Shear Force Feedback Control of a Single-Link Flexible Robot with a Revolute Joint

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***Abstract*— In this paper we present a shear force feedback control method for a single-link flexible robot arm with a revolute joint for which it has been shown that direct bending strain feedback can suppress its vibration. Our primary concern is the stability analysis of the closed-loop equation which has not appeared in the literature. We show the existence of a unique solution and the exponential stability of this solution by doing spectral analysis and estimating the norm of the resolvent operator associated with this equation. Some experiments are also conducted to verify these theoretical developments.**

***Index Terms*—Euler–Bernoulli beam equation, exponential stabilization, flexible robot, shear force control.**

I. Introduction

In recent years, there has been a great deal of interest in the control of flexible robot arms and flexible structures (see, for example, [1]–[14]). Among the many control methods developed, sensor output feedback control methods are particularly attractive due to their easy implementation in practice [6], [10]–[13].

For a one-link flexible arm with a revolute joint, it is now well known that a simplified linear dynamic model for the transverse vibration of the arm is given by [2], [3]

$$\begin{cases}\ddot{y}(t,x)+y''''(t,x)=-x\ddot{\theta}(t), & x\in(0,1)\\ y(t,0)=y'(t,0)=0\\ y''(t,1)=y'''(t,1)=0\\ y(0,x)=y_0(x),\dot{y}(0,x)=y_1(x)\end{cases} \tag{1}$$

where $y(t,x)$ denotes the transverse displacement of the arm at time t and position x along the arm length direction. $\ddot{y}(t,x)$ and $y''''(t,x)$ denote time and spacial derivatives, respectively. $\ddot{\theta}(t)$ is the angular acceleration of the joint motor which rotates the arm in the horizontal plane. Here, for notational simplicity, we normalized the arm length and the rigidity coefficient to be one.

To control vibration in the arm, it is proposed in [10] to measure the bending strain $y''(t,0)$ and to control the motion of the motor such that

$$\ddot{\theta}(t)=k\dot{y}''(t,0) \tag{2}$$

holds. Here $k>0$ is a constant feedback gain. This method is called direct strain feedback control and can be implemented easily using a motor driver of the velocity reference type. Substituting (2) into (1) yields the closed-loop equation of the direct strain feedback control

$$\begin{cases}\ddot{y}(t,x)+kx\dot{y}''(t,0)+y''''(t,x)=0, & x\in(0,1)\\ y(t,0)=y'(t,0)=0\\ y''(t,1)=y'''(t,1)=0\\ y(0,x)=y_0(x),\dot{y}(0,x)=y_1(x).\end{cases} \tag{3}$$

The well-posedness and asymptotic stability of the solution to (3) is analyzed in [10] using the concept of A-dependent operators. Actually, it was later found that (3) is exponentially stable in [11]. To see this, we formally introduce a new variable

$$w(t,x)=y''(t,1-x). \tag{4}$$

Then, (3) can be transformed into

$$\begin{cases}\ddot{w}(t,x)+w''''(t,x)=0, & x\in(0,1)\\ w(t,0)=w'(t,0)=0\\ w''(t,1)=0\\ w'''(t,1)=k\dot{w}(t,1).\end{cases} \tag{5}$$

In the sequel, the initial conditions associated with the partial differential equations such as (5) will be omitted for simplicity. It is easy to see that the energy stored in system (5) can be expressed by

$$E(t)=\frac{1}{2}\int_0^1[\dot{w}(t,x)]^2dx+\frac{1}{2}\int_0^1[w''(t,x)]^2dx \tag{6}$$

and is dissipating, since the time derivative of $E(t)$ along the solution of (5) is evaluated as

$$\dot{E}(t)=-w'''(t,1)\dot{w}(t,1)=-k[\dot{w}(t,1)]^2\le 0. \tag{7}$$

Based on this fact, the exponential stability of (5) can be shown using the energy multiplier method [15], [8]. Therefore, the solution $y(t,x)$ to (3) is also exponentially stable, since $y(t,x)$ is a double integration of $w(t,x)$.

