For N dimensions and $\Delta = \Delta_0$, (1.8) has to be replaced by [12]

$$\left[\sum_{i=1}^{N} \frac{\partial^2}{\partial(\partial_i)^2} - N\left(\frac{l_0}{s}\right)^2\right] \Delta_0^{(N)} = 0, \tag{1.9}$$

where

$$\lim_{l_0 \to 0} \Delta_0^{(N)} = 1.$$

From a solution of (1.9) one obtains the N dimensional finite difference operator by

$$\Delta_i^{(N)} = \left(\frac{2}{l_0}\right)^2 \frac{\mathrm{d}\Delta_0^{(N)}}{\mathrm{d}\partial_i}. \tag{1.10}$$

[1]Non-Archimedean is here meant in the sense of Archimedes' belief that the number π can be obtained by an unlimited progression of polygons inscribed inside a circle, impossible if there is a smallest length.

[2]Since the theory at this most fundamental level is exactly non-relativistic, the particle number is conserved, outlawing their change into other particles. This permits the permanent existence of negative besides positive Planck masses.

Introducing N-dimensional polar coordinates, (1.9) becomes

$$\left[\frac{1}{\partial^{N-1}}\frac{\mathrm{d}}{\mathrm{d}\partial}\left(\partial^{N-1}\frac{\mathrm{d}}{\mathrm{d}\partial}\right)-N\left(\frac{l_0}{2}\right)^2\right]\Delta_0^{(N)}=0, \quad (1.11)$$

where

$$\partial=\sqrt{\sum_{i=1}^{N}\partial_i^2}.$$

Putting $\partial\equiv x$, $\Delta_0^{(N)}\equiv y$, (1.11) takes the form

$$x^2y''+(N-1)xy'-N(l_0/2)^2x^2y=0 \quad (1.12)$$

with the general solution [13]

$$y=x^{\frac{3-N}{2}}Z_{\pm(3-N)/2}\left(i\sqrt{N}(l_0/2)x\right), \quad (1.13)$$

where Z_ν is a cylinder function.

Of interest for the three dimensional difference operator are solutions where $N=3$ and where $\lim_{l_0\to 0}\Delta_0^{(3)}=1$. One finds that

$$D_0\equiv\Delta_0^{(3)}=\frac{\sinh\left[\sqrt{3}(l_0/2)\partial\right]}{\sqrt{3}(l_0/2)\partial}, \quad (1.14)$$

and with (1.10) that

$$D_1\equiv\Delta_1^{(3)}=\left(\frac{2}{l_0}\right)^2\left[\cosh\left[\sqrt{3}\left(\frac{l_0}{2}\right)\partial\right]-\frac{\sinh\left[\sqrt{3}(l_0/2)\partial\right]}{\sqrt{3}(l_0/2)\partial}\right]\frac{\partial_i}{\partial^2}. \quad (1.15)$$

The expressions Δ_1 and D_1 will be used to obtain the dispersion relation for the finitistic Schrödinger equation in the limit of short wave lengths.

2. Planck Mass Plasma

The Planck mass plasma conjecture is the assumption that the vacuum of space is densely filled with an equal number of positive and negative Planck mass particles, with each Planck length volume in the average occupied by one Planck mass, with the Planck mass particles interacting with each other by the Planck force over a Planck length, and with Planck mass particles of equal sign repelling and those of opposite sign attracting each other[3]. The particular choice made for the sign of the Planck force is the only one which keeps the Planck mass plasma stable. While Newton's actio=reactio remains valid for the interaction of equal Planck mass particles, it is violated for the interaction of a positive with a negative Planck mass particle, even though globally the total linear momentum of the Planck mass plasma is conserved, with the recoil absorbed by the Planck mass plasma as a whole.

It is the local violation of Newton's actio=reactio which leads to quantum mechanics at the most fundamental level, as can be seen as follows: Under the Planck force $F_\mathrm{p}=m_\mathrm{p}c^2/r_\mathrm{p}$, the velocity fluctuation of a Planck mass particle interacting with a Planck mass particle of opposite sign is $\Delta v=(F_\mathrm{p}/m_\mathrm{p})t_\mathrm{p}=(c^2/r_\mathrm{p})(r_\mathrm{p}/c)=c$, and hence the momentum fluctuation $\Delta p=m_\mathrm{p}c$. But since $\Delta q=r_\mathrm{p}$, and because $m_\mathrm{p}r_\mathrm{p}c=\hbar$, one obtains Heisenberg's uncertainty relation $\Delta p\Delta q=\hbar$ for a Planck mass particle. Accordingly, the quantum fluctuations are explained by the interaction with hidden negative masses, with energy borrowed from the sea of hidden negative masses.

The conjecture that quantum mechanics has its cause in the interaction of positive with hidden negative masses is supported by its derivation from a variational principle first proposed by Fenyes [14]. Because Fenyes could not give a physical explanation for his variational principle, he was criticized by Heisenberg [15], but the Planck mass plasma hypothesis gives a simple explanation through the existence of negative masses.

According to Newtonian mechanics and Planck's conjecture, the interaction of a positive with a negative Planck mass particle leads to a velocity fluctuation $\dot{\delta}=a_\mathrm{p}t_\mathrm{p}=c$, with a displacement of the particle equal to $\delta=(1/2)a_\mathrm{p}t_\mathrm{p}^2=r_\mathrm{p}/2$, where $a_\mathrm{p}=F_\mathrm{p}/m_\mathrm{p}$. Therefore, a Planck mass particle immersed in the Planck mass plasma makes a stochastic quivering motion (Zitterbewegung) with the velocity

$$\boldsymbol{v}_D=-(r_\mathrm{p}c/2)(\nabla n/n), \quad (2.1)$$

where $n=1/2r_\mathrm{p}^3$ is the average number density of positive or negative Planck mass particles. The kinetic en-

[3] It was shown by Planck in 1911 that there must be a divergent zero point vacuum energy, by Nernst called an aether. It is for this reason that I have also called Planck's zero point vacuum energy, the Planck aether. Calling it a Planck mass plasma instead appears possible as well.

ergy of this diffusion process is given by

$$\left(\frac{m_p}{2}\right)\boldsymbol{v}_D^2 = \left(\frac{m_p}{8}\right) r_p^2 c^2 \left(\frac{\nabla n}{n}\right)^2 = \left(\frac{\hbar^2}{8m_p}\right)\left(\frac{\nabla n}{n}\right)^2 . \tag{2.2}$$

Putting

$$\boldsymbol{v} = \frac{\hbar}{m_p}\nabla S, \tag{2.3}$$

where S is the Hamilton action function and $\boldsymbol{v}$ the velocity of the Planck mass plasma, the Lagrange density for the Planck mass plasma is

$$\mathcal{L} = n\left[\hbar\frac{\partial S}{\partial t} + \frac{\hbar^2}{2m_p}(\nabla S)^2 + U + \frac{\hbar^2}{8m_p}\left(\frac{\nabla n}{n}\right)^2\right]. \tag{2.4}$$

Variation of (2.4) with regard to S according to

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial S/\partial t}\right) + \frac{\partial}{\partial r}\left(\frac{\partial \mathcal{L}}{\partial S/\partial r}\right) = 0 \tag{2.5}$$

leads to

$$\frac{\partial n}{\partial t} + \frac{\hbar}{m_p}\nabla(n\nabla S) = 0 \tag{2.6}$$

or

$$\frac{\partial n}{\partial t} + \nabla(n\boldsymbol{v}) = 0, \tag{2.7}$$

which is the continuity equation of the Planck mass plasma. Variation with regard to n according to

$$\frac{\partial \mathcal{L}}{\partial n} - \frac{\partial}{\partial r}\left(\frac{\partial \mathcal{L}}{\partial n/\partial r}\right) = 0 \tag{2.8}$$

leads to

$$\hbar\frac{\partial S}{\partial t} + U + \frac{\hbar^2}{2m_p}(\nabla S)^2 + \frac{\hbar^2}{4m_p}\left[\frac{1}{2}\left(\frac{\nabla n}{n}\right)^2 - \frac{\nabla^2 n}{n}\right] = 0 \tag{2.9}$$

or

$$\hbar\frac{\partial S}{\partial t} + U + \frac{\hbar^2}{2m_p}(\nabla S)^2 + \frac{\hbar^2}{2m_p}\frac{\nabla^2\sqrt{n}}{\sqrt{n}} = 0. \tag{2.10}$$

With the Madelung transformation

$$\psi = \sqrt{n}e^{iS}, \quad \psi^* = \sqrt{n}e^{-iS} \tag{2.11}$$

(2.6) and (2.10) is obtained from the Schrödinger equation

$$i\hbar\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m_p}\nabla^2\psi + U\psi. \tag{2.12}$$

Repeating the same with the non-Archimedean analysis by replacing $\partial/\partial t \to \Delta_1$ and $\nabla^2 \to D_1^2$, with Δ_1 given by (1.5) and D_1 by (1.15), leads to the finitistic form of (2.12):

$$i\hbar\Delta_1\psi = -\frac{\hbar^2}{2m_p}D_1^2\psi + U\psi. \tag{2.13}$$

In the limit of short wave lengths one can neglect the potential U. In this limit the dispersion relation for (2.13) differs from the on for (2.12), but goes over into the latter in the limit $r_p \to 0$, $t_p \to 0$. For a wave field $\psi = \psi(x,t)$ one obtains from (2.13)

$$\frac{2i\hbar}{t_p}\sinh\left[\frac{t_p}{2}\frac{\partial}{\partial t}\right]\psi = -\frac{8m_p c^2}{r_p^2}\left\{\cosh\left[\sqrt{3}(r_p/2)\partial/\partial x\right] - \frac{\sinh\left[\sqrt{3}(r_p/2)\partial/\partial x\right]}{\sqrt{3}(r_p/2)\partial/\partial x}\right\}^2\frac{1}{(\partial/\partial x)^2}\psi, \tag{2.14}$$

and for a plane wave

$$\psi = Ae^{i(kx-\omega t)} \tag{2.15}$$

one obtains the dispersion relation

$$\sin\left(\frac{\omega t_p}{2}\right) = \frac{3}{\left(\sqrt{3}(r_p/2)k\right)^2} \cdot\left\{\frac{\sin\left[\sqrt{3}(r_p/2)k\right]}{\sqrt{3}(r_p/2)k} - \cos\left[\sqrt{3}(r_p/2)k\right]\right\}^2. \tag{2.16}$$

Putting $x = \sqrt{3}(r_p/2)k$, this can be written as follows:

$$\sqrt{3\sin\left(\frac{\omega t_p}{2}\right)} = f(x), \quad f(x) = \frac{3}{x}\left[\frac{\sin x}{x} - \cos x\right]. \tag{2.17}$$

One has $\lim_{x\to 0} f(x) = x$ and $\lim_{x\to\infty} f(x) = 0$, with a maximum of $f(x)$ at $x \simeq 2.1$, where $f(x) \simeq 1.3$. In the limit $x = 0$ and $t_p \to 0$ one has $\omega = (cr_p/2)k^2$, which with $m_p r_p c = \hbar$ is $\omega = \hbar k^2/2m_p$, the dispersion relation in the limit $U = 0$. For $x \simeq 2.1$, where $f(x)$ has its maximum, one has $k \simeq 2.4/r_p$ and hence $\sin(\omega_{max} t_p/2) \simeq 0.57$. With the energy operator $E = i\hbar\Delta_1$ one then finds

$$E_{max} = \left(\frac{2\hbar}{t_p}\right) \sin\left(\frac{\omega_{max} t_p}{2}\right) \approx 1.14 m_p c^2. \tag{2.18}$$

For the transition from classical to quantum mechanics we set for the momentum operator in analogy to $\boldsymbol{p} = (\hbar/i)\partial/\partial\boldsymbol{r}$

$$\boldsymbol{p} = \frac{\hbar}{i} D_i. \tag{2.19}$$

To insure the integrity of the Poisson bracket relation $\{q, p\} = 1$ requires then a change in the position operator $\boldsymbol{q}$ to keep unchanged the commutation relation $\boldsymbol{pq} - \boldsymbol{qp} = \hbar/i$, and one has to set

$$\boldsymbol{q} = \left(\frac{\partial_i}{D_1}\right) \boldsymbol{r}. \tag{2.20}$$

The meaning of the nonlocal position operator (2.20) can be understood by the position eigenfunctions of this operator. If $\psi_n(\boldsymbol{q})$ is the position eigenfunction and $\boldsymbol{q}$ the position operator, the eigenvalues $\boldsymbol{q}_n$ and eigenfunctions $\psi_n(\boldsymbol{q})$ are determined by

$$\boldsymbol{q}\psi_n(\boldsymbol{q}) = \boldsymbol{q}_n \psi_n(\boldsymbol{q}). \tag{2.21}$$

If the position can be precisely measured, one has

$$\psi_n(\boldsymbol{q}) = \delta(|\boldsymbol{q} - \boldsymbol{q}_n|) \tag{2.22}$$

and

$$\int \boldsymbol{q}\delta(|\boldsymbol{q} - \boldsymbol{q}_n|)\mathrm{d}\boldsymbol{q} = \boldsymbol{q}_n, \tag{2.23}$$

for which one can write ($\int \equiv 1/\mathrm{d}$)

$$\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}} \boldsymbol{q}\delta(|\boldsymbol{q} - \boldsymbol{q}_n|) = \boldsymbol{q}_n. \tag{2.24}$$

If $\boldsymbol{q} = \boldsymbol{q}_n$ one has

$$\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}} \delta(|\boldsymbol{q} - \boldsymbol{q}_n|) = 1. \tag{2.25}$$

For the nonlocal position operator one likewise has

$$\frac{1}{D_1} D(|\boldsymbol{q} - \boldsymbol{q}_n|) = 1, \tag{2.26}$$

where $D(|\boldsymbol{q} - \boldsymbol{q}_n|)$ is the position eigenfunction with the limit

$$\lim_{r_p \to 0} D(|\boldsymbol{q} - \boldsymbol{q}_n|) = \delta(|\boldsymbol{q} - \boldsymbol{q}_n|). \tag{2.27}$$

By comparison with (2.24) and (2.25) it follows that with the position eigenfunction $D(|\boldsymbol{q} - \boldsymbol{q}_n|)$, the position operator is

$$\boldsymbol{q} \to \frac{1}{D_1} \left(\frac{\mathrm{d}}{\mathrm{d}q}\right) q, \tag{2.28}$$

which is the same as (2.20). For the potential U in (2.12) and (2.13), coming from all the Planck mass particles acting on one Planck mass particle $\pm m_p$ described by the wavefuntion $\psi_\pm$, we set

$$U_\pm = \pm 2\hbar c r_p^2 \langle \psi_\pm^* \psi_\pm - \psi_\mp^* \psi_\mp \rangle, \tag{2.29}$$

which we justify as follows: A dense assembly of positive and negative Planck mass particles, each of them occupying the volume r_p^3, has the expectation value $\langle \psi_\pm^* \psi_\pm \rangle = 1/2r_p^3$, whereby $2\hbar c r_p^2 \langle \psi_\pm^* \psi_\pm \rangle = m_p c^2$, implying an average potential $\pm m_p c^2$ for the positive or negative Planck mass particles within the Planck mass plasma, and consistent with $F_p r_p = m_p c^2$. We thus have for both the positive and negative Planck mass particles

$$i\hbar \frac{\partial \psi_\pm}{\partial t} = \mp \frac{\hbar^2}{2m_p} \nabla^2 \psi_\pm \pm 2\hbar c r_p^2 \langle \psi_\pm^* \psi_\pm - \psi_\mp^* \psi_\mp \rangle \psi_\pm. \tag{2.30}$$

To make the transition from the one particle Schrödinger equation (2.30) to the many particle equation of the Planck mass plasma we replace (2.30) by

$$i\hbar \frac{\partial \psi_\pm}{\partial t} = \mp \frac{\hbar^2}{2m_p} \nabla^2 \psi_\pm \pm 2\hbar c r_p^2 \langle \psi_\pm^\dagger \psi_\pm - \psi_\mp^\dagger \psi_\mp \rangle \psi_\pm, \tag{2.31}$$

where $\psi^\dagger, \psi$ are field operators satisfying the commutation relations

$$\begin{aligned} \left[\psi_\pm(\boldsymbol{r}) \psi_\pm^\dagger(\boldsymbol{r}')\right] &= \delta(\boldsymbol{r} - \boldsymbol{r}'), \\ [\psi_\pm(\boldsymbol{r}) \psi_\pm(\boldsymbol{r}')] &= \left[\psi_\pm^\dagger(\boldsymbol{r}) \psi_\pm^\dagger(\boldsymbol{r}')\right] = 0. \end{aligned} \tag{2.32}$$

Replacing in (2.31) $\psi^\dagger, \psi$ by the classical field functions φ^*, φ, (2.31) becomes a nonlinear Schrödinger equation which can be derived from the Lagrange density

$$\mathcal{L}_\pm = i\hbar\varphi_\pm^*\dot{\varphi}_\pm \mp \frac{\hbar^2}{2m_\mathrm{p}}(\nabla\varphi_\pm^*)(\nabla\varphi_\pm) \mp 2\hbar c r_\mathrm{p}^2\left[\frac{1}{2}\varphi_\pm^*\varphi_\pm - \varphi_\mp^*\varphi_\mp\right]\varphi_\pm^*\varphi_\pm \quad (2.33)$$

with the Hamilton density

$$\mathbf{H}_\pm = \pm\frac{\hbar^2}{2m_\mathrm{p}}(\nabla\varphi_\pm^*)(\nabla\varphi_\pm) \mp 2\hbar c r_\mathrm{p}^2\left[\frac{1}{2}\varphi_\pm^*\varphi_\pm - \varphi_\mp^*\varphi_\mp\right]\varphi_\pm^*\varphi_\pm. \quad (2.34)$$

With $H_\pm = \int \mathbf{H}_\pm \mathrm{d}\boldsymbol{r}$ one has the Heisenberg equation of motion

$$i\hbar\dot{\psi}_\pm = [\psi_\pm, H_\pm], \quad (2.35)$$

which agrees with (2.31) and shows that the classical and quantum equations have the same form. For the particle number operator $N_\pm = \int \psi_\pm^\dagger\psi_\pm \mathrm{d}\boldsymbol{r}$ one finds that

$$i\hbar\dot{N}_\pm = [N_\pm, H_\pm] = 0, \quad (2.36)$$

which shows that the particle numbers are conserved, permitting the permanent existence of negative masses.

The foregoing can be carried over to the non-Archimedean formulation in each step. With the finite difference operators the commutation relations (2.32) become

$$\left[\psi_\pm(\boldsymbol{r}), \psi_\pm^\dagger(|\boldsymbol{r}'|)\right] = D(|\boldsymbol{r}-\boldsymbol{r}'|), \quad [\psi_\pm(\boldsymbol{r}), \psi_\pm(\boldsymbol{r}')] = \left[\psi_\pm^\dagger(\boldsymbol{r})\psi^\dagger(\boldsymbol{r}')\right] = 0, \quad (2.37)$$

where $D(|\boldsymbol{r}-\boldsymbol{r}'|)$ is the generalized three-dimensional delta function

$$\lim_{r_\mathrm{p}\to 0} D(|\boldsymbol{r}-\boldsymbol{r}'|) = \delta(|\boldsymbol{r}-\boldsymbol{r}'|), \quad (2.38)$$

and where

$$\frac{1}{D_1}D(|\boldsymbol{r}-\boldsymbol{r}'|) = 1. \quad (2.39)$$

This changes (2.31) into the non-Archimedean form

$$i\hbar\Delta_1\psi_\pm = \mp\frac{\hbar^2}{2m_\mathrm{p}}D_1^2\psi_\pm \pm 2\hbar c r_\mathrm{p}^2(\psi_\pm^\dagger\psi_\pm - \psi_\mp^\dagger\psi_\mp)\psi_\pm. \quad (2.40)$$

In the quantum equation of motion

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{i}{\hbar}[H,F], \quad (2.41)$$

where $H = D_1^{-1}\mathbf{H}$, the r.h.s. remains unchanged because the Poisson bracket for any dynamical quantity can be reduced to a sum of Poisson brackets for position and momentum, with the operator $\mathrm{d}/\mathrm{d}t$ replaced by Δ_1. The equation of motion is therefore changed into

$$\Delta_1 F = \frac{i}{\hbar}[H,F]. \quad (2.42)$$

The Lagrange density now is

$$\mathcal{L}_\pm = i\hbar\varphi_\pm^*\Delta_1\varphi_\pm \mp \frac{\hbar^2}{2m_\mathrm{p}}(D_1\varphi_\pm^*)(D_1\varphi_\pm) \mp 2\hbar c r_\mathrm{p}^2\left[\frac{1}{2}\varphi_\pm^*\varphi_\pm - \varphi_\mp^*\varphi_\mp\right]\varphi_\pm^*\varphi_\pm. \quad (2.43)$$

Variation with regard to φ^* according to

$$\frac{\partial\mathcal{L}_\pm}{\partial\varphi_\pm^*} - D_1\frac{\partial\mathcal{L}_\pm}{\partial(D_1\varphi_\pm^*)} = 0 \quad (2.44)$$

leads to (2.40) replacing φ^*, φ by $\psi^\dagger$, ψ. Variation with regard to φ according to

$$\frac{\partial\mathcal{L}_\pm}{\partial\varphi_\pm} - D_1\frac{\partial\mathcal{L}_\pm}{\partial(D_1\varphi_\pm)} - \Delta_1\frac{\partial\mathcal{L}_\pm}{\partial(\Delta_1\varphi_\pm)} = 0 \quad (2.45)$$

leads to the conjugate complex equation.