On the other hand, for Cartesian or SCARA robots with long flexible arms in the Z-direction, it is found that shear force feedback is effective in suppressing vibrations arising from arm flexibility [12]. Considering, without loss of generality, only vibrations in one direction (for example, the X-direction in an XY robot), the dynamic equation is given by

$$\begin{cases}\ddot{y}(t,x)+y''''(t,x)=-\ddot{s}(t), & x\in(0,1)\\ y(t,0)=y'(t,0)=0\\ y''(t,1)=y'''(t,1)=0\end{cases} \tag{8}$$

where notations are as before except that $\ddot{s}(t)$ is the linear acceleration of the base at which one end of the flexible arm

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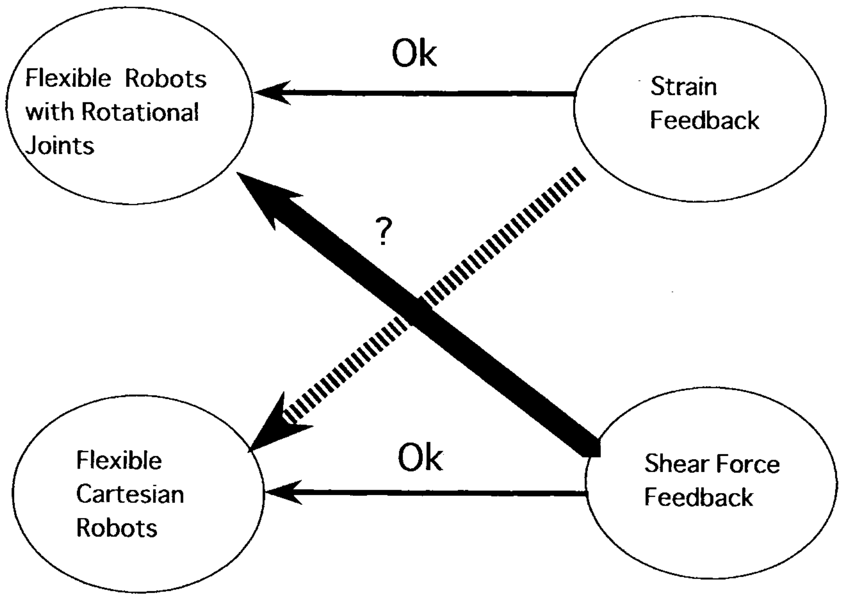

Fig. 1. Current status for sensor output feedback control of flexible robot arms.

is attached. The shear force feedback control is to measure the shear strain $y'''(t,0)$ at the root end of the arm and to control the motion of the robot joint motors such that

$$\ddot{s}(t) = -k\dot{y}'''(t,0), \quad k > 0 \tag{9}$$

holds. Here k is again a constant feedback gain. Substituting (9) into (8) yields the closed-loop equation of the shear force feedback control

$$\begin{cases} \ddot{y}(t,x) - k\dot{y}'''(t,0) + y''''(t,x) = 0, \quad x \in (0,1) \\ y(t,0) = y'(t,0) = 0 \\ y''(t,1) = y'''(t,1) = 0. \end{cases} \tag{10}$$

The well-posedness and asymptotic stability of the solution to (10) can also be analyzed using the concept of A-dependent operators in [10]. Actually, it is shown in [12] that the solution to (10) is exponentially stable. To see this, we formally introduce a new variable

$$w(t,x) = y''(t,1-x) \tag{11}$$

which is the same as (4). Then, (11) can be transformed into

$$\begin{cases} \ddot{w}(t,x) + w''''(t,x) = 0, \quad x \in (0,1) \\ w(t,0) = w'(t,0) = 0 \\ w''(t,1) = -k\dot{w}'(t,1) \\ w'''(t,1) = 0. \end{cases} \tag{12}$$

It is easy to see that the energy stored in (12) can be expressed by

$$E(t) = \frac{1}{2}\int_0^1 [\dot{w}(t,x)]^2 dx + \frac{1}{2}\int_0^1 [w''(t,x)]^2 dx \tag{13}$$

and is dissipating, since the time derivative of $E(t)$ along the solution of (12) is evaluated as

$$\dot{E}(t) = -w''(t,1)\dot{w}'(t,1) = -k[\dot{w}'(t,1)]^2 \le 0. \tag{14}$$

The exponential stability of the solution of (12) had been long unproved by using the energy multiplier method but was finally proven in [16] by making use of a result in [17]. Since the solution $w(t,x)$ to (12) is exponentially stable, the solution $y(t,x)$ to (10) is also exponentially stable.

The results we have obtained are summarized in Fig. 1. Strain feedback guarantees the closed-loop stability for vibration control of flexible robots with revolute joints, while shear force feedback guarantees the closed-loop stability for vibration control of flexible Cartesian or SCARA robots, symbolized by the "Ok's" in Fig. 1. The background of these phenomena lie in the fact that (w''', w) is an adjoint pair variable, and also (w'', w') is an adjoint pair variable.

Our objective in this research is to clarify whether shear force feedback can control the vibration of flexible robots with revolute joints, as indicated by the solid diagonal arrow in Fig. 1. Namely, we measure the shear force $y'''(t,0)$, and we control the revolute joint motor such that

$$\ddot{\theta}(t) = -k\dot{y}'''(t,0). \tag{15}$$

Substituting (15) into (1), we get the closed-loop equation

$$\begin{cases} \ddot{y}(t,x) - kx\dot{y}'''(t,0) + y''''(t,x) = 0, \quad x \in (0,1) \\ y(t,0) = y'(t,0) = 0 \\ y''(t,1) = y'''(t,1) = 0. \end{cases} \tag{16}$$

It should be noted that the second term in the first equation of (16) is different from the corresponding terms in (3) and (10). Unlike (3) and (10), the well-posedness and stability of this equation cannot be proven by making use of the concept of the A-dependent operator proposed in [10]. By the same transformation as (4), (16) is converted to

$$\begin{cases} \ddot{w}(t,x) + w''''(t,x) = 0, \quad x \in (0,1) \\ w(t,0) = w'(t,0) = 0 \\ w''(t,1) = 0 \\ w'''(t,1) = k\dot{w}'(t,1). \end{cases} \tag{17}$$

This equation is different from (5) and (12). Since the variables w''' and w' are not adjoint, it is even difficult to find an energy function for (17).

The motivation for us to consider shear force feedback control for flexible robots with revolute joints is twofold.

1) Practically, there are cases in which it is easier to measure shear force than bending strain. For example, load cells are easily attached to the flexible arm to measure the shear force needed for feedback, especially for robots with a flexible arm whose length can vary (imagining a polar robot), where it is impossible to cement strain gauge foils to measure the bending strain.
2) Theoretically, the well-posedness and stability of (16), or equivalently (17), is not known in the literature. Neither the energy multiplier method nor the theorem in [17] can be used directly to prove the exponential stability of (16). It is thus of interest to clarify these points.

In the sequel, we first perform spectral analysis for the system operator associated with (16) and estimate the norm of the resolvent of this operator. Based on this estimation, we are able to show the existence and uniqueness of the solutions of (16). Furthermore, we can show the exponential stability of this solution, for $k > 0$, which is the main contribution of this paper. These results imply that the system operator generates a one-time integrated semigroup. For the concept of integrated semigroups, the interested reader is referred to [18] and [19]. At this point, it should be understood that the one-time integrated semigroup is weaker than the usual strongly continuous semigroup. So we still do not know whether the system operator generates a strongly continuous semigroup on the state space, which is left an open question. Section III is devoted to studying the asymptotic behavior of the spectrum of the system operator. In Section IV, a polar robot with two degrees-of-freedom (one revolute joint and one prismatic joint) is constructed to demonstrate the effectiveness and applicability of the control law proposed in this paper. Conclusions are drawn in Section V.

Remark 1: We believe that, by the same reasoning used in this paper, strain feedback can control flexible Cartesian robots as indicated by the broken diagonal arrow in Fig. 1. We shall not discuss this problem here due to the space limitation.

II. Well-Posedness and Exponential Stability of (16)

To begin with, let $L^2(0,1)$ be the usual square integrable function space and $H^n(0,1),\ n = 1,\cdots$, the Sobolev space of order n. Let $H^2_E(0,1) = \{f(x) \in H^2(0,1)\,|\, f(0) = f'(0) = 0\}$ and $\mathcal{H} = H^2_E(0,1) \times L^2(0,1)$ be the state Hilbert space with inner product induced norm

$$\|(f,g)^T\|^2_{\mathcal{H}} = \int_0^1 [|f''(x)|^2 + |g(x)|^2]dx.$$

Then, (16) can be written as an evolution equation on $\mathcal{H}$

$$\frac{d}{dt}W(t) = \mathcal{A}W(t) \tag{18}$$

where $W(t) = (y, \dot{y} - kxy'''(t,0))^T$, and operator $\mathcal{A}$ is defined by

$$\mathcal{A}\begin{bmatrix} f(x) \\ g(x) \end{bmatrix} = \begin{bmatrix} g(x) + kxf'''(0) \\ -f''''(x) \end{bmatrix} \tag{19}$$

with domain

$$\begin{aligned} D(\mathcal{A}) = \{(f,g)^T \in H^4(0,1) \times H^2(0,1)\,|\, \\ f(0) = f'(0) = f''(1) = f'''(1) = 0 \\ g(0) = 0, g'(0) = -kf'''(0)\} \end{aligned} \tag{20}$$

which is dense in $\mathcal{H}$.