The momentum density conjugate to φ is

$$\pi_\pm = \frac{\partial\mathcal{L}_\pm}{\partial(\Delta_1\varphi_\pm)} = i\hbar\varphi_\pm^*, \quad (2.46)$$

and hence the Hamilton density

$$\begin{aligned}\mathbf{H}_\pm &= \pi_\pm\Delta_1\varphi_\pm - \mathcal{L}_\pm \\ &= \mp\frac{i\hbar}{2m_\mathrm{p}}(D_1\pi_\pm)(D_1\varphi_\mp) \\ &\quad - 2icr_\mathrm{p}^2\left[\frac{1}{2}\varphi_\pm^*\varphi_\pm - \varphi_\mp^*\varphi_\mp\right]\pi_\pm\varphi_\pm \\ &= \pm\frac{\hbar^2}{2m_\mathrm{p}}(D_1\varphi_\pm^*)(D_1\varphi_\pm) \\ &\quad + 2\hbar c r_\mathrm{p}^2\left[\frac{1}{2}\varphi_\pm^*\varphi_\pm - \varphi_\mp^*\varphi_\mp\right]\varphi_\pm^*\varphi_\pm.\end{aligned} \quad (2.47)$$

For the quantum equation of motion we have

$$i\hbar\Delta_1\psi_\pm = [\psi_\pm, H_\pm] \tag{2.48}$$

$$= \left[\psi_\pm, D_1^{-1'}\frac{\hbar^2}{2m_p}\left(D_1^1\psi_\pm^{\dagger'}\right)(D_1'\psi_\pm')\right] + \left[\psi_\pm, \pm D_1^{-1'}2\hbar c r_p^2\left(\frac{1}{2}\psi_\pm^{\dagger'}\psi_\pm' - \psi_\mp^{\dagger'}\psi_\mp'\right)\psi_\pm^{\dagger'}\psi_\pm'\right].$$

With (2.37) one has for the first commutator

$$\begin{aligned}
&\left[\psi, D_1^{-1'}\left(D_1'\psi^{\dagger'}\right)(D_1'\psi')\right] \\
&= -\left[\psi, D_1^{-1'}\psi^{\dagger'}D_1^{2'}\psi'\right] \\
&= -D_1^{-1'}\left[\psi', \psi^{\dagger'}\right]D_1^{2'}\psi' \\
&= -D^{-1'}D_1'^2\psi' D(|\boldsymbol{r}-\boldsymbol{r}'|) \\
&= -D_1^2\psi.
\end{aligned} \tag{2.49}$$

In evaluating the second commutator we can take the operator $\psi_-^\dagger\psi_-$ as a numerical (c-number) function with regard to the operators $\psi_+^\dagger$, ψ_+ and likewise the operator $\psi_+^\dagger\psi_+$ as a numerical function with regard to the operators $\psi_-^\dagger$, ψ_-. For these terms we have expressions in the integrand of the form

$$\psi\psi^{\dagger'}\psi' - \psi^{\dagger'}\psi'\psi = \psi'(\psi\psi^{\dagger'} - \psi^{\dagger'}\psi) = \psi' D(|\boldsymbol{r}-\boldsymbol{r}'|). \tag{2.50}$$

Applying to (2.50) the operator $D_1^{-1'}$ from the left, one obtains ψ. For the remaining terms one has

$$\begin{aligned}
&\psi\psi^{\dagger'}\psi'\psi^{\dagger'}\psi' - \psi^{\dagger'}\psi'\psi^{\dagger'}\psi'\psi \\
&= \psi^{\dagger'}\psi\psi'\psi^{\dagger'}\psi' + D(|\boldsymbol{r}-\boldsymbol{r}'|)\psi'\psi^{\dagger'}\psi' \\
&\quad - \psi^{\dagger'}\psi'\psi^{\dagger'}\psi'\psi \\
&= \psi^{\dagger'}\psi'\psi\psi^{\dagger'}\psi' - \psi^{\dagger'}\psi'\psi^{\dagger'}\psi'\psi \\
&\quad + D(|\boldsymbol{r}-\boldsymbol{r}'|)\psi'\psi^{\dagger'}\psi' \\
&= \psi^{\dagger'}\psi'\left[\psi\psi^{\dagger'}\psi' - \psi^{\dagger'}\psi'\psi\right] \\
&\quad + D(|\boldsymbol{r}-\boldsymbol{r}'|)\psi'\psi^{\dagger'}\psi' \\
&= \psi^{\dagger'}\psi'\psi' D(|\boldsymbol{r}-\boldsymbol{r}'|) + D(|\boldsymbol{r}-\boldsymbol{r}'|)\psi'\psi^{\dagger'}\psi' \\
&= 2\psi^{\dagger'}\psi'\psi' D(|\boldsymbol{r}-\boldsymbol{r}'|),
\end{aligned} \tag{2.51}$$

and applying to (2.51) $D_1^{-1'}$ one obtains $2\psi^\dagger\psi$. Inserting these results into (2.48) one obtains (2.40). This shows that the (non-Archimedean) classical and quantum equations have the same form.

With the particle number operator

$$N_\pm = D_1^{-1}\psi_\pm^\dagger\psi_\pm \tag{2.52}$$

one can show that

$$i\hbar\Delta_1 N_\pm = [N_\pm, H_\pm] = 0, \tag{2.53}$$

establishing the permanent existence of negative masses in the non- Archimedean formulation of the theory.

As before, $\psi_-^\dagger\psi_-$ can be treated as a c-number with regard to the operators $\psi_+^\dagger$, ψ_+ and vice versa. In these terms $\psi_-^\dagger\psi_-$ (resp. $\psi_+^\dagger\psi_+$) acts like an external potential. It is well known that for a nonrelativistic field theory without interaction, but in the presence of an external potential the particle number is conserved. We therefore have only to show that this is also true if the nonlinear self-interaction term is included. In the commutator it leads to integrals with integrands of the form

$$\begin{aligned}
&\psi^\dagger\psi\psi^{\dagger'}\psi'\psi^{\dagger'}\psi' - \psi^{\dagger'}\psi'\psi^{\dagger'}\psi'\psi^\dagger\psi \\
&= \psi^\dagger\psi^{\dagger'}\psi\psi'\psi^{\dagger'}\psi' + \psi^\dagger D(|\boldsymbol{r}-\boldsymbol{r}'|)\psi'\psi^{\dagger'}\psi' \\
&\quad - \psi^{\dagger'}\psi'\psi^{\dagger'}\psi'\psi^\dagger\psi \\
&= \psi^{\dagger'}\psi^\dagger\psi'\psi\psi^{\dagger'}\psi' + \psi^\dagger D(|\boldsymbol{r}-\boldsymbol{r}'|)\psi'\psi^{\dagger'}\psi' \\
&\quad - \psi^{\dagger'}\psi'\psi^{\dagger'}\psi'\psi^\dagger\psi \\
&= \psi^{\dagger'}\psi'\psi^\dagger\psi\psi^{\dagger'}\psi' - \psi^\dagger D(|\boldsymbol{r}-\boldsymbol{r}'|)\psi\psi^{\dagger'}\psi' \\
&\quad + \psi^\dagger D(|\boldsymbol{r}-\boldsymbol{r}'|)\psi'\psi^{\dagger'}\psi' - \psi^\dagger\psi'\psi^{\dagger'}\psi'\psi^\dagger\psi \\
&= \psi^{\dagger'}\psi'[\psi^\dagger\psi\psi^{\dagger'}\psi' - \psi^{\dagger'}\psi'\psi^\dagger\psi] \\
&\quad + \psi^\dagger D(|\boldsymbol{r}-\boldsymbol{r}'|)\psi'\psi^{\dagger'}\psi' \\
&\quad - \psi^{\dagger'}D(|\boldsymbol{r}-\boldsymbol{r}'|)\psi\psi^{\dagger'}\psi'
\end{aligned} \tag{2.54}$$

with the last two terms in (2.54) canceling each other by multiplication from the left with D_1^{-1}. The first term is zero as well, because it can be reduced to a term which would arise in the presence of an externally applied potential.

3. Hartree-Fock Approximation

For wave lengths large compared to the Planck length it is sufficient to solve (2.31) nonperturbatively with the Hartree-Fock approximation. For temperatures $kT \ll m_\mathrm{p}c^2$, the Planck mass plasma is a two-component superfluid, with each mass component described by a completely symmetric wave function. In the Hartree-Fock approximation one has

$$\begin{aligned} \langle \psi_\pm^\dagger \psi_\pm \psi_\pm \rangle &\cong 2\varphi_\pm^* \varphi_\pm^2, \\ \langle \psi_\mp^\dagger \psi_\mp \psi_\pm \rangle &\cong \varphi_\mp^* \varphi_\mp \varphi_\pm, \end{aligned} \tag{3.1}$$

whereby (2.31) becomes the two-component nonlinear Schrödinger equation:

$$i\hbar \frac{\partial \varphi_\pm}{\partial t} = \mp \frac{\hbar^2}{2m_\mathrm{p}} \nabla^2 \varphi_\pm \pm 2\hbar c r_\mathrm{p}^2 [2\varphi_\pm^* \varphi_\pm - \varphi_\mp^* \varphi_\mp] \varphi_\pm. \tag{3.2}$$

With the Madelung transformation

$$\begin{aligned} n_\pm &= \varphi_\pm^* \varphi_\pm, \\ n_\pm v_\pm &= \mp \frac{i\hbar}{2m_\mathrm{p}} [\varphi_\pm^* \nabla \varphi_\pm - \varphi_\pm \nabla \varphi_\pm^*], \end{aligned} \tag{3.3}$$

(3.2) is brought into the hydrodynamic form

$$\begin{aligned} \frac{\partial \boldsymbol{v}_\pm}{\partial t} + (\boldsymbol{v}_\pm \bullet \nabla)\boldsymbol{v}_\pm &= -\frac{1}{m_\mathrm{p}} \nabla (U_\pm + Q_\pm), \\ \frac{\partial n_\pm}{\partial t} + \nabla \bullet (n_\pm \boldsymbol{v}_\pm) &= 0, \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} U_\pm &= 2m_\mathrm{p}c^2 r_\mathrm{p}^3 (2n_\pm - n_\mp), \\ Q_\pm &= -\frac{\hbar^2}{2m_\mathrm{p}} \frac{\nabla^2 \sqrt{n_\pm}}{\sqrt{n_\pm}}, \end{aligned} \tag{3.5}$$

$$\begin{aligned} &\varphi_\pm = A_\pm e^{iS_\pm}, \ A_\pm > 0, \ 0 \le S_\pm \le 2\pi, \\ &n_\pm = A_\pm^2, \quad \boldsymbol{v}_\pm = \pm \frac{\hbar}{m_\mathrm{p}} \mathrm{grad} S_\pm, \end{aligned} \tag{3.6}$$

$$\oint \boldsymbol{v}_\pm \bullet \mathrm{d}r = \pm \frac{nh}{m_\mathrm{p}}, \quad n = 0, 1, 2 \cdots. \tag{3.7}$$

For small amplitude disturbances of long wave lengths one can neglect the quantum potential $Q_\pm$ against $U_\pm$, and one obtains from (3.4) and (3.5)

$$\begin{aligned} \frac{\partial}{\partial t}(\boldsymbol{v}_+ + \boldsymbol{v}_-) &= -2c^2 r_\mathrm{p}^3 \nabla (n'_+)(n'_-), \\ \frac{\partial}{\partial t}(\boldsymbol{v}_+ - \boldsymbol{v}_-) &= -6c^2 r_\mathrm{p}^3 \nabla (n'_+ - n'_-), \\ \frac{\partial n'_\pm}{\partial t} + n_\pm \nabla \boldsymbol{v}_\pm &= 0, \end{aligned} \tag{3.8}$$

where $n'_\pm$ is a disturbance of the equilibrium density $n_\pm = 1/2r_\mathrm{p}^3$. Eliminating $n'_\pm$ from (3.8) one obtains two wave equations:

$$\begin{aligned} \frac{\partial^2}{\partial t^2}(\boldsymbol{v}_+ + \boldsymbol{v}_-) &= c^2 \nabla^2 (\boldsymbol{v}_+ + \boldsymbol{v}_-), \\ \frac{\partial^2}{\partial t^2}(\boldsymbol{v}_+ - \boldsymbol{v}_-) &= 3c^2 \nabla^2 (\boldsymbol{v}_+ - \boldsymbol{v}_-), \end{aligned} \tag{3.9}$$

the first, for a wave propagating with c, with the oscillations of the positive and negative masses in phase, while for the second the wave is propagating with $\sqrt{3}c$, with the oscillations out of phase by $180°$. The first wave is analogous to the ion acoustic plasma wave while the second resembles electron plasma oscillations.

From (3.4), (3.5) and (3.7) one obtains furthermore two quantized, vortex solutions, for which

$$\begin{aligned} &\boldsymbol{v}_- = \pm \boldsymbol{v}_+, \\ &n_+ - n_- = 0, \\ &|\boldsymbol{v}_\pm| = v_\varphi = c\left(\frac{r_\mathrm{p}}{r}\right) \quad , \quad r > r_\mathrm{p}, \\ &\phantom{|\boldsymbol{v}_\pm|} = 0 \qquad\qquad\quad , \quad r < r_\mathrm{p}, \\ &n_\pm = \left(\frac{1}{2r_\mathrm{p}^3}\right)\left[1 - \frac{1}{2}\left(\frac{r_\mathrm{p}}{r}\right)^2\right]. \end{aligned} \tag{3.10}$$

The vortex core radius $r = r_\mathrm{p}$, where $|\boldsymbol{v}_\pm| = c$, is obtained by equating $U_\pm$ with $Q_\pm$.

4. Formation of a Vortex Lattice

In a frictionless fluid the Reynolds number is infinite. In the presence of some internal motion the flow is unstable, decaying into vortices. By comparison, the superfluid Planck mass plasma is unstable even in the absence of any internal motion, because with an equal number of positive and negative mass particles it can without the expenditure of energy by spontaneous symmetry breaking decay into a vortex sponge (resp. vortex lattice).

In nonquantized fluid dynamics the vortex core radius is about equal a mean free path λ, where the velocity reaches the velocity of sound, the latter about equal to the thermal molecular velocity v_t. With the kinematic viscosity $\upsilon \simeq v_t, \lambda$, the Reynolds number in the vortex core is

$$\mathrm{Re} = vr/\upsilon = v_t \lambda / \upsilon = 1. \tag{4.1}$$

Interpreting Schrödinger's equation as an equation with an imaginary quantum viscosity $\upsilon_Q = i\hbar/2m_\mathrm{p} \sim ir_\mathrm{p}c$, and defining a quantum Reynolds number

$$\mathrm{Re}^Q = ivr/\upsilon_Q, \tag{4.2}$$

one finds with $\upsilon_Q \sim ir_\mathrm{p}c$ that in the vortex core of a quantum fluid (where $v = c$), $\mathrm{Re}^Q \sim 1$. Because of this analogy one can apply the stability analysis for a lattice of vortices in nonquantum fluid dynamics to a vortex lattice in quantum fluid dynamics. For the two- dimensional Karman vortex street the stability was analyzed by Schlayer [16], who found that the radius r_0 of the vortex core must be related to the distance ℓ between two vortices by

$$r_0 \simeq 3.4 \times 10^{-3} \ell. \tag{4.3}$$

Setting $r_0 = r_\mathrm{p}$ and $\ell = 2R$, where R is the radius of a vortex lattice cell occupied by one line vortex, one has

$$\frac{R}{r_\mathrm{p}} \simeq 147. \tag{4.4}$$

No comparable stability analysis seems to have been performed for a three-dimensional vortex lattice made up of vortex rings. The instability leading to the decay into vortices apparently arises from the disturbance one vortex exerts on an adjacent vortex. At the distance R/r_p the velocity by a vortex ring is larger by the factor $\log(8R/r_\mathrm{p})$, compared to the velocity of a line vortex at the same distance. With $R/r_\mathrm{p} \simeq 147$ for a line vortex lattice, a value of R/r_p for a ring vortex lattice can be estimated by solving for R/r_p the equation

$$R/r_\mathrm{p} \simeq 147 \log\left(\frac{8R}{r_\mathrm{p}}\right), \tag{4.5}$$

and one finds that [17]

$$R/r_\mathrm{p} \simeq 1360. \tag{4.6}$$

5. The Origin of Charge

Through their zero point fluctuations Planck mass particles bound in vortex filaments have a kinetic energy density by order of magnitude equal to

$$\varepsilon \approx \frac{m_\mathrm{p} c^2}{r_\mathrm{p}^3} = \frac{\hbar c}{r_\mathrm{p}^4}. \tag{5.1}$$

By order of magnitude this is about equal the energy density g^2, where g is the Newtonian gravitational field of a Planck mass particle m_p at the distance $r = r_\mathrm{p}$. Because

$$g \cong \frac{\sqrt{G} m_\mathrm{p}}{r_\mathrm{p}^2}, \tag{5.2}$$

one has

$$g^2 \approx \frac{G m_\mathrm{p}^2}{r_\mathrm{p}^4} = \frac{\hbar c}{r_\mathrm{p}^4}. \tag{5.3}$$

The interpretation of this result is as follows: Through its zero-point fluctuations a Planck mass particle bound in a vortex filament becomes the source of virtual phonons setting up a Newtonian type attractive force field with the coupling constant $G m_\mathrm{p}^2 = \hbar c$. Charge is thus explained by the zero point fluctuations of the Planck mass particles bound in vortices. The smallness of the gravitational coupling constant is explained by the near cancellation of the kinetic energy coming from the positive and negative mass component of the Planck mass plasma. Such a cancellation does not happen for fields not coupled to the energy momentum tensor. Therefore, with the exception of the gravitational coupling constant, all other coupling constants are within a few orders of magnitude equal to $\hbar c$ [18].

6. Maxwell's and Einstein's Equations

There are two kinds of transverse waves propagating through a vortex lattice, one simulating Maxwell's electromagnetic and the other one Einstein's gravitational wave. For electromagnetic waves this was shown by Thomson [19].

Let $\boldsymbol{v} = \{v_x, v_y, v_z\}$ be the undisturbed velocity of the vortex lattice and $\boldsymbol{u} = \{u_x, u_y, u_z\}$ a small superimposed velocity disturbance. Only taking those disturbances for which $\mathrm{div}\boldsymbol{v} = \mathrm{div}\boldsymbol{u} = 0$, the x-component of the equation of motion for the disturbance is

$$\begin{aligned}\frac{\partial u_x}{\partial t} = &-(v_x+u_x)\frac{\partial(v_x+u_x)}{\partial x}\\ &-(v_y+u_y)\frac{\partial(v_x+u_x)}{\partial y}\\ &-(v_z+u_z)\frac{\partial(v_x+u_x)}{\partial z}.\end{aligned} \tag{6.1}$$

From the continuity equation div$\boldsymbol{v} = 0$ one has

$$v_x\frac{\partial v_x}{\partial x} + v_x\frac{\partial v_y}{\partial y} + v_x\frac{\partial v_z}{\partial z} = 0. \tag{6.2}$$

Subtracting (6.2) from (6.1) and taking the y-z average, one finds

$$\frac{\partial u_x}{\partial t} = -\frac{\partial(\overline{v_y v_x})}{\partial y} - \frac{\partial(\overline{v_z v_x})}{\partial z} \tag{6.3a}$$

and similarly, by taking the x-z and x-y averages:

$$\frac{\partial u_y}{\partial t} = -\frac{\partial(\overline{v_x v_y})}{\partial x} - \frac{\partial(\overline{v_z v_y})}{\partial z}, \tag{6.3b}$$

$$\frac{\partial u_z}{\partial t} = -\frac{\partial(\overline{v_x v_z})}{\partial x} - \frac{\partial(\overline{v_y v_z})}{\partial y}. \tag{6.3c}$$

With div$\boldsymbol{u}$ = 0 one obtains from (6.3a – c)

$$\overline{v_i v_k} = \overline{v_k v_i} \tag{6.4}$$

Taking the x-component of the equation of motion, multiplying it by v_y and then taking the y-z average, and the y-component multiplied by v_x and taking the x-z average, finally subtracting the first from the second of these equations one finds

$$\frac{\partial}{\partial t}(\overline{v_x v_y}) = -v^2\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right), \tag{6.5}$$

where $v^2 = v_x^2 = v_y^2 = v_z^2$ is the average microvelocity square of the vortex field.

Putting $\varphi_z = -v_x v_y/2v^2$, (6.5) is just the z-component of

$$\frac{\partial \varphi}{\partial t} = \frac{1}{2}\mathrm{curl}\,\boldsymbol{u}, \tag{6.6}$$

where $\varphi_x = -\overline{v_y v_z}/2v^2, \varphi_y = -\overline{v_z v_x}/2v^2$. The equations (6.3a – c) then take the form

$$\frac{\partial \boldsymbol{u}}{\partial t} = -2v^2\mathrm{curl}\,\varphi. \tag{6.7}$$

Elimination of φ from (6.6) and (6.7) results in a wave equation for $\boldsymbol{u}$

$$-(1/c^2)\partial^2\boldsymbol{u}/\partial t^2 + \nabla^2\boldsymbol{u} = 0. \tag{6.8}$$

In the continuum limit $R \rightarrow r_{\mathrm{p}}$, the microvelocity $|\boldsymbol{v}|$ has to be set equal to c. Then putting $\varphi = -(1/2c)\boldsymbol{H}, \boldsymbol{u} = \boldsymbol{E}$, (6.6) and (6.7) are Maxwell's vacuum field equations.

Adding (6.1) and (6.2) and taking the average over x, y and z one has

$$\frac{\partial u_x}{\partial t} = -\frac{\partial\overline{v_x^2}}{\partial x} - \frac{\partial\overline{v_x v_y}}{\partial y} - \frac{\partial\overline{v_x v_z}}{\partial z}, \tag{6.9a}$$

and similar

$$\frac{\partial u_y}{\partial t} = -\frac{\partial\overline{v_y^2}}{\partial y} - \frac{\partial\overline{v_y v_z}}{\partial z} - \frac{\partial\overline{v_y v_x}}{\partial x}, \tag{6.9b}$$

$$\frac{\partial u_z}{\partial t} = -\frac{\partial\overline{v_z^2}}{\partial z} - \frac{\partial\overline{v_z v_x}}{\partial x} - \frac{\partial\overline{v_z v_y}}{\partial y}. \tag{6.9c}$$

With div $\boldsymbol{u} = 0$ this leads to

$$\frac{\partial^2}{\partial x_i \partial x_k}(\overline{v_i v_k}) = 0. \tag{6.10}$$

For (6.9a – c) one can write

$$\frac{\partial u_k}{\partial t} = -\frac{\partial}{\partial x_i}(\overline{v_i v_k}). \tag{6.11}$$

Multiplying the v_i component of the equation of motion with v_k, and vice versa, its v_k-component with v_i, adding both and taking the average, one finds

$$\frac{\partial}{\partial t}(\overline{v_i v_k}) = -v^2\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right). \tag{6.12}$$

From (6.11) one has

$$\frac{\partial^2 u_k}{\partial t^2} = \frac{\partial}{\partial t \partial x_i}(\overline{v_i v_k}), \tag{6.13}$$

and from (6.12)

$$\frac{\partial^2}{\partial x_i \partial t}(\overline{v_i v_k}) = -v^2\left(\frac{\partial}{\partial x_k}\frac{\partial u_i}{\partial x_i} + \frac{\partial^2 u_k}{\partial x_i^2}\right) = -v^2\frac{\partial^2 u_k}{\partial x_i^2}, \tag{6.14}$$

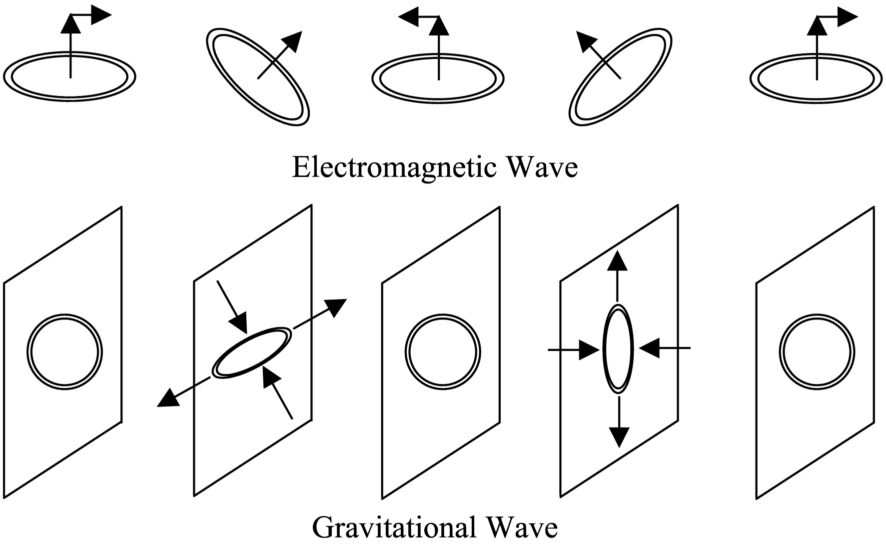


Fig. 1. Deformation of the vortex lattice for electromagnetic and gravitational waves.

the latter because div $\boldsymbol{u} = 0$. Eliminating $v_i v_k$ from (6.13) and (6.14) and putting as before $v^2 = c^2$, finally results in

$$\frac{\partial^2 u_k}{\partial t^2} = c^2 \frac{\partial^2 u_k}{\partial x_i^2}. \tag{6.15}$$

or

$$\nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0. \tag{6.16}$$

The line element of a linearized gravitational wave propagating into the x_1-direction is

$$\mathrm{d}s^2 = \mathrm{d}s_0^2 + h_{22}\mathrm{d}x_2^2 + 2h_{23}\mathrm{d}x_2\mathrm{d}x_3 + h_{33}\mathrm{d}x_3^2, \tag{6.17}$$

where $(x \equiv x_1)$

$$h_{22} = -h_{33} = f(t - x/c),\ h_{23} = g(t - x/c) \tag{6.18}$$

with f and g two arbitrary functions, and $\mathrm{d}s_0^2$ the line element in the absence of a gravitational wave. A deformation of an elastic body can likewise be described by a line element

$$\mathrm{d}s^2 = \mathrm{d}s_0^2 + 2\varepsilon_{ik}\mathrm{d}x_i\mathrm{d}s_k, \tag{6.19}$$

where

$$\varepsilon_{ik} = \frac{1}{2}\left(\frac{\partial \varepsilon_i}{\partial x_k} + \frac{\partial \varepsilon_k}{\partial x_i}\right). \tag{6.20}$$

In (6.19) and (6.20) $\vec{\varepsilon} = (\varepsilon_x, \varepsilon_y, \varepsilon_x)$ is the displacement vector, which is related to the velocity disturbance vector $\boldsymbol{u}$ by

$$\boldsymbol{u} = \frac{\partial \vec{\varepsilon}}{\partial t}. \tag{6.21}$$

In an elastic medium, transverse waves obey the wave equation

$$\nabla^2 \vec{\varepsilon} - \frac{1}{c^2}\frac{\partial^2 \vec{\varepsilon}}{\partial t^2} = 0. \tag{6.22}$$

Because of (6.21), this is the same as (6.16). From the condition div $\boldsymbol{u} = 0$ and (6.21) then also follows div $\vec{\varepsilon} = 0$.

For a transverse wave propagating into the x-direction, $\varepsilon_x = \varepsilon_1 = 0$. The condition div $\vec{\varepsilon}$ then leads to

$$\frac{\partial \varepsilon_2}{\partial x_2} + \frac{\partial \varepsilon_3}{\partial x_3} = \varepsilon_{22} + \varepsilon_{33} = 0. \tag{6.23}$$

The same is true for a gravitational wave putting $2\varepsilon_{ik} = h_{ik}$.

Figure 1 illustrates the deformation of the vortex lattice for electromagnetic and gravitational waves.

The totality of the vortex rings can be viewed as a fluid obeying an exactly nonrelativistic equation of motion. It therefore satisfies a nonrelativistic continuity equation which has the same form as the equation for charge conservation

$$\frac{\partial \rho_\mathrm{e}}{\partial t} + \mathrm{div}\boldsymbol{j}_\mathrm{e} = 0, \tag{6.24}$$

where ρ_e and $\boldsymbol{j}_e$ are the electric charge and current density. Because the charges are the source of the electromagnetic field, one has

$$\operatorname{div}\boldsymbol{E} = 4\pi\rho_e, \tag{6.25}$$

and in order to satisfy (6.24), a term must be added to the Maxwell equation if charges are present, which thereby becomes

$$\frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} + \frac{4\pi}{c}\boldsymbol{j}_e = \operatorname{curl}\boldsymbol{H}. \tag{6.26}$$

As the hydrodynamic form (6.6) shows, Maxwell's equation $(-1/c)\partial\boldsymbol{H}/\partial t = \operatorname{curl}\boldsymbol{E}$, is purely kinematic and unchanged by the presence of charges. Finally, because of div $\varphi = 0$, it follows that div $\boldsymbol{H} = 0$.

A gravitational wave propagating into the x-direction obeys the equation

$$\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2}\right)h_{ik} = 0, \quad i,k = 1,2,3,4. \tag{6.27}$$

By a space-time coordinate transformation it can be brought into the form

$$\Box\psi_i^k = 0, \quad i,k = 1,2,3,4 \tag{6.28}$$

with the subsidiary (gauge) condition

$$\frac{\partial\psi_i^k}{\partial x^k} = 0, \tag{6.29}$$

where $\psi_i^k = h_i^k - (1/2)\delta_i^k h$.

In the presence of matter, the gravitational field equation must have the form

$$\Box\psi_i^k = \kappa\Theta_i^k \quad \kappa = \text{const}, \tag{6.30}$$

where Θ_i^k is a four-dimensional tensor. Because of (6.29), it obeys the equation

$$\frac{\partial\Theta_i^k}{\partial x^k} = 0. \tag{6.31}$$

As it was shown by Gupta [20], splitting Θ_i^k in its matter part T_i^k and gravitational field part t_i^k,

$$\Theta_i^k = T_i^k + t_i^k, \tag{6.32}$$

(6.30) can be brought into Einstein's form

$$R_{ik} - \frac{1}{2}g_{ik}R = \kappa T_{ik}. \tag{6.33}$$

It has been argued that any kind of an aether theory with a compressible aether should lead to longitudinal compression waves. Since the Planck mass plasma as a compressible medium falls into this category, we must explain why longitudinal waves are not observed. The transverse vortex lattice waves describing Maxwell's electromagnetic and Einstein's gravitational waves, though, require longitudinal compression waves to couple the vortices of the vortex lattice. For this coupling to work there must be longitudinal waves at least in the wave length range $r_p < \lambda < R$. For short wave lengths approaching the Planck length, the wave equations are modified by the quantum potential. In the limit in which the quantum potential dominates, the equation of motion (3.4) is

$$\frac{\partial\boldsymbol{v}_\pm}{\partial t} = -\frac{1}{m_p}\nabla Q_\pm. \tag{6.34}$$

With the help of the expression for $\nabla Q_\pm$ and the linearized continuity equation one finds that

$$\frac{\partial^2\boldsymbol{v}_\pm}{\partial t^2} = -\frac{\hbar^2}{4m_p^2}\nabla^4\boldsymbol{v}_\pm, \tag{6.35}$$

leading to the nonrelativistic dispersion relation for the free Planck masses

$$\omega = \pm\hbar k^2/2m_p. \tag{6.36}$$

And with the inclusion of the quantum potential the first equation of (3.9) is modified as follows:

$$\frac{\partial^2}{\partial t^2}(\boldsymbol{v}_+ + \boldsymbol{v}_-) = c^2\nabla^2(\boldsymbol{v}_+ + \boldsymbol{v}_-) - \frac{\hbar^2}{4m_p^2}\nabla^4(\boldsymbol{v}_+ + \boldsymbol{v}_-), \tag{6.37}$$

possessing the dispersion relation

$$\omega = \left[c^2k^2 + \frac{\hbar^2}{4m_p}k^4\right]^{1/2} \tag{6.38}$$

first derived by Bogoliubov [23] for a dilute Bose gas. The influence of the quantum potential on the propagation of the waves for wave lengths larger than the Planck length can normally be neglected, but this is not the case if these waves propagate through a vortex lattice. There the influence of the quantum potential is estimated as follows: The quantum potential is largest near the vortex core, where according to (3.10) for

$r > r_p/\sqrt{2}$ the particle number density becomes zero. There we may put $\nabla^4 \approx (\sqrt{2}/r_p)^4 = 4/rp_p^4$, whereby

$$\frac{\hbar^2}{4m_p}\nabla^4 \boldsymbol{v}_\pm \simeq \frac{c^2}{r_p^2}\boldsymbol{v}_\pm . \tag{6.39}$$

If the vortex lattice consists of ring vortices with a ring radius R, and if it is evenly spaced by the same distance, the average number density of Planck masses bound in the vortices under these conditions is of the order of $(R/r_p)/R^3$. The average effect of the quantum potential on the vortex lattice is then obtained by multiplying (6.39) with the factor $r_p^3/R^2 r_p$, whereby one has for the average over the vortex lattice

$$\overline{\frac{\hbar^2}{4m_p}\nabla^4 \boldsymbol{v}_\pm} \simeq \left(\frac{c}{R}\right)^2 \boldsymbol{v}_\pm = \omega_0^2 \boldsymbol{v}_\pm ,\ \omega_0 = c/R. \tag{6.40}$$

With (6.40) the wave equation (6.37) is modified as follows:

$$\frac{\partial^2}{\partial t^2}(\boldsymbol{v}_+ + \boldsymbol{v}_-) = c^2\nabla^2(\boldsymbol{v}_+ + \boldsymbol{v}_-) - \omega_0^2(\boldsymbol{v}_+ + \boldsymbol{v}_-), \tag{6.41}$$

having the dispersion relation

$$\frac{\omega}{k} = \frac{c}{\sqrt{1-(\omega_0/\omega)^2}}. \tag{6.42}$$

It has a cut-off at $\omega = \omega_0 = c/R$, explaining why there are no longitudinal waves for wave lengths $\lambda > R$. Accordingly, these longitudinal waves could be detected only above energies corresponding to the length $R \sim 10^{-30}$ cm, that is above the GUT energy of $\sim 10^{16}$ GeV.

7. Dirac Spinors

A vortex ring of radius R has under elliptic deformations the resonance frequency [21]

$$\omega_v = \frac{cr_p}{R^2}, \tag{7.1}$$

and for the energy of its positive and negative mass component

$$\hbar\omega_v = \pm m_p c^2 \left(\frac{r_p}{R}\right)^2 = \pm m_v c^2. \tag{7.2}$$

For $R/r_p \simeq 1360$ one has $\hbar\omega \simeq 5\times 10^{12}$ GeV. The zero point fluctuations of the Planck mass particles bound in the vortex core are the source of a gravitational charge with Newton's coupling constant, but whereas the gravitational interaction energy of two positive masses is negative, it is positive for the interaction of a positive with a negative mass. Adding the mass m of the small positive gravitational interaction energy to m_v^+, whereby $m^+ = m_v^+ + m$ and $m^- = m_v^-$, one obtains a configuration which has been called a pole-dipole particle. The importance of this is that it can simulate Dirac spinors [22]. With $m^+ > |m^-|$, one has $m^+ - |m^-| \ll m^+$. The center of mass S of this two body configuration is still on the line connecting m^+ with m^-, but not located in between m^+ and m^-, with the distance from S to the position halfway in between m^+ and m^- equal to r_c (see Figure 2). If m^+ is separated by the distance r from m^-, and if the distance of m^+ from the center of mass is $r_c - r/2 \approx r_c$, then because of $m \ll m^+ \cong |m^-|$ one has $r \ll r_c$. Conservation of the center of mass requires that

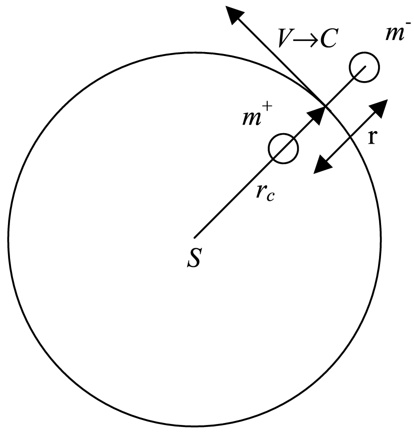


Fig. 2. A pole-dipole particle executes a circular motion around its center of mass S.

$$m^+ r_c = |m^-|(r_c + r). \tag{7.3}$$

The angular momentum of the pole-dipole particle is

$$J_z = \left[m^+ r_c^2 - |m^-|(r_c + r)^2\right]\omega, \tag{7.4}$$

where ω is the angular velocity around the center of mass. With $m = m^+ - |m^-|$ and $p = m^+ r \cong |m^-| \cong mr_c$, where m is the mass pole and p the mass dipole with p directed from m^+ to m^-, one finds with the help of (7.3) that

$$J_z = -m^+ r r_c \omega = -pv. \tag{7.5}$$

In the limit $v = c$, one has

$$J_z = -mcr_c. \tag{7.6}$$

The angular momentum is negative because m^- is separated by a larger distance from the center of mass than m^+.

Applying the solution of the well-known nonrelativistic quantum mechanical two- body problem with Coulomb interaction to the pole-dipole particle with Newtonian interaction, one can obtain an expression for m. For the Coulomb interaction, the groundstate energy is

$$W_0 = -\frac{1}{2}\frac{m^* e^4}{\hbar^2}, \tag{7.7}$$

where m^* is the reduced mass of the two-body system, with the potential energy $-e^2/r$ for two charges $\pm e$, of opposite sign. By comparison, the gravitational potential energy of two masses of opposite sign is $+Gm^+|m^-|/r \cong +G|m_v^\pm|^2/r$ instead, and one thus has to make the substitution $e^2 \to -G|m_v^\pm|^2$.

The reduced mass is

$$\frac{1}{m^*} = \frac{1}{m^+} + \frac{1}{m^-} = \frac{1}{m^+} + \frac{1}{|m^-|} \cong \frac{m}{|m_v^\pm|^2}, \tag{7.8}$$

and by putting $W_0 = mc^2$, one finds from (7.7) that

$$m = \frac{(1/\sqrt{2})|m_v^\pm|^3}{m_\mathrm{p}^2}. \tag{7.9}$$

Because of (7.2) this is

$$m/m_\mathrm{p} = (1/\sqrt{2})(r_\mathrm{p}/R)^6. \tag{7.10}$$

The Bohr radius for the hydrogen atom is

$$r_B = \frac{\hbar^2}{m^* e^2}, \tag{7.11}$$

which by the substitution for e^2 and m^* becomes

$$r_v = \hbar/\sqrt{2}|m_v^\pm|c = \left(r_\mathrm{p}/\sqrt{2}\right)\left(R/r_\mathrm{p}\right)^2. \tag{7.12}$$

With the above computed value $R/r_\mathrm{p} = 1360$ one finds that $m \sim 1$ GeV, about equal the proton mass, and $r_v \sim 2 \times 10^{-27}$ cm.

This result can in a very qualitative way also be obtained as follows: Equating the positive gravitational interaction energy with the rest mass one has $mc^2 = G|m_v|^2/r$, and from the uncertainty principle $|m_v|rc \cong \hbar$. Eliminating r from these two relations one obtains $m/m_\mathrm{p} = (|m_v|/m_\mathrm{p})^3$, and with $|m_v| = m_\mathrm{p}(r_\mathrm{p}/R)^2$, $m/m_\mathrm{p} = (r_\mathrm{p}/R)^6$.

For the electron mass one would have to set $R/r_\mathrm{p} \sim 5000$ instead of $R/r_\mathrm{p} \sim 1360$. This shows that the value for R/r_p obtained from a rough estimate of the vortex lattice dimensions is still very tentative.