Lemma 1: $\mathcal{A}^{-1}$ exists and is a compact operator on $\mathcal{H}$. Therefore, the spectrum $\sigma(\mathcal{A})$ consists entirely of isolated eigenvalues.

Proof: Define an operator A on $L^2(0,1)$ by

$$\begin{aligned} A\phi &= \phi''''(x) \\ D(A) &= \{\phi \in L^2\,|\, \phi(0) = \phi'(0) = \phi''(1) = \phi'''(1) = 0\}. \end{aligned} \tag{21}$$

It is well known that A is a self-adjoint and positive definite operator. Thus for any given $(f,g)^T \in \mathcal{H}$, it is easily shown that

$$\mathcal{A}^{-1}\begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} -A^{-1}g \\ f - kx\langle 1, g\rangle \end{bmatrix} \in H^4(0,1) \times H^2(0,1).$$

The conclusion then follows from the Sobolev embedding theorem [20]. □

We first state a lemma which concerns the properties of eigenvalues and eigenfunctions of operator A.

Lemma 2: Let $\{(\lambda_n, \phi_n(x))\}_{n=1}^{\infty}$ be the eigenpairs of operator A. Then we have the following.

i) $\lambda_n = \beta_n^4$, with β_n satisfying $1 + \cos(\beta_n)\cosh(\beta_n) = 0$ and $\beta_n = \mathcal{O}(n) > 0$.
ii) $\{\phi_n(x)\}$ forms an orthogonal basis on $L^2(0,1)$ and (x) as shown at the bottom of the page.
iii) Let $x = \sum_{n=1}^{\infty} b_n\phi_n(x) \in L^2(0,1)$. Then
$$\begin{aligned} \phi_n''(0) &= -2\beta_n^2 \\ \phi_n'''(0) &= -2\gamma_n\beta_n^3, b_n = -2\beta_n^{-2}\|\phi_n\|^{-2}. \end{aligned}$$
iv) $\gamma_n < 0$ for every n.

$$\begin{cases} \phi_n(x) = -\frac{1+\gamma_n}{2}\exp(\beta_n x) - \frac{1-\gamma_n}{2}\exp(-\beta_n x) + \gamma_n\sin(\beta_n x) + \cos(\beta_n x), \\ \gamma_n = -\frac{\exp(\beta_n) - \sin(\beta_n) + \cos(\beta_n)}{\exp(\beta_n) + \sin(\beta_n) + \cos(\beta_n)} \to -1, \text{ as } n \to \infty. \end{cases} \tag{x}$$

$$\begin{aligned} \gamma_n &= -\frac{\exp(\beta_n) - \sin(\beta_n) + \cos(\beta_n)}{\exp(\beta_n) + \sin(\beta_n) + \cos(\beta_n)} \\ &= -\frac{[\exp(\beta_n) - \sin(\beta_n) + \cos(\beta_n)][\exp(\beta_n) + \sin(\beta_n) + \cos(\beta_n)]}{[\exp(\beta_n) + \sin(\beta_n) + \cos(\beta_n)]^2} \end{aligned} \tag{y}$$

Proof: Assertions i) and ii) are well known. Claim iii) comes from the fact that

$$b_n\|\phi_n\|^2 = \langle x, \phi_n\rangle = \frac{1}{\lambda_n}\int_0^1 x\phi_n''''(x)dx = -\frac{1}{\lambda_n}\int_0^1 \phi_n'''(x)dx = \frac{1}{\lambda_n}\phi_n''(0) = -2\beta_n^{-2}.$$

As for claim iv), since we have (y) shown at the bottom of the previous page, it is needed only to show that

$$g(x) = (e^x - \sin(x) + \cos(x))(e^x + \sin(x) + \cos(x)) = e^{2x} + 2e^x\cos(x) + \cos^2(x) - \sin^2(x) > 0$$

for all $x > 0$ satisfying $1 + \cos(x)\cosh(x) = 0$. This is easily done using fundamental calculus. Thus, the details are omitted. □

We now proceed to analyze the spectrum of operator $\mathcal{A}$. Obviously, the eigenvalue λ and eigenfunction $\phi \neq 0$ of $\mathcal{A}$ must satisfy

$$\begin{cases}\lambda^2\phi(x) - k\lambda x\phi'''(0) + \phi''''(x) = 0\\ \phi(0) = \phi'(0) = \phi''(1) = \phi'''(1) = 0.\end{cases} \tag{22}$$

Throughout the paper, we assume that $k > 0$.