With the gravitational interaction of two counter-rotating masses reduced by the factor $\gamma^2 = 1 - v^2/c^2$, where v is the rotational velocity [24], the mass for the vortices in counter-rotating motion is much smaller. By order of magnitude one has $\gamma m_v Rc \sim \hbar$, and hence $\gamma \sim R/r_\mathrm{p}$, whereby instead of (7.10) one has

$$m/m_\mathrm{p} \approx (r_\mathrm{p}/R)^8. \tag{7.13}$$

For $R/r_\mathrm{p} \cong 5 \times 10^3$, this leads to a mass of $\sim 2 \times 10^{-2}$ eV, making it a possible candidate for the neutrino mass.

The result expressed by (7.10) amounts to a computation of the renormalization constant, which in this theory is finite and equal to $\kappa = m_\mathrm{p}\left[1 - \left(1/\sqrt{2}\right)\left(r_\mathrm{p}/R\right)^6\right]$, where $m = m_\mathrm{p} - \kappa$ is the difference between two very large masses.

In the pole-dipole particle configuration the spin angular momentum is the orbital angular momentum of the motion around the center of mass located on the line connecting m^+ with m^-. The rules of quantum mechanics permit radial s-wave oscillations of m^+ against m^-. They lead to the correct angular momentum quantization for the nonrelativistic pole-dipole particle configuration, as can be seen as follows: From Bohr's angular momentum quantization rule $m^* r_v v = \hbar$, where $v = r_v\omega$, one obtains by inserting the values for m^* and r_v that $mc^2 = -(1/2)\hbar\omega$, and because of $\omega = c/r_\mathrm{c}$ that $mr_\mathrm{c}c = -(1/2)\hbar$, or that $|r_\mathrm{c}| = \hbar/2mc$. Inserting this result into (7.6) leads to $J_z = (1/2)\hbar$. Therefore, even in the nonrelativistic limit the correct angular momentum quantization rule is obtained, the only one consistent with Dirac's relativistic wave equation. For the mutual oscillating velocity one finds $v/c = \left(|m_v^\pm|/m_\mathrm{p}\right)^2 = \left(r_\mathrm{p}/R\right)^4 \ll 1$, showing that a nonrelativistic approximation appears well justified.

To reproduce the Dirac equation, the velocity of the double vortex ring, moving as an excitonic quasiparticle on a circle with radius r_c, must become equal to the velocity of light. Because $r_\mathrm{c} = \hbar/2mc$ one has in the limit $m \to 0$, $r_\mathrm{c} \to \infty$ and a straight line for the trajectory.

As shown above the presence of negative masses leads to a "Zitterbewegung," by which a positive mass is accelerated. According to Bopp [25], the presence of negative masses can be accounted for in a generalized dynamics where the Lagrange function also depends on the acceleration. The equations of motion are there derived from the variational principle

$$\delta \int L(q_k, \dot{q}_k, \ddot{q}_k)\mathrm{d}t = 0, \tag{7.14}$$

or from

$$\delta \int \Lambda(x_\alpha, u_\alpha, \dot{u}_\alpha)\mathrm{d}s = 0, \tag{7.15}$$

where $u_\alpha = \mathrm{d}x_\alpha/\mathrm{d}s$, $\dot{u}_\alpha = \mathrm{d}u_\alpha/\mathrm{d}s$, $\mathrm{d}s = \left(1-\beta^2\right)^{1/2}\mathrm{d}t$, $\beta = v/c$, $x_\alpha = (x_1, x_2, x_3, ict)$, and where $L = \Lambda\left(1-\beta^2\right)^{1/2}$. With the subsidiary condition $F = u_\alpha^2 = -c^2$, the Euler-Lagrange equation for (7.15) is

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial\Lambda+\lambda F}{\partial u_\alpha} - \frac{\mathrm{d}}{\mathrm{d}s}\frac{\partial\Lambda}{\partial\dot{u}_\alpha}\right) - \frac{\partial\Lambda}{\partial x_\alpha} = 0 \tag{7.16}$$

with λ a Lagrange multiplier. In the absence of external forces, Λ can only depend on $\dot{u}_\alpha^2$. The most simple assumption is a linear dependence

$$\Lambda = -k_0 - (1/2)k_1\dot{u}_\alpha^2, \tag{7.17}$$

whereby (7.16) becomes

$$\frac{\mathrm{d}}{\mathrm{d}s}(2\lambda u_\alpha + k_1\ddot{u}_\alpha) = 0 \tag{7.18}$$

or

$$2\dot{\lambda}u_\alpha + 2\lambda\dot{u}_\alpha + k_1\dddot{u}_\alpha = 0. \tag{7.19}$$

Differentiating the subsidiary condition $u_\alpha^2 = -c^2$, one has

$$u_\alpha\dot{u}_\alpha = 0,\ u_\alpha\ddot{u}_\alpha + \dot{u}_\alpha^2 = 0,\ u_\alpha\dddot{u}_\alpha + 3\dot{u}_\alpha\ddot{u}_\alpha = 0, \tag{7.20}$$

by which (7.19) becomes

$$-2\dot{\lambda} - 3k_1\dot{u}\ddot{u}_\alpha = -2\dot{\lambda} - \frac{3}{2}k_1\frac{\mathrm{d}}{\mathrm{d}s}(\dot{u}_\alpha^2) = 0. \tag{7.21}$$

This equation has the integral (summation over v)

$$2\lambda = k_0 - (3/2)k_1\dot{u}_v^2, \tag{7.22}$$

where k_0 appears as a constant of integration. By inserting (7.22) into (7.18) the Lagrange multiplier is eliminated and one has

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[\left(k_0 - \frac{3}{2}k_1\dot{u}_v^2\right)u_\alpha + k_1\ddot{u}_\alpha\right] = 0. \tag{7.23}$$

That this is the equation of motion of a pole-dipole particle can be seen by writing it as follows:

$$\frac{\mathrm{d}\boldsymbol{P}_\alpha}{\mathrm{d}s} = 0, \qquad \boldsymbol{P}_\alpha = \left(k_0 - \frac{3}{2}k_1\dot{u}_v^2\right)u_\alpha + k_1\ddot{u}_\alpha. \tag{7.24}$$

By the angular momentum conservation law

$$\frac{\mathrm{d}J_{\alpha\beta}}{\mathrm{d}s} = 0, \tag{7.25}$$

where

$$J_{\alpha\beta} = [\boldsymbol{x}, \boldsymbol{P}]_{\alpha\beta} + [\boldsymbol{u}, \boldsymbol{p}]_{\alpha\beta}, \tag{7.26}$$

the mass dipole moment is equal to

$$p_\alpha = -k_1\dot{u}_\alpha. \tag{7.27}$$

For a particle at rest with $P_k = 0, k = 1,2,3$, one obtains

$$J_{kl} = [\boldsymbol{u}, \boldsymbol{p}]_{kl} = u_k p_l - u_1 p_k, \tag{7.28}$$

which is the spin angular momentum.

The energy of a pole-dipole particle at rest, for which $u_4 = ic\gamma$, is determined by the fourth component of (7.24):

$$P_4 = imc = i\left(k_0 - \frac{3}{2}k_1\dot{u}_v^2\right)c\gamma. \tag{7.29}$$

From (7.24) and (7.27) for $P_k = 0$, $k = 1, 2, 3$, one obtains for the dipole moment p:

$$p = \left(k_0 - \frac{3}{2}k_1\dot{u}_v^2\right)r_\mathrm{c}, \tag{7.30}$$

and because of (7.29) one has

$$p = \frac{mr_\mathrm{c}}{\gamma}. \tag{7.31}$$

With $\boldsymbol{u} = \gamma\boldsymbol{v}$, one obtains for the spin angular momentum

$$J_z = -pu = mvr_\mathrm{c} \cong -mcr_\mathrm{c}, \tag{7.32}$$

which is the same as (7.6).

For the transition to wave mechanics, one needs the equation of motion in canonical form. From $L =$

$\Lambda\,\mathrm{d}s/\mathrm{d}t$ one obtains, by separating space and time parts (using units in which $c=1$):

$$L=-\left(k_0+\frac{1}{2}k_1\dot{u}_\alpha^2\right)(1-v^2)^{1/2},$$
$$\dot{u}_\alpha^2=\frac{1}{[(1-v^2)^{1/2}]^4}\left[\dot{\boldsymbol{v}}^2+\left(\frac{\boldsymbol{v}\cdot\boldsymbol{v}}{(1-v^2)^{1/2}}\right)^2\right], \tag{7.33}$$

where $L=L(\boldsymbol{r},\dot{\boldsymbol{r}},\ddot{\boldsymbol{r}})$. From

$$\boldsymbol{P}=\frac{\partial L}{\partial \boldsymbol{v}}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\boldsymbol{v}}},\quad \boldsymbol{s}=\frac{\partial L}{\partial \dot{\boldsymbol{v}}} \tag{7.34}$$

one can compute the Hamilton function

$$H=\boldsymbol{v}\cdot\boldsymbol{P}+\dot{\boldsymbol{v}}\cdot\boldsymbol{s}-L. \tag{7.35}$$

From $\boldsymbol{s}=\partial L/\partial\dot{\boldsymbol{v}}$ one obtains

$$\boldsymbol{s}=-\frac{k_1}{\sqrt{1-v^2}^3}\left[\dot{\boldsymbol{v}}+\frac{(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})\boldsymbol{v}}{1-v^2}\right],$$
$$\dot{\boldsymbol{v}}=-\frac{\sqrt{1-v^2}^3}{k_1}[\boldsymbol{s}-(\boldsymbol{v}\cdot\boldsymbol{s})\boldsymbol{v}], \tag{7.36}$$

by which, with the help of (7.33), $\dot{\boldsymbol{v}}\cdot\boldsymbol{s}$ can be expressed in terms of $\boldsymbol{v}$ and $\boldsymbol{s}$. In these variables the angular momentum conservation law (7.25) assumes the form

$$\boldsymbol{r}\times\boldsymbol{P}+\boldsymbol{v}\times\boldsymbol{s}=\text{const}, \tag{7.37}$$

with the vector $\boldsymbol{s}$ equal the dipole moment. For the Hamilton function (7.35) one then finds

$$H=\boldsymbol{v}\cdot\boldsymbol{P}+k_0(1-v^2)^{1/2}$$
$$-(1/2k_1)(1-v^2)^{3/2}[\boldsymbol{s}^2-(\boldsymbol{s}\cdot\boldsymbol{v})^2]. \tag{7.38}$$

Putting

$$\boldsymbol{P}=\frac{\hbar}{i}\frac{\partial}{\partial\boldsymbol{r}},\ \ \boldsymbol{v}=\alpha,\ \ (1-v^2)^{1/2}=\alpha_4, \tag{7.39}$$

where $\alpha=\{\alpha,\alpha_4\}$ are the Dirac matrices, one finally obtains the Dirac equation

$$\frac{\hbar}{i}\frac{\partial\psi}{\partial t}+H\psi=0, \tag{7.40}$$

where

$$H=\alpha_1\mathbf{P}_1+\alpha_2\mathbf{P}_2+\alpha_3\mathbf{P}_3+\alpha_4 m,$$
$$\alpha_\mu\alpha_\nu+\alpha_\nu\alpha_\mu=2\delta_{\mu\nu} \tag{7.41}$$

with the mass given by

$$m=k_0-(1/2k_1)(1-k_1)(1-v^2)[\boldsymbol{s}^2-(\boldsymbol{s}\cdot\boldsymbol{v})^2]. \tag{7.42}$$

Higher particle families result from internal excited states of the pole-dipole configuration. They can also be obtained by Bopp's generalized mechanics putting

$$\Lambda=-f(Q),\quad Q=\dot{u}_\alpha^2, \tag{7.43}$$

where $f(Q)$ is an arbitrary function of Q which depends on the internal structure of the mass dipole. With (7.43) one has for (7.16)

$$\frac{\mathrm{d}}{\mathrm{d}s}\left\{[f(Q)-4Qf'(Q)]u_\alpha+2\frac{\mathrm{d}}{\mathrm{d}s}[f'(Q)]\dot{u}_\alpha\right\}=0, \tag{7.44}$$

where

$$p_\alpha=-2f'(Q)\frac{\mathrm{d}u_\alpha}{\mathrm{d}s} \tag{7.45}$$

is the dipole moment. For the simple pole-dipole particle one has according to (7.17) $f(Q)=k_0+(1/2)k_1Q$, whereby $p_\alpha=-k_1\dot{u}_\alpha$ as in (7.27). Instead of (7.36) one now has

$$\mathbf{s}=-2\frac{f'(Q)}{\sqrt{1-v^2}^3}\left[\dot{\boldsymbol{v}}+\frac{(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})\boldsymbol{v}}{1-v^2}\right],$$
$$\dot{\boldsymbol{v}}=-\frac{1}{2}\frac{\sqrt{1-v^2}^3}{f'(Q)}[\boldsymbol{s}-(\boldsymbol{v}\cdot\boldsymbol{s})\boldsymbol{v}]. \tag{7.46}$$

Computation of $\dot{\boldsymbol{v}}\cdot\boldsymbol{s}$ from both of these equations leads to the identity

$$4Qf'(Q)^2=R=(1-v^2)[\boldsymbol{s}^2-(\boldsymbol{v}\cdot\boldsymbol{s})^2], \tag{7.47}$$

from which the function $Q=Q(R)$ can be obtained and by which $\dot{v}$ can be eliminated from H:

$$H=\boldsymbol{v}\cdot\boldsymbol{P}+\dot{\boldsymbol{v}}\cdot\boldsymbol{s}-L=\boldsymbol{v}\cdot\boldsymbol{P}+\sqrt{1-v^2}F(R), \tag{7.48}$$

where

$$F(R)=f(Q)-2Qf'(Q). \tag{7.49}$$

For the wave mechanical treatment of this problem it is convenient to use the four-dimensional representation by making a canonical transformation from the variables $(\boldsymbol{v},\boldsymbol{v}_0;\mathbf{s},\mathbf{s}_0)$ to (u_α,p_α):

$$\boldsymbol{s}\cdot\mathrm{d}\boldsymbol{v}+\boldsymbol{s}_0\mathrm{d}\boldsymbol{v}_0+u_\alpha\mathrm{d}p_\alpha=\mathrm{d}\Phi(\boldsymbol{v},\boldsymbol{v}_\alpha,p_\alpha) \tag{7.50}$$

with the generating function

$$\Phi = \frac{v_0}{\sqrt{1-v^2}}(\boldsymbol{v}\cdot\boldsymbol{p}+ip_4) \quad (7.51)$$

and where v_0, s_0 are superfluous coordinates. Expressed in the new variables, one has

$$R = -\frac{1}{2}M^2_{\alpha\beta}, \quad M_{\alpha\beta} = u_\alpha p_\beta - u_\beta p_\alpha. \quad (7.52)$$

With $P_\alpha = \{\boldsymbol{P}, iH\}$, (7.48) is replaced by

$$K = u_\alpha P_\alpha + \sqrt{-u^2_\alpha}F(R) = 0. \quad (7.53)$$

Because $u^2_\alpha = -1$ and $u_\alpha p_\alpha = 0$, the superfluous coordinates v_o and s_o can be eliminated. Putting

$$P_\alpha = \frac{\hbar}{i}\frac{\partial}{\partial x_\alpha}, \quad p_\alpha = \frac{\hbar}{i}\frac{\partial}{\partial u_\alpha}, \quad (7.54)$$

one obtains the wave equation

$$K\psi \equiv \left[\left(u_\alpha, \frac{\hbar}{i}\frac{\partial}{\partial x_\alpha}\right) + F(R)\right]\psi(x,u) = 0, \quad (7.55)$$

where

$$R = -\frac{1}{2}M^2_{\alpha\beta},\ M_{\alpha\beta}\frac{\hbar}{i}\left(u_\alpha\frac{\partial}{\partial u_\beta} - u_\beta\frac{\partial}{\partial u_\alpha}\right). \quad (7.56)$$

For $\boldsymbol{P} = 0$, the wave function has the form

$$\psi(x,u) = \psi(u)e^{-i\varepsilon t/\hbar} \quad (7.57)$$

with the wave equation for $\psi(u)$

$$F(R)\psi(u) = \frac{\varepsilon}{\sqrt{1-v^2}}\psi(u) \quad (7.58)$$

or, if G is the inverse function of F:

$$R\psi(u) = G\left(\frac{\varepsilon}{\sqrt{1-v^2}}\right)\psi(u). \quad (7.59)$$

From the condition $u_\alpha p_\alpha = 0$ follows that

$$R = -\hbar^2\frac{\partial^2}{\partial u^2_\alpha}. \quad (7.60)$$

With (θ, ϕ spherical polar coordinates)

$$u_\alpha = [\sinh\alpha\sin\theta\cos\phi, \sinh\alpha\sin\theta\sin\phi, \sinh\alpha\cos\theta, \cosh\alpha],$$

$$\psi = \frac{\psi_0}{\sinh\alpha}, \quad \tanh\alpha = v, \quad (7.61)$$

the wave equation becomes

$$-\left[\frac{\partial^2}{\partial\alpha^2} - 1 - \frac{M^2}{\sinh^2\alpha}\right]\psi_0 = G(\varepsilon\cosh\alpha)\psi_0, \quad (7.62)$$

where

$$M^2 = (\boldsymbol{v}\times\boldsymbol{s})^2 = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}, \quad (7.63)$$

having the eigenvalues $j(j+1)$, where j is an integer. The wave equation, therefore, finally becomes

$$\frac{d^2\psi_0}{d\alpha^2} = V(\alpha)\psi_0 = \left[1 + \frac{j(j+1)}{\sinh^2\alpha} - G(\varepsilon\cosh\alpha)\right]\psi_0. \quad (7.64)$$

The eigenvalues can be obtained by the WKB method, with the factor $j(j+1)$ replaced by $(j+1/2)^2$ to account for the singularity at $\alpha = 0$. The eigenvalues are then determined by the equation

$$j = \frac{1}{\pi}\int_{a_1}^{a_2}\sqrt{-V(\alpha)}d\alpha = n + \frac{1}{2},\ n = 0,1,2,\cdots \quad (7.65)$$

with

$$V(\alpha) = 1 + \frac{(j+1/2)^2}{\sinh^2\alpha} - G(\varepsilon\cosh\alpha). \quad (7.66)$$

Of special interest are the cases where $j = -1/2$, because they correspond to the correct angular momentum quantization rule for the Zitterbewegung. For $j = -1/2$ one simply has

$$V(\alpha) = 1 - G(\varepsilon\cosh\alpha). \quad (7.67)$$

To obtain an eigenvalue requires a finite value of the phase integral (7.65). The function $G(x)$, ($x = \varepsilon\cosh\alpha$), must therefore qualitatively have the form of a parabola cutting the line $G = 1$ at two points x_1, x_2, in between which $G(x) > 1$. One can then distinguish two limiting cases. First if $\varepsilon \ll 1$ and a second if $\varepsilon \gg 1$. In both cases one may approximate (7.65) as follows:

$$J \cong (1/\pi)\overline{\sqrt{-V(\alpha)}}(\alpha_2 - \alpha_1). \quad (7.68)$$

In the first case $\alpha \gg 1$ and one has

$$\alpha \cong \ln\left(\frac{x}{\varepsilon} + \sqrt{\frac{x^2}{\varepsilon^2} - 1}\right) \approx \ln\left(\frac{2x}{\varepsilon}\right) - \frac{\varepsilon^2}{4x^2} + \cdots, \tag{7.69}$$

hence

$$\alpha_2 - \alpha_1 \cong \ln\left(\frac{x_2}{x_1}\right) + \frac{x_2^2 - x_1^2}{x_1^2 x_2^2}\frac{\varepsilon^2}{4} + \cdots. \tag{7.70}$$

In the second case one has

$$\alpha \cong \sqrt{2}\left(\frac{x}{\varepsilon} - 1\right)^{1/2}, \tag{7.71}$$

or if $x/\varepsilon \gg 1$ simply

$$\alpha \cong \sqrt{\frac{2x}{\varepsilon}}, \tag{7.72}$$

hence

$$\alpha_2 - \alpha_1 \cong \sqrt{\frac{2}{\varepsilon}}(x_2^{1/2} - x_1^{1/2}). \tag{7.73}$$

One, therefore, sees that the phase integral has for $\varepsilon \ll 1$ the form $J = a + b\varepsilon^2$, but for $\varepsilon \gg 1$ the form $J = a/\sqrt{\varepsilon}$. The $J(\varepsilon)$ curve can for this reason cut twice the lines $J = 1/2 (n = 0)$, and $J = 3/2 (n = 1)$.

In Fig. 3, we have adjusted the phase integral to account for the electron, muon and tau, where $\varepsilon = mc^2$, with m the electron mass. The fact that the mass ratio of the tau and muon are so much smaller than the mass ratio of the muon and electron suggests that both the muon and tau result from cuts of the line $J = 3/2$. Because of the proximity on the $J = 3/2$ line, it is unlikely that the phase integral would cut the line $J = 5/2$ or higher. Since for large values of $\varepsilon, J \propto 1/\sqrt{\varepsilon}$, it follows that there must be one more eigenvalue for which $J = 1/2$, which from the position of the first three families is guessed to be around 80,000 mc$^2 \cong 40$ GeV. This result suggests that there are no more than four particle families. In the framework of the proposed model a more definite conclusion has to await a determination of the structure function $f(Q)$ for the mass distribution of the exciton made up from the positive-negative mass vortex resonance. The finiteness of the number of possible families is here a consequence of Bopp's nonlinear dynamics involving negative masses, not as in superstring theories, where it results from a topological constraint.

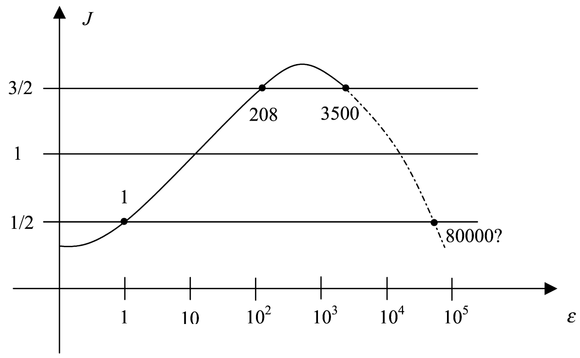


Fig. 3. Qualitative form of the phase integral $J = J(\varepsilon)$ to determine the number of families and their masses.

One may ask what kind of mass distributions generate half integer spin quantization. The answer to this question is that noninteger angular momentum quantization is possible if the rotational motion is accompanied by a simultaneous time-dependent deformation of the positive-negative mass distribution. A superposition of a rotational motion with a simultaneous periodic deformation can for example, occur in rotating molecules if the rotation is accompanied by a time-dependent deformation. As it was demonstrated by Delacretaz et al. [26], it there can lead to half-integer angular momentum quantization. The same must be possible for the positive-negative mass vortex configuration.

8. Finestructure Constant

According to Wilczek [27] the ratio of the baryon mass m to the Planck mass m_p can be expressed in terms of the finestructure constant α at the unification energy of the electroweak and strong interaction:

$$\frac{m}{m_\text{p}} = e^{-\frac{k}{\alpha}}, \tag{8.1}$$

where $k = 11/2\pi$ is a calculable factor computed from the antiscreening of the strong force. One thus has

$$\frac{1}{\alpha} = \frac{2\pi}{11}\log\left(\frac{m_\text{p}}{m}\right). \tag{8.2}$$

With the help of (7.10) one can then compute α [17]:

$$\frac{1}{\alpha} = \frac{12\pi}{11}\log\left(\frac{R}{r_\text{p}}\right). \tag{8.3}$$

For $R/r_\text{p} = 1360$, one finds that $1/\alpha = 24.8$, in surprisingly good agreement with the empirical value, $1/\alpha = 25$.

9. Quantum Mechanics and Lorentz Invariance

According to Einstein and Hopf the friction force acting on a charged particle moving with the velocity v through an electromagnetic radiation field with a frequency dependent spectrum $f(\omega)$ is given by [28]

$$F = -\text{const}\left[f(\omega) - \frac{\omega}{3}\frac{\mathrm{d}f(\omega)}{\mathrm{d}\omega}\right]v. \tag{9.1}$$

This force vanishes if

$$f(\omega) = \text{const}\,\omega^3. \tag{9.2}$$

It is plausible that this is universally true, not only for electromagnetic interactions. But now, the spectrum (9.2) is not only frictionless, but it is also Lorentz invariant.

In the Planck mass plasma the zero point energy results from the "Zitterbewegung" caused by the interaction of positive with negative Planck mass particles. To relax it into a frictionless state as is the case for a superfluid, the spectrum must assume the form (9.2). With $4\pi\omega^2\mathrm{d}\omega$ modes of oscillation in between ω and $\omega+\mathrm{d}\omega$ the energy for each mode must be proportional to ω to obtain, as in quantum mechanics, the ω^3 dependence (9.2). Because the spectrum (9.2) is generated by collective oscillations of the discrete Planck mass particles, it has to be cut off at the Planck frequency $\omega_\mathrm{p} = c/r_\mathrm{p} = 1/t_\mathrm{p}$, where the zero point energy is equal to $(1/2)\hbar\omega_\mathrm{p} = (1/2)m_\mathrm{p}c^2$. It thus follows that the zero point energy of each mode with a frequency $\omega < \omega_\mathrm{p}$ must be $E = (1/2)\hbar\omega$. A cut-off of the zero point energy at the Planck frequency destroys Lorentz invariance, but only for frequencies near the Planck frequency and hence only at extremely high energies. The nonrelativistic Schrödinger equation, in which the zero point energy is expressed through the kinetic energy term $-(\hbar^2/2m)\nabla^2\psi$, therefore remains valid for masses $m < m_\mathrm{p}$, to be replaced by Newtonian mechanics for masses $m > m_\mathrm{p}$.

A cut-off at the Planck frequency generates a distinguished reference system in which the zero point energy spectrum is isotropic and at rest. In this distinguished reference system the scalar potential from which the forces are to be derived satisfies the inhomogeneous wave equation

$$-\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} + \nabla^2\Phi = -4\pi\rho(r,t), \tag{9.3}$$

where $\rho(r,t)$ are the sources of this field. For a body in static equilibrium, at rest in the distinguished reference system for which the sources are those of the body itself, one has

$$\nabla^2\Phi = -4\pi\rho(r). \tag{9.4}$$

if set into absolute motion with the velocity v along the x-axis, the coordinates of the reference system at rest with the moving body are obtained by the Galilei transformation:

$$x' = x - vt,\; y' = y,\; z' = z,\; t' = t, \tag{9.5}$$

transforming (9.3) into

$$-\frac{1}{c^2}\frac{\partial^2\Phi'}{\partial t'^2} + \frac{2v}{c^2}\frac{\partial^2\Phi'}{\partial x'\partial t'} + \left(1-\frac{v^2}{c^2}\right)\frac{\partial^2\Phi'}{\partial x'^2} + \frac{\partial^2\Phi'}{\partial y'^2} + \frac{\partial^2\Phi'}{\partial z'^2} = 4\pi\rho' r' t'. \tag{9.6}$$

After the body has settled into a new equilibrium in which $\partial/\partial t' = 0$, one has instead of (9.4)

$$\left(1-\frac{v^2}{c^2}\right)\frac{\partial^2\Phi'}{\partial x'^2} + \frac{\partial^2\Phi'}{\partial y^2} + \frac{\partial^2\Phi'}{\partial z^2} = -4\pi\rho(x',y,z). \tag{9.7}$$

Comparison of (9.7) with (9.4) shows that the l.h.s. of (9.7) is the same it one sets $\Phi' = \Phi$ and $\mathrm{d}x' = \mathrm{d}x\sqrt{1-v^2/c^2}$. This implies a uniform contraction of the body by the factor $\sqrt{1-v^2/c^2}$, because the sources within the body are contracted by the same factor, whereby the r.h.s. of (9.7) becomes equal to the r.h.s. of (9.4). Since all clocks can be viewed as light clocks (made up from rods with light signals reflected back and forth along the rod), the length contraction leads to slower going clocks, going slower by the same factor. Thus using contracted rods and slower going clocks as measuring devices, (9.3) is Lorentz invariant.

With the repulsive quantum force having its cause in the zero point energy and with the zero point energy Lorentz invariant, it follows that the attractive electric force is balanced by the repulsive quantum force always in such a way that Lorentz invariance appears to be true. Departures from Lorentz invariance could then only be noticed for particle energies near the Planck energy, where Lorentz invariance is violated through the cut-off of the zero point energy spectrum.

10. Gauge Invariance

In Maxwell's equations the electric and magnetic fields can be expressed through a scalar potential $\boldsymbol{\Phi}$ and a vector potential $\boldsymbol{A}$:

$$\boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} - \operatorname{grad}\boldsymbol{\Phi}, \quad \boldsymbol{H} = \operatorname{curl}\boldsymbol{A}. \tag{10.1}$$

$\boldsymbol{E}$ and $\boldsymbol{H}$ remain unchanged under the gauge transformation of the potentials

$$\boldsymbol{\Phi}' = \boldsymbol{\Phi} - \frac{1}{c}\frac{\mathrm{d}f}{\mathrm{d}t}, \quad \boldsymbol{A}' = \boldsymbol{A} + \operatorname{grad} f, \tag{10.2}$$

where f is called the gauge function. Imposing on $\boldsymbol{\Phi}$ and $\boldsymbol{A}$ the Lorentz gauge condition

$$\frac{1}{c}\frac{\partial \boldsymbol{\Phi}}{\partial t} + \operatorname{div}\boldsymbol{A} = 0, \tag{10.3}$$

the gauge function must satisfy the wave equation

$$-\frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} + \nabla^2 f = 0. \tag{10.4}$$

In an electromagnetic field the force on a charge e is

$$\begin{aligned}\boldsymbol{F} &= e\left[\boldsymbol{E} + \frac{1}{c}\boldsymbol{v}\times\mathbf{H}\right]\\ &= e\left[-\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} - \operatorname{grad}\boldsymbol{\Phi} + \frac{1}{c}\boldsymbol{v}\times\operatorname{curl}\boldsymbol{A}\right].\end{aligned} \tag{10.5}$$

By making a gauge transformation of the Hamilton operator in the Schrödinger wave equation, the wave function transforms as

$$\psi' = \psi \exp\left[\frac{ie}{\hbar c} f\right], \tag{10.6}$$

leaving invariant the probability density $\psi^*\psi$.

To give gauge invariance a hydrodynamic interpretation, we compare (10.5) with the force acting on a test body of mass m placed into the moving Planck mass plasma. This force follows from Euler's equation and is

$$F = m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = m\left[\frac{\partial \boldsymbol{v}}{\partial t} + \operatorname{grad}\left(\frac{v^2}{2}\right) - \boldsymbol{v}\times\operatorname{curl}\boldsymbol{v}\right]. \tag{10.7}$$

Complete analogy between (10.5) and (10.7) is established if one sets

$$\boldsymbol{\Phi} = -\frac{m}{2e}v^2, \quad \boldsymbol{A} = -\frac{mc}{e}\boldsymbol{v}. \tag{10.8}$$

According to (10.2) and (10.6) $\boldsymbol{\Phi}$ and $\mathbf{A}$ shift the phase of a Schrödinger wave by

$$\delta\varphi = \frac{e}{\hbar}\int_{t_1}^{t_2}\boldsymbol{\Phi}\,\mathrm{d}t, \quad \delta\varphi = \frac{e}{\hbar c}\oint \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{s}. \tag{10.9}$$

The corresponding expressions for a gravitational field can be directly obtained from the equivalence principle [29]. If $\partial\boldsymbol{v}/\partial t$ is the acceleration and $\vec{\omega}$ the angular velocity of the universe relative to a reference system assumed to be at rest, the inertial forces in this system are

$$\boldsymbol{F} = m\left[\frac{\partial \boldsymbol{v}}{\partial t} + \dot{\vec{\omega}}\times\boldsymbol{r} - \vec{\omega}\times(\vec{\omega}\times\boldsymbol{r}) - \dot{\boldsymbol{r}}\times 2\vec{\omega}\right]. \tag{10.10}$$

For (10.10) we write

$$\boldsymbol{F} = m\left[\hat{\boldsymbol{E}} + \frac{1}{c}\boldsymbol{v}\times\hat{\boldsymbol{H}}\right], \tag{10.11}$$

where

$$\begin{aligned}\hat{\boldsymbol{E}} &= \frac{\partial \boldsymbol{v}}{\partial t} + \dot{\vec{\omega}}\times\boldsymbol{r} - \vec{\omega}\times(\vec{\omega}\times\boldsymbol{r}),\\ \hat{\mathbf{H}} &= -2c\vec{\omega}.\end{aligned} \tag{10.12}$$

With

$$\begin{aligned}&\operatorname{curl}(\dot{\vec{\omega}}\times\boldsymbol{r}) = 2\dot{\omega}\\ &\operatorname{div}(-\vec{\omega}\times(\vec{\omega}\times\boldsymbol{r})) = 2\vec{\omega}^2,\end{aligned} \tag{10.13}$$

one has

$$\operatorname{div}\hat{\boldsymbol{H}} = 0, \quad \frac{1}{c}\frac{\partial \hat{\boldsymbol{H}}}{\partial t} + \operatorname{curl}\hat{\boldsymbol{E}} = 0. \tag{10.14}$$

$\hat{\boldsymbol{E}}$ and $\hat{\boldsymbol{H}}$ can be derived from a scalar and vector potential

$$\hat{\boldsymbol{E}} = -\frac{1}{c}\frac{\partial \hat{\boldsymbol{A}}}{\partial t} - \operatorname{grad}\hat{\boldsymbol{\Phi}}, \quad \hat{\boldsymbol{H}} = \operatorname{curl}\hat{\boldsymbol{A}}. \tag{10.15}$$

Applied to a rotating reference system one has

$$\hat{\boldsymbol{\Phi}} = -\frac{1}{2}(\vec{\omega}\times\boldsymbol{r})^2, \quad \hat{\boldsymbol{A}} = -c(\vec{\omega}\times\boldsymbol{r}), \tag{10.16}$$

or

$$\hat{\boldsymbol{\Phi}} = -\frac{v^2}{2}, \quad \hat{\boldsymbol{A}} = -c\boldsymbol{v}. \tag{10.17}$$

Apart from the factor m/e this is the same as (10.8).

For weak gravitational fields produced by slowly moving matter, Einstein's linearized gravitational field equations permit the gauge condition (replacing the Lorentz gauge)

$$\frac{4}{c}\frac{\partial\hat{\boldsymbol{\Phi}}}{\partial t}+\mathrm{div}\hat{\boldsymbol{A}}=0, \qquad \frac{\partial\hat{\boldsymbol{A}}}{\partial t}=0 \tag{10.18}$$

with the gauge transformation for $\hat{\boldsymbol{\Phi}}$ and $\hat{\boldsymbol{A}}$

$$\hat{\boldsymbol{\Phi}}'=\hat{\boldsymbol{\Phi}}, \qquad \hat{\boldsymbol{A}}'=\hat{\boldsymbol{A}}+\mathrm{grad}\, f, \tag{10.19}$$

where f has to satisfy the potential equation

$$\nabla^2 f=0. \tag{10.20}$$

For a stationary gravitational field the vector potential changes the phase of the Schrödinger wave function according to

$$\psi'=\psi\exp\left[\frac{im}{\hbar c}f\right], \tag{10.21}$$

leading to a phase shift on a closed path

$$\delta\varphi=-\frac{m}{\hbar c}\oint\hat{\boldsymbol{A}}\cdot\mathrm{d}\boldsymbol{s}=\frac{m}{\hbar}\oint\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{s}. \tag{10.22}$$

11. Principle of Equivalence

According to (5.1), Newton's law of gravitational attraction, and with it the property of gravitational mass, has its origin in the zero point fluctuations of the Planck mass particles bound in quantized vortex filaments. For the attraction to make itself felt, both the attracting and attracted mass must be composed of Planck mass particles bound in vortex filaments. The gravitational field generated by a mass M, different from m_p, is the sum of all masses m_p bound in vortex filaments, whereby fractions of m_p are possible for an assembly consisting of an equal number of positive and negative Planck masses, with the positive kinetic energy of the positive Planck masses different from the absolute value of the negative kinetic energy of the negative Planck masses. The gravitational charge of the mass M is $\sqrt{G}M$ and the gravitational field generated by M

$$g=\frac{\sqrt{G}M}{r^2}. \tag{11.1}$$

The force exerted by g on another mass m (of charge $\sqrt{G}m$), which like M is composed of masses m_p bound in vortex filaments, is

$$F=\sqrt{G}m\times g=\frac{GmM}{r^2}, \tag{11.2}$$

but because the vortex lattice propagates tensorial waves, simulating those derived from Einstein's gravitational field equations, the gravitational field appears to be tensorial in the low energy limit.

The equivalence of the gravitational and inertial masses can most easily be demonstrated for the limiting case of an incompressible Planck mass plasma. To prove the principle of equivalence in this limit, we use the equations for an incompressible frictionless fluid

$$\mathrm{div}\,\boldsymbol{v}=0, \quad \rho\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}=-\mathrm{grad}\,p. \tag{11.3}$$

Applied to the positive mass component of (11.3), we have $\rho=nm_\mathrm{p}$, $n=1/2r_\mathrm{p}^3$. In the limit of an incompressible fluid, the pressure p plays the role of a Lagrange multiplier, with which the incompressibility condition $\mathrm{div}\,\boldsymbol{v}=0$ in the Lagrange density function has to be multiplied [37]. The force resulting from a pressure gradient is for this reason a constraint force. The same is true for the inertial forces in general relativity, where they are constraint forces imposed by curvilinear coordinates in a noninertial reference system.

The inertial force $\rho\mathrm{d}\boldsymbol{v}/\mathrm{d}t$ in Euler's equation can be interpreted as a constraint force resulting from the interaction with the Planck masses filling all of space. Apart from those regions occupied by the vortex filaments, the Planck mass plasma is everywhere superfluid and must obey the equation

$$\mathrm{curl}\,\boldsymbol{v}=0. \tag{11.4}$$

With the incompressibility condition $\mathrm{div}\,\boldsymbol{v}=0$, the solutions of Euler's equation for the superfluid regions are solutions of Laplace's equation for the velocity potential ψ

$$\nabla^2\psi=0. \tag{11.5}$$

where $\boldsymbol{v} = -\operatorname{grad}\psi$. A solution of (11.5) is solely determined by the boundary conditions on the surface of the vortex filaments, as can be seen more directly in the following way: With the help of the identity

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \frac{\partial \boldsymbol{v}}{\partial t} + \nabla\left(\frac{v^2}{2}\right) - \boldsymbol{v}\times\operatorname{curl}\boldsymbol{v} \tag{11.6}$$

one can obtain an equation for p by taking the divergence on both sides of Euler's equation

$$\nabla^2\left(\frac{p}{\rho} + \frac{\boldsymbol{v}^2}{2}\right) = \operatorname{div}(\boldsymbol{v}\times\operatorname{curl}\boldsymbol{v}) \tag{11.7}$$

Solving for p one has (up to a function of time depending on the initial conditions but which is otherwise of no interest)

$$\frac{p}{\rho} = -\frac{v^2}{2} - \int \frac{\operatorname{div}(\boldsymbol{v}\times\operatorname{curl}\boldsymbol{v})}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}\boldsymbol{r}'. \tag{11.8}$$

Inserting this expression for p back into Euler's equation, it becomes an integro-differential equation [38]

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \nabla\left(\frac{v^2}{2}\right) + \nabla\int \frac{\operatorname{div}(\boldsymbol{v}\times\operatorname{curl}\boldsymbol{v})}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}\boldsymbol{r}'. \tag{11.9}$$

If the superfluid Planck mass plasma could be set into uniform rotational motion of angular velocity ω, one would have

$$v = r\omega \tag{11.10}$$

and hence

$$|\nabla(v^2/2)| = r\omega^2, \tag{11.11}$$

but because for a superfluid $\operatorname{curl}\boldsymbol{v} = 0$, the velocity field (11.10) is excluded. If set into uniform rotation, a superfluid rather sets up a lattice of parallel vortex filaments, possessing the same average vorticity as a uniform rotation. If space would be permeated by such an array of vortices, it would show up in an anisotropy which is not observed. Other nonrotational motions leading to nonvanishing $\nabla(v^2/2)$-terms involve expansions or dilations, excluded for an incompressible fluid. Therefore, the only remaining term on the r.h.s. of (11.9) is the integral term. It is nonlocal and purely kinematic. Through it a "field" is transmitted to the position $\boldsymbol{r}$ over the distance $|\boldsymbol{r}-\boldsymbol{r}'|$. One can therefore write for the incompressible Planck mass plasma

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \nabla\int \frac{\operatorname{div}(\boldsymbol{v}\times\operatorname{curl}\boldsymbol{v})}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}\boldsymbol{r}'. \tag{11.12}$$

As Einstein's equation of motion for a test body, which is the equation for a geodesic in a curved four-dimensional space, equation (11.12) does not contain the mass of the test body. It is for this reason purely kinematic, giving a kinematic interpretation of inertial mass.