Lemma 3: Let $\sigma(\mathcal{A})$ denote the spectrum of operator $\mathcal{A}$. $\lambda \in \sigma(\mathcal{A})$ if and only if λ is a solution of $G(\lambda) = 0$, where $G(\lambda)$ is an entire function of λ and is given by

$$G(\lambda) = 4 + (1+k)\left(e^{\sqrt{2\lambda}} + e^{-\sqrt{2\lambda}}\right) + (1-k)\left(e^{i\sqrt{2\lambda}} + e^{-i\sqrt{2\lambda}}\right). \tag{23}$$

Proof: Let (λ, ϕ) satisfy (22). Let $\tilde{\phi}(x) = \phi''(x)$. It is clear that $\tilde{\phi}$ should satisfy

$$\begin{cases}\lambda^2\tilde{\phi}(x) + \tilde{\phi}''''(x) = 0\\ \tilde{\phi}(0) = \tilde{\phi}'(0) = \tilde{\phi}''(1) = 0, \tilde{\phi}'''(1) = k\lambda\tilde{\phi}'(1).\end{cases} \tag{24}$$

The solution of (24) is given by

$$\tilde{\phi}(x) = c_1e^{\sqrt{\lambda i}x} + c_2e^{-\sqrt{\lambda i}x} + c_3e^{i\sqrt{\lambda i}x} + c_4e^{-i\sqrt{\lambda i}x}$$

where $c_i, i = 1, 2, 3, 4$ should satisfy (25), as shown at the bottom of the page. The boundary value problem (24) has a nontrivial solution if and only if the determinant of the coefficient matrix of c_i in (25) is equal to zero, i.e.,

$$G(\lambda) = 4 + (1+k)\left(e^{\sqrt{2\lambda}} + e^{-\sqrt{2\lambda}}\right) + (1-k)\left(e^{i\sqrt{2\lambda}} + e^{-i\sqrt{2\lambda}}\right) = 0.$$

Since $k > 0$, it is seen that $G(\lambda) \neq 0$ for all $\lambda = i\omega, \omega \in R^1$. In other words, $\text{Re}(\lambda) \neq 0$, for any $\lambda \in \sigma(\mathcal{A})$. □

We present an alternative characterization of the spectrum of $\mathcal{A}$.

Lemma 4: The spectrum $\sigma(\mathcal{A})$ of operator $\mathcal{A}$ consists of all zeros of the following entire function:

$$F(\lambda) = 1 + \frac{k}{2}\lambda + 4k\sum_{n=1}^{\infty}\frac{\lambda^3}{\lambda^2+\lambda_n}\gamma_n\beta_n^{-3}\|\phi_n\|^{-2} \tag{26}$$

where λ_n and $\phi_n(x)$ are given in Lemma 2.

Proof: From Lemma 3, for all $\lambda \in \sigma(\mathcal{A}), \text{Re}(\lambda) \neq 0$. This implies $(\lambda^2 + A)^{-1}$ exists. Solving (22) for $\phi(x)$, we obtain

$$\phi(x) = k\lambda\phi'''(0)(\lambda^2 + A)^{-1}x. \tag{27}$$

Integrating with respect to x from zero to one on both sides of the first equation in (22), one gets

$$\lambda^2\int_0^1 \phi(x)dx - \frac{k}{2}\lambda\phi'''(0) - \phi'''(0) = 0. \tag{28}$$

Substituting (27) into (28) yields

$$\left[k\lambda^3\int_0^1[(\lambda^2+A)^{-1}x]dx - \frac{k}{2}\lambda - 1\right]\phi'''(0) = 0.$$

Since $\phi'''(0) = 0$ implies $\phi(x) = 0$, $\lambda \in \sigma(\mathcal{A})$ if and only if λ satisfies