The integral on the r.h.s. of (11.12) must be extended over all regions where $\operatorname{curl}\boldsymbol{v} \neq 0$. These are the regions occupied by vortex filaments. In the Planck mass plasma, the vortex core radius is the Planck length r_p, with the vortex having the azimuthal

$$v_\phi = c(r_\mathrm{p}/r) = 0, \quad r > r_\mathrm{p}. \tag{11.13}$$

Using Stokes' theorem for a surface cutting through the vortex core and having a radius equal to the core radius r_p, one finds that at $r = r_\mathrm{p}$

$$\operatorname{curl}_z\boldsymbol{v} = \frac{2c}{r_\mathrm{p}}. \tag{11.14}$$

A length element of the vortex tube equal to r_p, makes the contribution

$$\int_{r_\mathrm{p}} \frac{\operatorname{div}(\boldsymbol{v}\times\operatorname{curl}\boldsymbol{v})}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}\boldsymbol{r}' \cong \frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}\oint(\boldsymbol{v}\times\operatorname{curl}\boldsymbol{v})\cdot\mathrm{d}\boldsymbol{f}, \tag{11.15}$$

where $\int \mathrm{d}f \cong 2\pi r_\mathrm{p}^2$ is the surface element for a length r_p of the vortex tube. Therefore, each such element r_p contributes

$$\int_{r_\mathrm{p}} \frac{\operatorname{div}(\boldsymbol{v}\times\operatorname{curl}\boldsymbol{v})}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}\boldsymbol{r}' \cong \frac{r_\mathrm{p}c^2}{|\boldsymbol{r}-\boldsymbol{r}'|}. \tag{11.16}$$

The contribution to $\mathrm{d}\boldsymbol{v}/\mathrm{d}t$ of an element located at $\boldsymbol{r}'$ then is

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \frac{r_\mathrm{p}c^2}{|\boldsymbol{r}-\boldsymbol{r}'|} \tag{11.17}$$

Because of $Gm_\mathrm{p}^2 = \hbar c$ and $m_\mathrm{p}r_\mathrm{p}c = \hbar$, this can be written as

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \frac{Gm_\mathrm{p}}{|\boldsymbol{r}-\boldsymbol{r}'|^2}, \tag{11.18}$$

demonstrating the origin of the inertial mass in the gravitational mass of all the Planck masses in the universe bound in vortex filaments.

To obtain a value for the inertial mass of a body, its interaction with all those Planck masses in the universe must be taken into account. By order of magnitude, this sum can be estimated by placing all these Planck masses at an average distance R, with R given by

$$R = \frac{GM}{c^2} \tag{11.19}$$

and where M is the mass of the universe, equal to the sum of all Planck masses. Relation (11.19) follows from general relativity, but it can also be justified if one assumes that the total energy of the universe is zero, with the positive rest mass energy Mc^2 equal to the negative gravitational potential energy $-GM^2/R$. Summing up all contributions to $\mathrm{d}\mathbf{v}/\mathrm{d}t$, one obtains from (11.18) and (11.19)

$$\left|\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\right| \approx \frac{GM}{R^2} = \frac{c^2}{R}, \tag{11.20}$$

as in Mach's principle. Because the gravitational interaction is solely determined by the gravitational mass, the equivalence of the inertial and gravitational mass is obvious.

According to Mach's principle inertia results from a "gravitational field" set up by an accelerated motion of all the masses in the universe relative to an observer. It is for this reason that the hypothetical gravitational field of Mach's principle must act instantaneously with an infinite speed. By contrast, the inertia in the Planck mass plasma is due to the presence of the Planck masses filling all of space relative to which an absolute accelerated motion generates inertia as a purely local effect, not as a global effect as in Mach's principle. Notwithstanding this difference in explaining the origin of inertia in the Planck mass plasma model and in Mach's principle, the latter can be recovered from the Planck mass plasma model provided not only all the masses in the universe are set into an accelerated motion relative to an observer, but with them the Planck mass plasma as well, or what is the same, the physical vacuum of quantum field theory. Only then is complete kinematic equivalence established, and only then would Mach's principle not require action at a distance.

12. Gravitational Self-Shielding of Large Masses

Einstein's gravitational field equations are in the Planck mass plasma nonlinear field equations in flat space-time, permitting to obtain an expression for the field energy density. For a static field this energy density is negative with the negative mass density [29]

$$\rho_{\mathrm{g}} = -\frac{(\nabla\phi)^2}{4\pi Gc^2}, \tag{12.1}$$

where ϕ is the scalar gravitational potential.

With the inclusion of this negative mass density the Poisson equation becomes

$$\nabla^2\phi + \frac{1}{c^2}(\nabla\phi)^2 = 4\pi G\rho. \tag{12.2}$$

Making the substitution

$$\psi = e^{\phi/c^2}, \tag{12.3}$$

(12.2) is transformed into

$$\nabla^2\psi = \frac{4\pi G\rho}{c^2}\psi. \tag{12.4}$$

In spherical coordinates with $\phi = c^2 \ln\psi = 0$ for $r = 0$, (12.4) has for a sphere of constant density the solution

$$\psi = \frac{\sinh kr}{kr}, \quad k^2 = \frac{4\pi G\rho}{c^2}, \tag{12.5}$$

hence

$$\phi = c^2 \ln\left[\frac{\sinh kr}{kr}\right]. \tag{12.6}$$

For small values of r one has

$$\phi = (2\pi/3)G\rho r^2 \tag{12.7}$$

with the force per unit mass

$$g = -\nabla\phi = (4\pi/3)G\rho r \tag{12.8}$$

as in Newton's theory. In general on has

$$g = c^2\left[\frac{1}{r} - \frac{\sqrt{4\pi G\rho}}{c}\operatorname{ctnh}\left(\frac{\sqrt{4\pi G\rho}}{c}r\right)\right], \tag{12.9}$$

which for $r \to \infty$ yields the constant value

$$g = -\sqrt{4\pi G\rho}c. \tag{12.10}$$

This result implies the shielding of a large mass by the negative mass of the gravitational field surrounding the mass. The shielding becomes important at the distance

$$R = \frac{c}{\sqrt{4\pi G\rho}}. \tag{12.11}$$

Inserting into this expression the average mass density of the universe, the distance becomes about equal the radius of the universe as a closed curved space obtained from general relativity. The great similarity of the shielding effect by the negative mass of the gravitational field, with the electrostatic shielding of the positive ions by the negative electrons in a plasma, there leading to the Debye shielding length, suggests that our universe is in reality a Debye-like cell, surrounded by an infinite number of likewise cells, all of them separated from each other by the shielding length (12.11).

In Einstein's curved space-time interpretation of the gravitational field equations the field energy cannot be localized. This is possible in the flat space-time interpretation of the same equations. The negative value of this field energy implies negative masses, which is one of the two components of the Planck mass plasma.

13. Black Hole Entropy

Up to a numerical factor of the order unity, the entropy of a black hole is given by the Bekenstein-Hawking [30, 31] formula

$$S = Akc^3/G\hbar, \tag{13.1}$$

where A is the area of the black hole (k Boltzmann constant). As long as one is only interested in the order of magnitude for S, one can set

$$A = R_0^2, \tag{13.2}$$

where

$$R_0 = \frac{GM}{c^2} \tag{13.3}$$

is the gravitational radius of a mass M. With the Planck length one can write for (13.1)

$$S = (R_0/r_\mathrm{p})^2 k. \tag{13.4}$$

For a solar mass black hole, one has $R_0 \sim 10^5$ cm. With $r_\mathrm{p} \sim 10^{-33}$ cm, the Bekenstein-Hawking entropy (in units in which $k = 1$) gives $S \sim 10^{76}$.

By comparison, the statistical mechanical expression for an assembly of N particles is (up to a logarithmic factor) given by

$$S = Nk. \tag{13.5}$$

Containing about $\sim 10^{57}$ baryons, the statistical mechanical entropy of the sun is $S \sim 10^{57}$, that is 19 orders of magnitude smaller than the entropy given by (13.4). This large discrepancy makes it difficult to understand the Bekenstein-Hawking black hole entropy in the framework of statistical mechanics. An expression for the entropy of a black hole, like the one given by the Bekenstein-Hawking formula can be obtained from string theory, but in the absence of any other verifiable prediction of string theory this should not be taken too seriously. The non-observation of Hawking radiation from disintegrating mini-black holes left over from the early universe, rather should cast doubt on the very existence of general relativistic black holes. General relativity tells us that inside the event horizon of a black hole the time goes in a direction perpendicular to the direction of the time outside of the event horizon, a prediction which makes little sense and cannot be verified. In the Planck mass plasma conjecture, Einstein's field equations are true outside the event horizon for gravitational potentials corresponding to a velocity $v < c$. In approaching the event horizon where $v = c$, infalling matter acquires a large kinetic energy, and if the infalling matter is composed of Dirac spinor quasiparticles, these quasiparticles disintegrate in approaching the event horizon. Inside the event horizon Einstein's equation becomes invalid and physical reality is better described by nonrelativistic Newtonian mechanics.

With Einstein's field equation valid outside the event horizon, Schwarzschild's solution of Einstein's gravitational field equation

$$\begin{aligned} \mathrm{d}s^2 = {} & \frac{\mathrm{d}r^2}{1 - \dfrac{2Gm}{c^2 r}} + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right) \\ & - c^2 \left(1 - \frac{2Gm}{c^2 r}\right) \mathrm{d}t^2 \end{aligned} \tag{13.6}$$

still holds and can be given a simple interpretation in terms of special relativity and Newtonian gravity. According to Newtonian mechanics, a body falling in the gravitational field of a mass M assumes a velocity v given by

$$v^2 = \frac{2GM}{r}. \tag{13.7}$$

In combination with special relativity, it leads to the length contraction

$$\mathrm{d}r = \mathrm{d}r' \sqrt{1 - \frac{v^2}{c^2}} = \mathrm{d}r' \sqrt{1 - \frac{2GM}{c^2 r}} \tag{13.8}$$

and the time dilation

$$dt = dt'/\sqrt{1-v^2/c^2} = dt'/\sqrt{1-2GM/c^2r}. \quad (13.9)$$

Inserting (13.8) and (13.9) into the line element of special relativity $ds^2 = dr'^2 - c^2dt'^2$, one obtains Schwarzschild's line element (13.6).

Unlike dr and dt which are measured at $r \approx \infty$ by an outside observer far away from the mass M, dr' and dt' are measured in the inertial system carried along the infalling body. According to (13.7), the body reaches the velocity $v = c$ at the Schwarzschild radius $R_s = 2GM/c^2 = 2R_0$. The time needed to reach $r = R_s$ is for the outside observer infinite, even though it is finite in the inertial system carried along the infalling body. Since the same is true for a body as a whole collapsing under its own gravitational field, the gravitational collapse time for an outside observer is infinite. In the Planck mass plasma model, this time is finite because in approaching $r = R_s$ the kinetic energy of all the elementary particles, of which the body is composed, rises in proportion to $1/\sqrt{1-v^2/c^2} = 1/\sqrt{1-2GM/c^2r}$, until the energy of the elementary particles reaches the Planck energy, at which the particles disintegrate. This happens at a radius somewhat larger than R_s.

In approaching the Schwarzschild radius R_s, an infalling particle assumes relativistic velocities which permits to set in (13.6) $ds = 0$. The velocity seen by an outside observer therefore is approximately given by

$$\frac{dr}{dt} = -c\left(1 - \frac{R_s}{r}\right) \quad (13.10)$$

with the velocity of the infalling body measured in its own inertial reference system about equal to c. From (13.10) one has

$$-ct = \int \frac{dr}{1-R_s/r} \cong R_s \int \frac{dr}{r-R_s} \quad (13.11)$$

and hence

$$r - R_s = \text{const.}\ e^{-ct/R_s}. \quad (13.12)$$

If at $t = 0$, $r = (a+1)R_s$, $a \gg 1$, it follows from (13.12) that

$$\frac{r-R_s}{R_s} = ae^{-ct/R_s}. \quad (13.13)$$

For an elementary particle of mass m to reach the Planck energy and to disintegrate, it must have an absolute velocity equal to

$$v/c = \sqrt{1-(m/m_p)^2}, \quad (13.14)$$

hence

$$m/m_p = \sqrt{1-v^2/c^2} = \sqrt{1-R_s/r} \quad (13.15)$$

and therefore

$$\frac{r-R_s}{R_s} \cong \left(\frac{m}{m_p}\right)^2. \quad (13.16)$$

Inserting this into (13.13) and solving for $t = t_0$, the time needed to reach the distance r at which the kinetic energy of an infalling particle becomes equal to the Planck energy, one finds

$$t_0 = \frac{R_s}{c}\ln\left[a\left(\frac{m_p}{m}\right)^2\right]. \quad (13.17)$$

After having disintegrated into Planck mass objects, and reached the Schwarzschild radius at $r = R_s$ where $v \approx c$, the time needed to reach $r = 0$ is of the order R_s/c. The logarithmic factor therefore represents the increase of the gravitational collapse time over the nonrelativistic collapse time R_s/c. Only in the limit $m_p \to \infty$ does this time become infinitely long as in general relativity. For an electron one has $m_p/m \sim 10^{22}$ and hence

$$\ln\left(\frac{m_p}{m}\right)^2 \cong 100, \quad (13.18)$$

making the collapse time about 100 times longer. In the example of a solar mass black hole one has $R_s \sim 1$ km, and hence for the nonrelativistic collapse time $R_s/c \sim 3\times10^{-6}$ sec. The gravitational collapse time for a solar mass black hole, with the collapse starting from $r = 100R_s (a \sim 10^2)$, would therefore be $\sim 100R_s/c \sim 3\times10^{-4}$ sec.

During the gravitational collapse, the location of the event horizon defined as the position where $v = c$, develops from "inside out" [32]. Assuming a spherical body of radius R and constant density ρ, the Newtonian potential inside and outside the body is

$$\begin{aligned} \phi_{in} &= -\frac{GM}{2R}\left[3 - \left(\frac{r}{R}\right)^2\right], \quad r < R; \\ \phi_{out} &= -\frac{GM}{r}, \qquad r > R. \end{aligned} \quad (13.19)$$

An infalling test particle would at $r = 0$ reach the maximum velocity v given by

$$\frac{v^2}{2} = -\phi_{in}(0) = \frac{3GM}{2R}. \quad (13.20)$$

Therefore, it would reach the velocity of light if the sphere has contracted to the radius

$$R_1 = 3GM/c^2 = (3/2)R_s = 3R_0. \qquad (13.21)$$

Accordingly, the disintegration of the matter inside the sphere begins at its center after the sphere has collapsed to (3/2) of its Schwarzschild radius.

A qualitatively similar result is obtained from general relativity by taking the Schwarzschild interior solution for an incompressible fluid [33].

In passing through the event horizon (in going from the subluminal to the superluminal state), the Planck mass plasma looses its superfluidity, and the vortex rings responsible for the formation of Dirac spinor quasiparticles disintegrate. What is left are rotons and phonons, the quasiparticles in the Hartree approximation.

To compute the black hole entropy, we make order of magnitude estimates. If the negative gravitational energy of a spherical body of mass M and radius R, which by order of magnitude is GM^2/R, is set equal its rest mass energy Mc^2, the body has contracted to the gravitational radius $R_0 = GM/c^2$. After the body has passed through the event horizon, its internal energy is confined and cannot be lost by radiation of otherwise. In passing through the event horizon, the matter of the collapsing body has disintegrated into N_0 Planck mass objects (like rotons), the number of which is estimated by setting

$$N_0 m_p c^2 = \frac{GM^2}{R_0} = Mc^2 \qquad (13.22)$$

and therefore

$$N_0 = M/m_p. \qquad (13.23)$$

For a solar mass $M \sim 10^{33}$ g (with $m_p \sim 10^{-5}$ g), one has $N_0 \sim 10^{38}$. At the beginning of the collapse, one would have (in units in which $k = 1$) for the entropy of the sun $S \sim N_0 \sim 10^{38}$. If the collapse proceeds to a radius $r < R_0$, the number N of Planck mass objects increases according to

$$N m_p c^2 = \frac{GM^2}{r} \qquad (13.24)$$

or as

$$N/N_0 = R_0/r. \qquad (13.25)$$

This increase is possible because the vacuum has an unlimited supply of Planck mass particles in the superfluid groundstate. With the negative energy of the gravitational field, interpreted by an excess of negative over positive Planck mass objects, the gravitational collapse below $r = R_0$ implies that a pair of positive and negative Planck mass objects are generated out of the superfluid groundstate, with the positive Planck masses accumulating in the collapsing body, and with the negative Planck masses, from which the positive Planck masses have been separated, surrounding the body.

The Planck mass objects, consisting of rotons etc., cannot interact gravitationally for a distance of separation smaller than a Planck length. It is for this reason that a spherical assembly of N Planck masses cannot collapse through the action of gravitational forces below a radius less than

$$r_0 = N^{1/3} r_p. \qquad (13.26)$$

The core of a solar mass black hole, for example, would have a radius $r_0 \sim 10^{19} r_p \sim 10^{-14}$ cm. Inserting (13.26) into (13.24), one has

$$N = (R_0/r_p)^{3/2}. \qquad (13.27)$$

In statistical mechanics, the entropy of a gas composed of N Planck mass particles m_p with a temperature T is given by the Sackur- Tetrode formula [34]

$$S = Nk \ln\left[(V/N)(2\pi m_p kT)^{3/2} e^{5/2}/h^3\right], \qquad (13.28)$$

where $V/N = r_p^3$, with the smallest phase space volume $(\Delta p \Delta q)^3 = (m_p c r_p)^3 = \hbar^3$. The temperature T in the core of the black hole is the Davies-Unruh temperature [35,36]

$$T = \hbar a / 2\pi kc, \qquad (13.29)$$

where $a = GM/r_0^2$ is the gravitational acceleration at $r = r_0$. With the help of (13.26) and (13.27) one obtains from (13.28)

$$S = (5/2)(R_0/r_p)^{3/2} k = (5/2)Nk. \qquad (13.30)$$

The entropy of a black hole therefore goes here in proportion of the (3/4) power of the black hole area, instead in proportion of the area as in the Bekenstein-Hawking entropy. The entropy of two black holes with area A_1 and A_2 prior to their merger is here proportional to

$$S_0 = A_1^{3/4} + A_2^{3/4}, \qquad (13.31)$$

and after their merger proportional to

$$S_1 = A^{3/4}, \tag{13.32}$$

where

$$A^{3/4} = (A_1^{3/4} + A_2^{3/4})^{3/2} > A_1^{3/4} + A_2^{3/4} \tag{13.33}$$

as required by the second law of thermodynamics.

With the black hole entropy proportional to the (3/4) power of its area, the entropy of a solar mass black hole is reduced from the Bekenstein-Hawking value $S \sim 10^{76}$ to $S \sim 10^{57}$, about equal the statistical mechanical value if N is set equal the number of baryons in the sun.

14. Analogies between General Relativity, Non-Abelian Gauge Theories and Superfluid Vortex Dynamics

In Einstein's gravitational field theory the force on a particle is expressed by the Christoffel symbols which are obtained from first order derivatives of the ten potentials of the gravitational field represented by the ten components of the metric tensor. From the Christoffel symbols the Riemann curvature tensor is structured by the symbolic equation

$$\mathbf{R} = \mathrm{Curl}\,\mathbf{\Gamma} + \mathbf{\Gamma} \times \mathbf{\Gamma}. \tag{14.1}$$

The expression for the field strength, and hence force, in Yang-Mills field theories is symbolically given by (g coupling constant with the dimension of electric charge)

$$\boldsymbol{F} = \mathrm{Curl}\,\boldsymbol{A} - g^{-1}\boldsymbol{A} \times \boldsymbol{A}. \tag{14.2}$$

It too has the form of a curvature tensor, albeit not in space-time, but in internal charge space, in QCD for example, in color space. It was Riemann who wondered if in the small there might be a departure from the Euclidean geometry. The Yang-Mills field theories have answered this question in a quite unexpected way, not as a non-Euclidean structure in space-time but rather one in charge-space, making itself felt only in the small.

Comparing (14.1) with (14.2) one can from a gauge-field theoretic point of view consider the Christoffel symbols Γ_{kl}^i as gauge fields. If the curvature tensor vanishes, they can be globally eliminated by a transformation to a pseudoeuclidean Minkowski space-time metric. One may for this reason call the Γ_{kl}^i pure gauge fields, which for a vanishing curvature tensor can always be transformed away by a gauge transformation. Likewise, if the curvature tensor in (14.2) vanishes, one can globally transform away the gauge potentials.

Table 1. Hierachical displacement of Einstein and Yang-Mills field theories as related to the hierachical displacement of Newton point-particle, and Helmholtz line-vortex dynamics.

Newton point-particle dynamics and Einstein's gravitational field theory		Kinematic Quantities	Helmholtz line-vortex dynamics and Yang-Mills field theories	
		$\boldsymbol{r}$	ψ f	velocity potential, gauge functions
Newtonian potential, metric tensor	ϕ g_{ik}	$\dot{\boldsymbol{r}}$	$-\nabla\psi$ A	Force on line vortex, gauge potentials
Force on point particle, gravitational force field expressed by Christoffel symbols	$-\nabla\phi$ Γ	$\ddot{\boldsymbol{r}}$	$W = \nabla \times A - g^{-1}A \times A$	Yang-Mills force field expressed by charge space curvature tensor
Einstein's field equations expressed by metric-space curvature tensor	$R = \nabla \times \Gamma + \Gamma \times \Gamma$			

From the Newtonian point of view, contained in Einstein's field equations, the force is the first derivative of a potential. Apart from the nonlinear term in (14.2) this is also true for a Yang-Mills field theory. But with the inclusion of the nonlinear terms, (14.2) has also the structure of a curvature tensor. In Einstein's theory the curvature tensor involves second order derivatives of the potentials, whereas in a Yang-Mills field theory the curvature tensor in charge space involves only first order derivatives of the potentials. This demonstrates a displacement of the hierarchy for the potentials with regard to the forces. A displacement of hierarchies also occurs in fluid dynamics by comparing Newton's point particle dynamics with Helmholtz's line vortex dynamics. Whereas in Newton's point particle dynamics the equation of motion is $m\ddot{\boldsymbol{r}} = \boldsymbol{F}$, the corresponding equation in Helmholtz's vortex dynamics is $\mu\dot{\boldsymbol{r}} = \boldsymbol{F}$, where μ is an effective mass [37]. Therefore, whereas in Newtonian mechanics a body moves with constant velocity in the absence of a force, it remains at rest in vortex dynamics. And whereas what is at rest remains undetermined in Newtonian mechanics, it is fully determined in vortex dynamics, where at rest means at rest with regard to the fluid. The same would have to be true with regard to the Planck mass plasma.

The hydrodynamics of the Planck mass plasma suggests that the hierarchical displacement of the curvature tensor for Einstein and Yang-Mills fields is related to the hierarchical displacement of the vortex equation of motion and the Newtonian point particle equation of motion. The hierarchical displacement and analogies to hydrodynamics is made complete by recognizing that the gauge function f is related to the velocity potential of an irrotational flow. A gauge transformation leaving the forces unchanged corresponds in the hydrodynamic picture to the addition of an irrotational flow field. These analogies and hierarchical displacements of Newtonian point mechanics and Einstein gravity, versus Helmholtz's vortex dynamics and Yang-Mills gauge field theories are shown in Table 1 [38].

15. Quark-Lepton Symmetries

For the Planck mass plasma conjecture to be a credible candidate as a model of a theory of elementary particles it should be able to make at least plausible the existence of the quark-lepton symmetries actually observed. The fact that quarks cannot exist as free particles indicates that they are bound like phonons are bound in a solid. In accordance with this analogy, quarks may be the result of localized condensates of the Planck mass plasma and may only occur within these localized regions. This interpretation receives support from condensed matter physics, where fractional electron charges are observed in thin layers and are responsible for the anomalous quantum Hall effect. At least three points define a plane, making it plausible why a minimum of three quarks are needed to form a stable configuration.

As Laughlin [39] has shown, an electron gas confined within a thin sheet can be described by the wave function

$$\psi(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots \boldsymbol{r}_N) = \Big[\prod_{(j<k)} (z_j - z_k)^{\mu}\Big] \Big[\prod_{(j)} e^{-|z_j|^2/4\ell^2}\Big], \tag{15.1}$$

where $z_j = x_j - iy_j$ is the coordinate of the j^{th} electron in complex notation, and $\ell^2 = \hbar c/eH$ (obtained from $m\ell^2\omega = \hbar$ and $\omega = eH/mc$ for the lowest Landau level) with the magnetic field H directed perpendicular to the sheet. If μ is an odd integer, the wave function is completely antisymmetric, obeying Fermi statistics, made up from states of the first Landau level with the kinetic energy equal to $(1/2)\hbar\omega$ per electron. For the square of the wave function one has

$$|\psi|^2 = e^{-\beta H},$$

$$\beta H = 2\mu \sum_{(j<k)} \ln|\boldsymbol{r}_j - \boldsymbol{r}_k| + (1/2\ell^2)\sum_j |\boldsymbol{r}_j|^2, \tag{15.2}$$

which is the probability distribution $|\psi|^2$ of a one-component two-dimensional plasma.

For $\mu = 1$, the wave function is a Slater determinant, but this wave function does not describe the situation actually observed. Numerical calculations for four to six electrons done by Laughlin rather show that the wave function (15.1) for $\mu = 3$ gives a much better agreement. It is this wave function which satisfactorily explains the fractional quantized Hall effect, in which plateaus in the conductivity are found to occur in multiple steps of $(1/3)e^2/h$.

The meaning of the wave function can be understood if one keeps all electrons, except one, fixed in their position and carries out a closed loop motion of the one electron around a point at which the wave function vanishes. This displacement produces the phase shift

$$\Delta\phi = (e/\hbar c)\oint \boldsymbol{A}\cdot \mathrm{d}\boldsymbol{s} = (e/\hbar c)\int \boldsymbol{H}\cdot \mathrm{d}\boldsymbol{f} \tag{15.3}$$

where $\boldsymbol{H} = \operatorname{curl}\boldsymbol{A}$. Accordingly, there should be

$$Z = (e/\hbar c)\int \boldsymbol{H}\cdot \mathrm{d}\boldsymbol{f} \tag{15.4}$$

vortices within the area $\int \mathrm{d}\boldsymbol{f}$. To satisfy the Pauli principle there must be at least one vortex at the position of each electron. In Laughlin's wave function there are exactly μ vortices for each electron. We therefore have to put $Z = \mu$. The fractional quantized Hall effect then simply means that the charge of one vortex is $e/3$ provided $\mu = 3$. It follows that in the two-dimensional electron fluid each electron splits into three vortices of charge $e/3$. The quantization condition for the vortices in the presence of a magnetic field is given by

$$\oint \boldsymbol{v}\cdot \mathrm{d}\boldsymbol{s} = \frac{h}{m}\nu - \frac{e}{mc}\int \boldsymbol{H}\cdot \mathrm{d}\boldsymbol{f}, \quad \nu = 1, 2, 3, \ldots, \tag{15.5}$$

which shows that the presence of a magnetic field generates the vortices in the electron fluid. If a magnetic field is adiabatically applied to the electron fluid, the Helmholtz theorem

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint \boldsymbol{v}\cdot \mathrm{d}\boldsymbol{s} = 0 \tag{15.6}$$

states that if the circulation $\oint \boldsymbol{v} \bullet \mathrm{d}\boldsymbol{s}$ is zero before a magnetic field is applied, it remains zero thereafter.

This, of course, does not imply that the circulation inside the contour taken in (15.6) cannot differ from zero, because the circulation of different vortices can add up to zero, as it would be the case for four vortices with equal and opposite circulation. A vortex configuration with the total angular momentum $(1/2)\hbar$ could be constructed from two vortices with opposite circulation quantum number $v = \pm 1$, and one vortex with $v = 1$ with the spin quantum numbers adding up to a total angular momentum $(1/2)\hbar$.

To explain the fractionally charged quarks, we propose that they result from the splitting up of the electron- and neutrino-wave functions into vortices, with the splitting up caused by a strong field acting like a magnetic field. Leaving for the moment aside the question what that field might be, the occurrence of three-quark configurations suggests that this field acts within a thin sheet, with the vortices perpendicular to and confined within this sheet and with a minimum of three vortices needed to define the orientation of a planar sheet.

If the electron- and neutrino-wave functions split up in a similar way as in the fractionally quantized Hall effect, the quark-lepton symmetries can easily be understood. The angular momentum of a vortex in units of $\hbar$ is equal to the circulation quantum number v. The vortices for which $v = 1$ we call A, those for which $v = 0$ we call B, and finally those for which $v = -1$ we call C. The neutrino (v) and positron (e_+) wave functions are then to be represented by six vortex states, with the lower indices giving the value for the electric charge of these vortex states:

$$(v) = \begin{cases} A_0 \\ B_0 \\ C_0 \end{cases} , \quad (\mathrm{e}_+) = \begin{cases} A_{1/3} \\ B_{1/3} \\ C_{1/3} \end{cases} . \qquad (15.7)$$

A second set of six vortex states is obtained by replacing the neutrino and positron by their antiparticles. We claim that the first six vortex states can reproduce all the six u and d quarks of the first family. How the neutrino and positron are composed of these vortex states is already shown in (15.7). With the index r, g, b (red, green, blue) identifying what is called the color, we have for the colors of the three u quarks

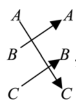
$$u_\mathrm{r} = \begin{cases} A_{1/3} \\ B_{1/3} \\ C_0 \end{cases} , u_\mathrm{g} = \begin{cases} A_{1/3} \\ B_0 \\ C_{1/3} \end{cases} , u_\mathrm{b} = \begin{cases} A_0 \\ B_{1/3} \\ C_{1/3} \end{cases} . \qquad (15.8)$$

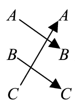
For the three d quarks we have

$$d_\mathrm{r} = \begin{cases} A_0 \\ B_0 \\ C_{-1/3} \end{cases} , d_\mathrm{g} = \begin{cases} A_0 \\ B_{-1/3} \\ C_0 \end{cases} , d_\mathrm{b} = \begin{cases} A_{-1/3} \\ B_0 \\ C_0 \end{cases} . \qquad (15.9)$$

Because the vortices are substates of the leptons, color confinement simply means that only those vortex configurations which can be combined into leptons are able to assume the form of free particles. Mesons are made up from quark–antiquark configurations, each containing three vortices and three antivortices.

If the vortices interact, they do this by the exchange of bosons. But because they are confined within a thin sheet, the bosons are massive, very much as an electromagnetic wave in a wave guide where the photons are massive and have a longitudinal component in addition to their transverse component. It is then possible to explain the eight gluons of the standard model. The gluons are bosons transmitting angular momentum. To change an A vortex into a B vortex, a B vortex into a C vortex, or vice versa, requires a change in the angular momentum $\Delta L = \pm 1$, and to change an A vortex into a C vortex, or vice versa, a change by $\Delta L = \pm 2$ is needed. These changes can be made by just two angular momentum operators with $L = 1$ and $L = 2$, having $\sum(2L+1) = 2+1+4+1 = 8$ states, equal to the number of the gluons in quantum chromodynamics (QCD). The transitions in QCD, identified by red-green (r–g), red-blue (r–b), and green-blue (g–b), lead to changes in the angular momentum of the vortices in the following way

r → g = g → b : 

, r → b : 

. (15.10)

These transitions require four changes $\Delta L = \pm 1$, and two changes $\Delta L = \pm 2$, in total 6 changes. In addition, there is the change for which $\Delta L = 0$:

$$\mathrm{r} \to \mathrm{r} = \mathrm{g} \to \mathrm{g} = \mathrm{b} \to \mathrm{b} : \quad \begin{matrix} \mathrm{A} \to \mathrm{A} \\ \mathrm{B} \to \mathrm{B} \\ \mathrm{C} \to \mathrm{C} \end{matrix} \qquad (15.11)$$

realized with the two $L_z = 0$ -components of the angular momentum operators for $\Delta L = \pm 1$ and $\Delta L = \pm 2$, to be counted twice. Together with the 6 changes involving $\Delta L = \pm 1, \pm 2$ one has a total of 8 possible changes involving the exchange of angular momentum. The eight gluons of QCD are not to be identified with these eight angular momentum transmitting bosons, but rather with certain combinations of them. A color-changing gluon would be always a superposition of a spin 1 and a spin 2 boson. A transition leaving the color unchanged would involve the superposition of spin 2 or spin 1 bosons. The color charge is thereby reduced to angular momentum and through angular momentum quantization to the zero point fluctuations of the Planck masses, like the other charges. The weak interaction phenomenon is explained by the exchange of bosons made up from spin 1 and spin 2 angular momentum transitions in between the vortices of the leptons given by (15.7). The same decomposition into vortex states done here for the first family can be repeated for the following generations.

We remark that the model can be compared with the rishon model, with the rishons turning out to be vortex states. The three hypercolor charges of the rishon model [40,41] are the three angular momentum states $L = 1, 0, -1$ of the vortices. The prescription of the rishon model that only those configurations are possible which are color neutral with regard to the hypercolor, is explained by the requirement that the vortex states must add up to zero angular momentum.

In the standard model the weak vector boson mass can be expressed by the "weak magnetic vector potential" **A** of the Weinberg-Salam-Glashow (WSG) theory as follows (with the vacuum gauge $\boldsymbol{A} = 0$):

$$m_{\mathrm{W}}c^2 = eA. \tag{15.12}$$

We suggest that the "weak magnetic field" of WSG theory is the cause for the lepton wave functions to split up into the vortices representing the wave functions of the quarks, implying that

$$m_{\mathrm{W}}c^2 = 2^{1/4} g\sqrt{\hbar c/G_{\mathrm{F}}} \cong 85\mathrm{GeV}, \tag{15.13}$$

where g is the semiweak coupling constant and G_{F} the Fermi weak interaction constant. If the vortices into which a lepton splits up are line vortices in a sheet of thickness δ, with the vortices ending at the two surfaces of the sheet, the "weak magnetic field" in the sheet must be of the order $H \sim A/\delta$. And because of $m_{\mathrm{W}}c\delta \cong \hbar$, one finds from an expression for H:

$$H = \left(m_{\mathrm{W}}c^2\right)^2 / e\hbar c \cong 10^{26}\,[\mathrm{esu}], \tag{15.14}$$

corresponding to a huge "weak magnetic field" of $\sim 10^{26}$ Gauss.

In the WSG theory the particles receive their mass from the hypothetical Higgs particle, whereas in the Planck mass plasma conjecture they have their cause in a vortex resonance of an energy of the order $\sim 10^{12}$ GeV.

16. Quantum Mechanical Nonlocality

According to the Copenhagen interpretation, the wave function is only an expression of our knowledge, and any attempt to explain the nonlocal effects by some underlying "hidden parameters" is doomed to fail. With the Copenhagen interpretation ultimately requiring conscious observers, absent from most places of the physical universe, not all physicists have adopted the Copenhagen interpretation as gospel truth. The most widely known attempt for a different interpretation is the pilot wave theory by de Broglie and Bohm, viewing the Schrödinger wave as a guiding field for the particle motion. Apart from its violation of Newton's actio = reactio, with the particle guided by the wave not exerting a recoil on the wave, the pilot wave theory leaves unanswered the question of the physical character of the wave. In keeping the whole mathematical structure of quantum theory, the pilot wave theory was called by Einstein as "too cheap". With physics having its origin in very high energies near the Planck energy, attempts to understand quantum mechanics should be undertaken at these high energies. In the Planck mass plasma conjecture, physics has its foundation in the Newtonian mechanics of positive and negative Planck masses, locally interacting over short distances. The nonlocal effects of quantum mechanics should for this reason be explained by the local interaction of all the Planck masses. In the Planck mass plasma all particles, save and except the Planck mass particles themselves, are quasiparticles, that is they are collective modes involving a very large number of Planck mass particles. The occurrence of nonlocal (resp. with infinite velocity) transmitted actions is not so uncommon to classical physics. In the theory of incompressible fluid dynamics, pressure forces are transmitted with infinite speed, giving the wrong physical perception of nonlocality. The nonlocality of incompressible fluid dynamics is a result of the mathematical model of an incompressible fluid which in reality is an approximation, valid for velocities small compared to the velocity of sound. The same argument can be made for quantum mechanics, if one recognizes quantum mechanics as a model-dependent approximation. With the Planck energy so very much larger than the energy scale of everyday life, quantum mechanics would remain a very good approximation. The superluminal nonlocal quantum mechanical actions are perhaps the strongest case for a background medium which may have wave modes with a phase velocity exceeding the velocity of light. Because the nonlocal actions of quantum mechanics cannot transmit any information, the existence of superluminal phase velocities is all what is needed.

We will now show that with the Planck mass plasma conjecture, many of the mysteries surrounding quantum mechanics don't look so mysterious after all. If physics has its root at the extremely high Planck energy, it should be of no surprise that effects projected from this energy scale down to the energy scale of everyday life may look incomprehensible.

A good example for quantum mechanical nonlocality is the Aharonov-Bohm effect, and it is instructive to compare it with the Sagnac effect. The outcome of his rotating interferometer experiment was used by Sagnac as a decisive argument against Einstein's claim that physics could do without the aether hypothesis [42]. Most text-books treating the Sagnac effect, like the well-known theoretical physics course by Sommerfeld [43], explain the effect by Sagnac's original argument, that it is caused by a whirling motion of the aether felt in a rotating reference system, but it is also often stated that the effect somehow is caused by the centrifugal- and Coriolis-forces and for this reason should be treated in the framework of general relativity. A discussion of the Sagnac effect within general relativity can be found in the theoretical physics lectures by Landau and Lifshitz [44], where it is alleged that the effect is caused by the special stationary gravitational field set up in a rotating reference frame, even though there are no masses present as the source of a true gravitational field. Already Ives [45], had given convincing arguments that the effect has nothing to do with the inertial forces in a rotating reference frame, keeping Sagnac's original argument to be valid as ever.

The interpretation of the Sagnac effect, which says that it is caused by an aether wind in the rotating reference system, would make in this system the velocity of light anisotropic. Such a conclusion seems to contradict the outcome of the Michelson-Morley and other experiments which claim to have proven the constancy of the velocity of light, which is why Einstein discarded the aether hypothesis. A close examination of all these experiments, however, shows that one there never measures the one-way velocity of light, but rather always the to and fro velocity. For this reason only the scalar c^2, not the vector $\boldsymbol{c}$, enters into the Lorentz transformation formulas, and Sagnac's interpretation is not subject to any logical contradiction if the dynamic interpretation of special relativity is adopted.

An effect where a phase shift on an electron wave occurs in the absence of any electromagnetic forces has been described by Aharonov and Bohm [46]. There, a change in the magnetic vector potential alone can produce a shift. The effect is explained as a direct consequence of Schrödinger's wave equation, into which the potentials and not the force fields enter. For this reason a case was made by Aharonov and Bohm to give the potentials a more direct physical meaning rather than to be just a convenient mathematical tool. However, to elevate the potentials to true physical significance has the problem that these potentials can always be changed by a gauge transformation without affecting the physical results derived from them. Because the potentials can measurably influence the phase of an electron wave in the absence of any electromagnetic force fields, it has alternatively been claimed that the effect is a proof for action at a distance in quantum mechanics, where the magnetic force field inside a magnetic solenoid can influence the phase of the electron wave in the force-free region outside the solenoid. Instead we will show that both the Sagnac and the Aharonov-Bohm effect can be understood to result from a rotational aether motion, revealing a close relationship between these effects. This picture not only gives a full physical explanation of the potentials and the meaning of a gauge transformation but also eliminates the need for any hypothetical action at a distance.

In the Sagnac effect, a light beam is split into two separate beams, each one following a half circle of radius r along the periphery of a rotating table. One of the two beams is propagating in the same direction as the velocity of the rotating table, the other one in the opposite direction. The rotating table shall have the angular velocity $-\Omega$, making its velocity at the radius r equal to $u = -r\Omega$.