$$\begin{aligned}F(\lambda) &= 1 + \frac{k}{2}\lambda - k\lambda^3\int_0^1[(\lambda^2+A)^{-1}x]dx\\ &= 1 + \frac{k}{2}\lambda - k\lambda^3\sum_{n=1}^{\infty}\frac{b_n}{\lambda^2+\lambda_n}\int_0^1\phi_n(x)dx\\ &= 1 + \frac{k}{2}\lambda - k\lambda^3\sum_{n=1}^{\infty}\frac{b_n}{\lambda^2+\lambda_n}\frac{1}{\lambda_n}\int_0^1\phi_n''''(x)dx\\ &= 1 + \frac{k}{2}\lambda + k\lambda^3\sum_{n=1}^{\infty}\frac{b_n}{\lambda^2+\lambda_n}\frac{1}{\lambda_n}\phi_n'''(0)\\ &= 1 + \frac{k}{2}\lambda - 2k\lambda^3\sum_{n=1}^{\infty}\frac{b_n}{\lambda^2+\lambda_n}\frac{\gamma_n}{\beta_n}\\ &= 1 + \frac{k}{2}\lambda + 4k\sum_{n=1}^{\infty}\frac{\lambda^3}{\lambda^2+\lambda_n}\gamma_n\beta_n^{-3}\|\phi_n\|^{-2}\\ &= 0.\end{aligned}$$

□

We shall use both $F(\lambda)$ and $G(\lambda)$ to exploit the properties of the spectrum of operator $\mathcal{A}$.

Lemma 5: Let λ be a zero of $F(\lambda)$. Then $\text{Re}(\lambda) < 0$.

Proof: Let $\lambda = a + bi, a, b \in R^1$ be a zero of $F(\lambda)$. By noting that $\lambda^3 = (a+bi)^3 = a^3 - 3ab^2 + (3a^2b - b^3)i$ and

$$\frac{\lambda^3}{\lambda^2+\lambda_n} = \frac{\lambda^3(\bar{\lambda}^2+\lambda_n)}{|\lambda^2+\lambda_n|^2} = \frac{|\lambda|^4\lambda + \lambda_n\lambda^3}{|\lambda^2+\lambda_n|^2} = \frac{|\lambda|^4a + \lambda_n(a^3-3ab^2) + i[|\lambda|^4b + \lambda_n(3a^2b - b^3)]}{|\lambda^2+\lambda_n|^2}$$

$$\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -1 & i & -i\\ e^{\sqrt{\lambda i}} & e^{-\sqrt{\lambda i}} & -e^{i\sqrt{\lambda i}} & -e^{-i\sqrt{\lambda i}}\\ (i-k)e^{\sqrt{\lambda i}} & (k-i)e^{-\sqrt{\lambda i}} & (1-ki)e^{i\sqrt{\lambda i}} & (ki-1)e^{-i\sqrt{\lambda i}}\end{bmatrix} \times \begin{bmatrix}c_1\\ c_2\\ c_3\\ c_4\end{bmatrix} = 0 \tag{25}$$

we have

$$\begin{cases} 1+\frac{k}{2}a+4k\sum_{n=1}^{\infty}\frac{|\lambda|^4a+\lambda_n(a^3-3ab^2)}{|\lambda^2+\lambda_n|^2}\gamma_n\beta_n^{-3}\|\phi_n\|^{-2}=0\\ \frac{k}{2}b+4k\sum_{n=1}^{\infty}\frac{|\lambda|^4b+\lambda_n(3a^2b-b^3)}{|\lambda^2+\lambda_n|^2}\gamma_n\beta_n^{-3}\|\phi_n\|^{-2}=0.\end{cases} \tag{29}$$

From the first equality above, it can be inferred that $a \neq 0$. Suppose $b \neq 0$. Then from the second equation in (29)

$$\frac{k}{2}+4k\sum_{n=1}^{\infty}\frac{|\lambda|^4+\lambda_n(3a^2-b^2)}{|\lambda^2+\lambda_n|^2}\gamma_n\beta_n^{-3}\|\phi_n\|^{-2}=0.$$

Therefore

$$\begin{aligned}\frac{k}{2}+4k\sum_{n=1}^{\infty}\frac{|\lambda|^4}{|\lambda^2+\lambda_n|^2}\gamma_n\beta_n^{-3}\|\phi_n\|^{-2}\\ =-4k\sum_{n=1}^{\infty}\frac{\lambda_n(3a^2-b^2)}{|\lambda^2+\lambda_n|^2}\gamma_n\beta_n^{-3}\|\phi_n\|^{-2}.\end{aligned}$$

By use of (29), we see that

$$\begin{aligned}0&=\frac{1}{a}+\frac{k}{2}+4k\sum_{n=1}^{\infty}\frac{|\lambda|^4+\lambda_n(a^2-3b^2)}{|\lambda^2+\lambda_n|^2}\gamma_n\beta_n^{-3}\|\phi_n\|^{-2}\\ &=\frac{1}{a}+4k\sum_{n=1}^{\infty}\frac{\lambda_n(a^2-3b^2)-\lambda_n(3a^2-b^2)}{|\lambda^2+\lambda_n|^2}\gamma_n\beta_n^{-3}\|\phi_n\|^{-2}\\ &=\frac{1}{a}-4k\sum_{n=1}^{\infty}\frac{2\lambda_n|\lambda|^2}{|\lambda^2+\lambda_n|^2}\gamma_n\beta_n^{-3}\|\phi_n\|^{-2}.\end{aligned}$$