According to the hypothesis of an aether at rest in the unaccelerated laboratory, the velocity of light judged from a co-rotating reference system would be $c - v$ for the beam propagating in the same direction as the rotating table and $c + v$ for the beam propagating in the opposite direction, where $v = -u$ is the aether velocity in the rotating reference system. The time difference for both beams leaving from the position A and arriving at position B follows from

$$c\delta t = \int_{A}^{B} (c+v)\mathrm{d}t - \int_{A}^{B} (c-v)\mathrm{d}t, \tag{16.1}$$

hence

$$\delta t = (1/c) \oint v \mathrm{d}t. \tag{16.2}$$

If we are only interested in first order effects in v/c, we can put $\mathrm{d}t \cong (1/c)\mathrm{d}s$, where $\mathrm{d}s$ is the line element along the circular path. If the light propagates along an arbitrary but simply connected curve we then have

$$\delta t = (1/c^2) \oint \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{s}, \tag{16.3}$$

for which we can also write

$$\delta t = (1/c^2) \oint \mathrm{curl}\,\boldsymbol{v} \cdot \mathrm{d}\boldsymbol{f}. \tag{16.4}$$

where $F = |\int \mathrm{d}\boldsymbol{f}|$ is the surface enclosed by the light path. Because $\Omega = (1/2)|\mathrm{curl}\,\boldsymbol{v}|$ we find

$$\delta t = 2\Omega F/c^2, \tag{16.5}$$

which agrees with the result obtained by Sagnac.

For the phase shift $\delta\varphi = \omega\delta t$, where ω is the circular frequency of the wave, we have

$$\delta\varphi = (\omega/c^2) \oint \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{s}. \tag{16.6}$$

Alternatively, the phase shift can be seen to be caused by the gravitational vector potential resulting from a circular flow of the Planck mass plasma, with the principle of equivalence precisely relating this circular flow to the angular velocity of a rotating platform. According to (10.17) one has for the gravitational vector potential in a rotating frame of reference

$$\hat{\boldsymbol{A}} = -\boldsymbol{v}c = -\Omega c\boldsymbol{r} \tag{16.7}$$

with the phase shift given by (10.22). For photons of frequency ν, and $mc^2 = h\nu = \hbar\omega$, one has

$$\delta\varphi = (\omega/c^2) \oint \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{s} = 2\Omega(\pi r^2)(\omega/c^2), \tag{16.8}$$

the same as predicted by Sagnac without quantum mechanics.

The formula (10.22) can also be applied to a neutron interferometer placed on a rotating platform. An experiment of this kind, using the rotating earth as in the Michelson-Gale version of the Sagnac experiment, was actually carried out [47], confirming the theoretically predicted phase shift.

Next we compute the phase shift (10.9) by a magnetic vector potential. To make a comparison with the gravitational vector potential in the Sagnac effect, we consider the magnetic field produced by an infinitely long cylindrical solenoid of radius R. Inside the solenoid the field is constant, vanishing outside. If the magnetic field inside the solenoid is H, the vector potential is (magnetic field directed downwards):

$$\begin{aligned} A_\varphi &= -\frac{1}{2}Hr \quad , \quad r < R, \\ &= -\frac{1}{2}\frac{HR^2}{r} \quad , \quad r > R. \end{aligned} \tag{16.9}$$

According to (10.9) the vector potential on a closed path leads to the phase shift

$$\begin{aligned} \delta\varphi &= -\frac{e}{\hbar c}H\pi r^2 \quad , \quad r < R, \\ &= -\frac{e}{\hbar c}H\pi R^2 \quad , \quad r > R. \end{aligned} \tag{16.10}$$

As noted by Aharonov and Bohm [46], there is a phase shift for $r > R$, even though $H = 0$ (because for $r > R$, $\mathrm{curl}\mathbf{A} = 0$).

Expressing $\mathbf{A}$ by (10.8) through v, the hypothetical circular aether velocity, one has

$$\begin{aligned} v_\varphi &= \frac{e}{2mc}Hr \quad , \quad r < R, \\ &= \frac{e}{2mc}\frac{HR^2}{r} \quad , \quad r > R. \end{aligned} \tag{16.11}$$

One sees that inside the coil the velocity profile is the same as in a rotating frame of reference, having outside the coil the form of a potential vortex. If expressed in terms of the aether velocity, the phase shift becomes

$$\delta\varphi = \frac{m}{\hbar} \oint \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{s}, \tag{16.12}$$

the same as (10.22) for the vector potential created by a gravitational field, and hence the same as in the Sagnac and the neutron interference experiments. For the magnetic vector potential the aether velocity can easily become much larger than in any rotating platform experiment. According to (16.11) the velocity reaches a maximum at $r = R$, where it is

$$\frac{|v_{\max}|}{c} = \frac{eHR}{2mc^2}. \tag{16.13}$$

For electrons this is $|v_{\max}|/c \cong 3 \times 10^{-4} HR$, where H is measured in Gauss. Assuming that $H = 10^4$ G, this would mean that $|v_{\max}| \gtrsim c$ for $R \gtrsim 0.3$ cm. It thus seems to follow that the aether can reach superluminal velocities for rather modest magnetic fields. In this regard it must be emphasized that in the Planck mass plasma all relativistic effects are explained dynamically, with the aether, which is here the Planck mass plasma, obeying an exactly nonrelativistic law of motion. It can for this reason assume superluminal velocities, and if this would be the same velocity felt on a rotating platform, it would lead to an enormous centrifugal and Coriolis-field inside the coil, obviously not observed.

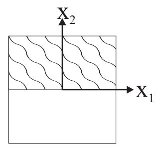
The Planck mass plasma can give a simple explanation for this paradox because it consists of two superfluid components, one composed of positive Planck masses and the other one of negative Planck masses. The two components can freely flow through each other making possible two configurations, one where both components are co-rotating, and one where they are counter-rotating. The co-rotating configuration is obviously realized on a rotating platform where it leads to the Sagnac and neutron interference effects. This suggests that in the presence of a magnetic vector potential the two superfluid components are counter-rotating. Outside the coil, where curl $\boldsymbol{A}$ = 0, the magnetic energy density vanishes, implying that the magnitude of both velocities is exactly the same. Inside the coil, where curl $\boldsymbol{A} \neq 0$, there must be a small imbalance in the velocity of the positive over the negative Planck masses to result in a positive energy density.

Whereas in the Sagnac effect $\omega = \mathrm{curl}\, \boldsymbol{v} \neq 0$, according to (10.7) resulting in observable inertial forces, no magnetic forces are present in the Aharonov-Bohm effect where curl $\boldsymbol{A}$ = 0. In the Sagnac effect the aether makes a uniform rotational motion, whereas in the Aharonov- Bohm effect the aether motion is a potential vortex. With the kinetic energy of the positive Planck masses cancelled by the kinetic energy of the negative Planck masses, the energy of the vortex is zero. Since for shifting the phase no energy is needed, there can be no contradiction.

An even more serious problem of quantum mechanical nonlocality is the strange phenomenon of phase entanglement in the many body Schrödinger equation in configuration space. In the Planck mass plasma this problem is avoided because it views all particles as quasiparticles of this plasma, and it is incorrect to visualize a many-body wave function to be composed of the same particles which are observed before an interaction between the particles is turned on. In the presence of an interaction it rather leads to a new set of quasiparticles into which the wave function can be factorized. This can be demonstrated for two identical particles moving in a harmonic oscillator well. The well shall have its coordinate origin at $x = 0$, with the first particle having the coordinate x_1 and the second on the coordinate x_2. Considering two oscillator wave functions $\psi_0(x)$ and $\psi_1(x)$, with ψ_0 having no and $\psi_1(x)$ having one node, there are two two-particle wave functions

$$\underline{\psi}(x_1, x_2) = \psi_0(x_1)\psi_1(x_2) = \sqrt{\frac{2}{\pi}} x_2 e^{-(x_1^2+x_2^2)/2},$$

=



(16.14a)

$$\underline{\psi}(x_1, x_2) = \psi_1(x_1)\psi_0(x_2) = \sqrt{\frac{2}{\pi}} x_1 e^{-(x_1^2+x_2^2)/2},$$

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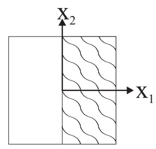


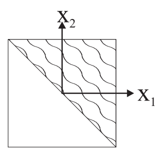
(16.14b)

graphically displayed in the x_1, x_2 configuration space, with the nodes along the lines $x_2 = 0$ and $x_1 = 0$. By a linear superposition of these wave functions we get a symmetric and an antisymmetric combination:

$$\underline{\psi}_s(x_1, x_2) = \frac{1}{\sqrt{2}} [\psi_0(x_1)\psi_1(x_2) + \psi_1(x_1)\psi_0(x_2)]$$

$$= \frac{1}{\sqrt{\pi}} (x_2 + x_1) e^{-(x_1^2+x_2^2)/2},$$

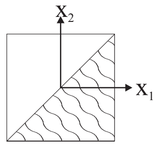
=



(16.15a)

$$\underline{\psi}_a(x_1,x_2) = \frac{1}{\sqrt{2}}\left[\psi_0(x_1)\psi_1(x_2) - \psi_1(x_1)\psi_0(x_2)\right] = \frac{1}{\sqrt{\pi}}(x_2 - x_1)e^{-(x_1^2+x_2^2)/2},$$

$$=$$



(16.15b)

If a perturbation is applied, whereby the two particles slightly attract each other, the degeneracy for the two wave-functions is removed, with the symmetric wave function leading to a lower energy eigenvalue. For a repulsive force between the particles the reverse is true. As regards the wave functions (16.14), one may still think of it in terms of two particles, because the wave functions can be factorized, with the quantum potential becoming a sum of two independent terms:

$$-\frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{\underline{\psi}^*\underline{\psi}}}{\sqrt{\underline{\psi}^*\underline{\psi}}} = -\frac{\hbar^2}{2m}\frac{1}{\sqrt{\psi_1^*\psi_1}}\frac{\partial^2\sqrt{\psi_1^*\psi_1}}{\partial x_1^2} - \frac{\hbar^2}{2m}\frac{1}{\sqrt{\psi_2^*\psi_2}}\frac{\partial^2\sqrt{\psi_2^*\psi_2}}{\partial x_2^2}. \tag{16.16}$$

Such a decomposition into parts is not possible for the wave functions (16.15), and it is there then not more possible to think of the two particles which are placed into the well. This, however, is possible by making a 45° rotation in configuration space. Putting

$$y = x_2 + x_1, \qquad x = x_2 - x_1, \tag{16.17}$$

one obtains the factorized wave functions

$$\underline{\psi}_s = \frac{1}{\sqrt{\pi}}ye^{-(x^2+y^2)/2}, \qquad \underline{\psi}_a = \frac{1}{\sqrt{\pi}}xe^{-(x^2+y^2)/2}, \tag{16.18}$$

for which the quantum potential separates into a sum of two independent terms, one depending only on x and the other one only on y. This means that the addition of a small perturbation, in form of an attraction or repulsion between the two particles, transforms them into a new set of two quasiparticles, different from the original particles. With the identification of all particles as quasiparticles of the Planck mass plasma, the abstract notion of configuration space and inseparability into parts disappears, because any many-body system can, in principle, at each point always be expressed as a factorizable wave function of quasiparticles, where the quasiparticle configuration may change from point to point. This can be shown quite generally. For an N-body system, the potential energy can in each point of configuration space be expanded into a Taylor series

$$U = \sum_{k,i}^{N} a_{ki}x_k x_i. \tag{16.19}$$

Together with the kinetic energy

$$T = \sum_i^N \frac{m_i}{2}\dot{x}_i^2. \tag{16.20}$$

one obtains the Hamilton function $H = T + U$, and from there the many- body Schrödinger equation. Introducing the variables $\sqrt{m_i}x_i = y_i$, one has

$$T = \sum_i^N \frac{1}{2}\dot{y}_i^2, \quad U = \sum_i^N b_{ki}y_k y_i, \tag{16.21}$$

which by a principal axis transformation of U becomes

$$T = \sum_i^N \frac{1}{2}\dot{z}_i^2, \quad U = \sum_i^N \frac{\omega_i^2}{2}z_i^2. \tag{16.22}$$

Unlike the Schrödinger equation with the potential (16.19), the Schrödinger equation with the potential (16.22) leads to a completely factorizable wave function, with a sum of quantum potentials each depending only on one quasiparticle coordinate. The transformation from (16.21) to (16.22) is used in classical mechanics to obtain the normal modes for a system of coupled oscillators. The quasiparticles into which the many-body wave function can be factorized are then simply the quantized normal modes of the corresponding classical system.

For the particular example of two particles placed in a harmonic oscillator well, the normal modes of the classical mechanical system are those where the particles either move in phase or out of phase by 180°. In quantum mechanics, the first mode corresponds to the symmetric, the second one to the antisymmetric wave function. It is clear that the quasiparticles representing the symmetric and antisymmetric mode cannot be localized at the position of the particles placed into the well.

The decomposition of a many-body wave function into a factorizable set of quasiparticles can in the course of an interaction continuously change, but it can also abruptly change, if the interaction is strong, in what is known as the collapse of the wave function.

17. Planck Mass Rotons as Cold Dark Matter and Quintessence

With greatly improved observational techniques a number of important facts about the physical content and large scale structure of our universe has emerged. They are:

1. About 70% of the material content of the universe is a negative pressure energy,
2. About 26% nonbaryonic cold dark matter,
3. About 4% ordinary matter and radiation,
4. The universe is Euclidean flat,
5. It's cosmological constant very small,
6. It's expansion slightly accelerated.

These are the basic facts which have to be explained, and no model which at least can make them plausible can be considered credible.

In the Planck mass plasma model with an equal number of positive and negative Planck mass particles the cosmological constant is zero and the universe Euclidean flat. In its groundstate the Planck mass plasma is a two component positive-negative mass superfluid with a phonon-roton energy spectrum for each component. Assuming that the phonon-roton spectrum measured in superfluid helium is universal, this would mean that in the Planck mass plasma this spectrum has the same shape, with the Planck energy taking the place of the Debye energy in superfluid helium (see Fig. 4), with the roton mass close to the Planck mass.

Rotons can be viewed as small vortex rings with the ring radius of the same order as the vortex core radius. A fluid with cavitons is in a state of negative pressure, and the same is true for a fluid with vortex rings [48]. In vortices the centrifugal force creates a vacuum in the vortex core, making a vortex ring to behave like a caviton.

The kinetic roton energy is bound by the height of the potential well in frequency space. From Fig. 4 it follows that the ratio of the energy gap (which is equal the roton rest mass energy), to the maximum kinetic roton energy is about 70 to 25, close to the observed ratio of the negative pressure energy to the cold dark matter energy. The roton hypothesis can therefore explain both the cold dark matter and negative pressure energy, the latter mimicking a cosmological constant [49].

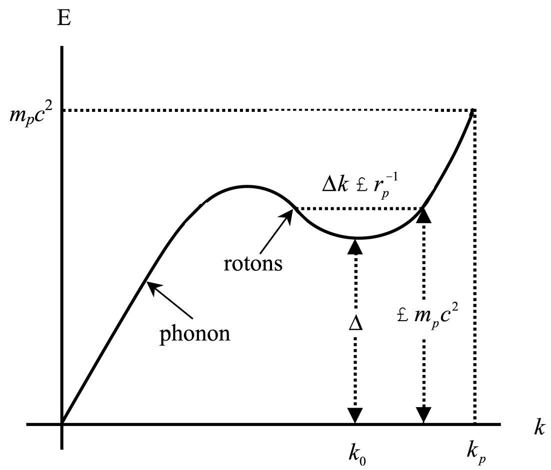


Fig. 4. The phonon-roton energy spectrum of the hypothetical Planck aether.

18. Conclusion

Finally, I try to compare the presented model with other attempts to formulate an unified model of elementary particles.

I begin with Heisenberg's nonlinear spinor theory [50]. As with Einstein's nonlinear gravitational field theory, it is massless and selfcoupled, and like Einstein's gravitational field it resists quantization by established rules. In his theory Heisenberg tries to overcome this problem by a novel kind of regularization. Because this leads to a Hilbert space with an indefinite metric, a high prize has to be paid. To obtain solutions describing elementary particles Heisenberg uses perturbation theory, which contradicts the spirit of the entire theory, because perturbation theory should come into play only after a spectrum of elementary particles had been obtained non-perturbatively. Perturbation theory would have to describe the interactions between the non- perturbatively obtained particles. As noted by Heisenberg, one fundamental problem of the theory is that there should be at least one fermion with a nonzero rest mass, absent from his nonlinear spinor wave equation.

By comparing Heisenberg's theory with the Planck mass plasma vacuum model, the following can be said: For a linear classical field without interaction, quantization is like a postscript, obtained by quantizing the normal modes of the classical field with Planck's harmonic oszillator quantization rule. The fundamental field equation of the Planck mass plasma is nonlinear but nonrelativistic. By the Madelung transformation it assumes the form of an Euler equation. As in classical fluid dynamics it is nonrelativistic and nonlinear. The fundamental solutions of this Euler equation are waves and vortices and they can be readily quantized. And with elementary particles understood as quasiparticle configurations made up from the waves and vortices, perturbation theory can there be done in the usual way. The prize to be paid is to give up the kinematic space-time symmetry of Einstein's special theory of relativity, replacing it by the dynamic pre-Einstein theory of relativity of Lorentz and Poincaré, which still assumed the existence of an aether, here identified with the Planck mass plasma.

Presently, the most popular attempt to formulate a unified theory of elementary particles, and to solve the problem of quantum gravity, is supersymmetric string theory in 10 space-time dimensions, or in its latest version 11 dimensional M-theory. But M-theory, and by implication string theory, stands or falls with supersymmetry. There are two main features which speak in favor of supersymmetry:

First, supersymmetric theories are less divergent because often, but not always, divergent Feynman diagrams cancel each other out. Second, supersymmetry makes the strong, the weak and the electromagnetic coupling constants to converge at very high energies almost into one point. Without supersymmetry the convergence is much less perfect.

What speaks against supersymmetry is the following: First, the supersymmetric extension of the standard model leads to electric dipole moments of the electron and neutron which are not observed. Supersymmetric string theories lead to even larger dipole moments [51]. Second, in spite of enormous efforts made, no supersymmetric particles have ever been found, neither in particle accelerators up to $F \sim$ 200 GeV (tevatron), nor in the cosmic radiation at much higher energies.

As an extension of Einstein's general theory of relativity to higher dimensions, string theory leads to all the pathological solutions of general relativity, like the Gödel-type travel back in time solution, and even worse, to the "NUT" solution but Newman, Unti and Tamborini [52]. These solutions result from the topological oddities of a Riemannian curved space.

As Heisenberg had told me, because of the topological problems in Einstein's theory, gravity should be formulated as a nonlinear field theory in flat space-time.

Comparing string theory with the Planck mass plasma model, one is wondering if closed strings are misunderstood quantized vortex rings at the Planck scale, and if supersymmetry is mimicked by the hidden existence of negative masses. According to Schrödinger, Hönl, and Bopp, Dirac spinors are explained as being composed of positive and negative masses, dynamically explaining supersymmetry. In the Planck mass plasma the problem of quantum gravity is reduced to the quantization of a non-relativistic many body problem.

Most recently a different attempt to formulate a fundamental theory of elementary particles has been made by 'tHooft [53]. As in the Planck mass plasma model it is assumed that at the Planck scale there are two types of particles, called "beables" and "changeables." The "beables" are massless non-interacting fermions. Because they form a complete set of observables that commute at all times, the "beables" are deterministic and can be described by an infinite sheet moving in a direction perpendicular to this sheet with the velocity of light. The quantum fuzziness enters through the "changeables," interacting with the "beables." The theory is thus a classical-quantum hybrid.

Comparing 'tHooft's theory with the Planck mass plasma, where fermions are composed of positive and negative masses, the "Zitterbewegung" length (which is $1/2$ the Compton wave length of a fermion) diverges with a vanishing rest mass, making understandable why the "beables" are extended over an infinite sheet. A further difference between t'Hooft's theory and the Planck mass plasma is that 't'Hooft's theory sustains the kinematic interpretation of the special theory of relativity, and by implication of the general theory of relativity, with the existence of black holes inside event horizons, while in the Planck mass plasma, where Lorentz invasiance is a dynamic symmetry, elementary particles disintegrate in approaching the event horizon, leading to the formation of non-relativistic red holes.

A theory in 3 space and 1 time dimension, inspired by condensed matter physics, appears to be much more plausible than a string theory with 6 more space dimensions, M-Theory with 7 more space dimensions or F-Theory with 7 more space- and one more time dimension, and also more plausible than the hybrid classical-quantum theory of "beables" and "changeables." My theory is based on the simple conjecture that the vacuum of space is a kind of plasma of locally interacting positive and negative Planck masses, leading to quantum mechanics and the theory of relativity as low energy approximations.

Acknowledgement

One purpose of this report is to remind the physics community of the almost forgotten work by Hönl and Bopp, whose work was stimulated by the work of Schrödinger. Because all these authors had published in German, their work did not get the attention it deserved.

Finally, I would like to express my sincere thanks to my colleagues H. Stumpf and H. Dehnen, for their critical reading of my manuscript and their many valuable comments.

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