Consequently

$$\frac{1}{a}=8k|\lambda|^2\sum_{n=1}^{\infty}\frac{\beta_n\gamma_n}{|\lambda^2+\lambda_n|^2}\|\phi_n\|^{-2}<0$$

since $\gamma_n < 0$ as stated in iv) of Lemma 2. The proof is complete if one can show that $b \neq 0$ for any $a > 0$. In fact, if $a > 0$ and $b = 0$, then $\sqrt{2\lambda}$ is a positive real number. Therefore

$$\begin{aligned}4+(1+k)\left(e^{\sqrt{2\lambda}}+e^{-\sqrt{2\lambda}}\right)&\geq 4+2(1+k)>2|1-k|\\ &\geq \left|(1-k)\left(e^{i\sqrt{2\lambda}}+e^{-i\sqrt{2\lambda}}\right)\right|\end{aligned}$$

which means $G(\lambda) \neq 0$, a contradiction. □

Lemma 6: Let $R(\lambda, \mathcal{A}) = (\lambda I - \mathcal{A})^{-1}$ denote the resolvent operator of $\mathcal{A}$. Then

$$\|R(\lambda,\mathcal{A})\|=\mathcal{O}(|\lambda|^{-1/2}) \quad \text{for } \mathrm{Re}(\lambda)\geq 0. \tag{30}$$

Proof: Since the proof is lengthy, it is given in the Appendix.

We are ready to state and prove Theorem 1.

Theorem 1:

i) There exists a unique classical solution to (18) for each initial condition $W_0 \in D(\mathcal{A}^2)$.
ii) There exists a unique absolutely continuous L^2-solution $W(t)$ to (18) for each initial condition $W_0 \in D(\mathcal{A})$ such that $W(t)$ satisfies

$$\begin{cases}\frac{d}{dt}\langle W(t),Z\rangle_{\mathcal{H}}=\langle W(t),\mathcal{A}^*Z\rangle_{\mathcal{H}},\\ \langle W(0),Z\rangle_{\mathcal{H}}=\langle W_0,Z\rangle_{\mathcal{H}}, \quad \text{for all } Z\in D(\mathcal{A}^*), t\geq 0 \text{ a.e.}\end{cases}$$

Here a solution $W(t)$ is called an L^2-solution if $\int_0^T \|W(t)\|^2 dt < \infty$ holds for each $T > 0$.

Proof:

i) As stated in [19, Proposition 2.3], if for some $-1 \leq \alpha < 0$ and $\mathrm{Re}(\lambda) > \omega$, a constant, there holds

$$\|R(\lambda,\mathcal{A})\|\leq M(1+|\mathrm{Im}(\lambda)|)^{\alpha} \tag{31}$$

then (18) has a unique classical solution for $W_0 \in D(\mathcal{A}^2)$. We shall show that (31) is true. To this end, let us define a continuous function $f(x) = \frac{(1+x)^{1/2}}{(1+x^2)^{1/4}} > 0$. Obviously, $f(0) = 1$ and $\lim_{x\to\infty} f(x) = 1$. Hence there exists a constant C such that $f(x) \leq C$ for any $x \in [0, \infty)$. Using this fact and (30), we have, for $\mathrm{Re}(\lambda) \geq 1$

$$\begin{aligned}\|R(\lambda,\mathcal{A})\|&\leq C_0([\mathrm{Re}(\lambda)]^2+[\mathrm{Im}(\lambda)]^2)^{-1/4}\\ &\leq C_0(1+[\mathrm{Im}(\lambda)]^2)^{-1/4}\\ &\leq C_0C(1+|\mathrm{Im}(\lambda)|)^{-1/2}.\end{aligned}$$

Thus we have proven (31) for $\alpha = -\frac{1}{2}, M = C_0C$ and $\omega = 1$.

ii) From (30), $\|R(\lambda, \mathcal{A})\| \leq C|\lambda|^{-1/2}$ for $C > 0$ and $\mathrm{Re}(\lambda) \geq 0$. Obviously, $\|R(\lambda, \mathcal{A})\| \leq \frac{C}{\sqrt{|\lambda|}} \leq C$ for $|\lambda| \geq 1$. For $|\lambda| \leq 1$ and $\mathrm{Re}(\lambda) \geq 0$, since $\|R(\lambda, \mathcal{A})\|$ is a continuous function, there exists a constant C_0 such that

$$\sup_{\substack{|\lambda|\leq 1\\ \mathrm{Re}(\lambda)\geq 0}}\|R(\lambda,\mathcal{A})\|\leq C_0.$$

Combining i) and ii), we have

$$\sup_{\mathrm{Re}(\lambda)\geq 0}\|R(\lambda,\mathcal{A})\|\leq\max\{C_0,C\}<\infty.$$

Therefore, the conditions in [21, Th. 3] are satisfied, and we conclude that (18) has a unique L^2-solution for each $W_0 \in D(\mathcal{A})$. □

The following theorem proves that the solution to (18) is exponentially stable, which is important in practical terms because it shows that shear force feedback can introduce damping for the vibration of flexible arms with revolute joints.

Theorem 2: Let $W(t)$ be the solution of (18) with initial condition W_0. Then, there exist constants $M > 0$ and $\omega > 0$, independent of W_0, such that

$$\begin{aligned}&\|W(t)\|_{\mathcal{H}}\leq Me^{-\omega t}\|W_0\|_{D(\mathcal{A})},\\ &\qquad \text{for all } t\geq 1, W_0\in D(\mathcal{A}^2)\subset D(\mathcal{A}).\end{aligned}$$

Proof: Let $\lambda = \sigma + i\tau$ with $\sigma, \tau \in R^1$. From the resolvent equation

$$R(\sigma+i\tau,\mathcal{A})-R(i\tau,\mathcal{A})=-\sigma R(\sigma+i\tau,\mathcal{A})R(i\tau,\mathcal{A})$$

it is seen that $\|\sigma R(i\tau, \mathcal{A})\| \leq \frac{1}{2}$, for $0 \geq \sigma \geq -\delta = -\frac{1}{2D}$. Hence $[1+\sigma R(i\tau, \mathcal{A})]^{-1}$ exists and $\|[1+\sigma R(i\tau, \mathcal{A})]^{-1}\| \leq 2$, where $D = \sup_{\tau\in R^1} \|R(i\tau, \mathcal{A})\| < \infty$. Thus, for $-\delta \leq \sigma \leq 0$

$$\begin{aligned}\|R(\sigma+i\tau,\mathcal{A})\|&\leq\|[1+\sigma R(i\tau,\mathcal{A})]^{-1}\|\|R(i\tau,\mathcal{A})\|\\ &\leq 2\|R(i\tau,\mathcal{A})\|=\mathcal{O}((\tau^2+\sigma^2)^{-1/4})\end{aligned}$$

i.e.,

$$\|R(\lambda,\mathcal{A})\|=\mathcal{O}(|\lambda|^{-1/2}), \quad \text{for all } \mathrm{Re}(\lambda)=\sigma\geq -\delta. \tag{32}$$

Now, let $\sigma \geq -\varepsilon > -\delta, \varepsilon > 0$ and $W_0 \in D(\mathcal{A})$. Then

$$\begin{aligned} R(\sigma + i\tau, \mathcal{A})W_0 &= R(\sigma + i\tau, \mathcal{A})R(-\delta, \mathcal{A})(-\delta - \mathcal{A})W_0 \\ &= \frac{1}{\sigma + \delta + i\tau}[R(-\delta, \mathcal{A}) - R(\sigma + i\tau, \mathcal{A})] \\ &\quad \times (-\delta - \mathcal{A})W_0 \\ &= \frac{1}{\sigma + \delta + i\tau}W_0 \\ &\quad - \frac{1}{\sigma + \delta + i\tau}R(\sigma + i\tau, \mathcal{A})(-\delta - \mathcal{A})W_0. \end{aligned} \tag{33}$$

Thus, there exists a constant $C > 0$ such that

$$\|R(\sigma + i\tau, \mathcal{A})W_0\|^2 \leq \frac{C}{(\delta + \sigma)^2 + \tau^2} \leq \frac{C}{(\delta - \varepsilon)^2 + \tau^2}$$

which means

$$\begin{aligned} &\sup_{\sigma \geq -\varepsilon} \int_{-\infty}^{\infty} \|R(\sigma + i\tau, \mathcal{A})W_0\|^2 d\tau \\ &\quad \leq \int_{-\infty}^{\infty} \frac{C}{(\delta - \varepsilon)^2 + \tau^2} d\tau < \infty. \end{aligned}$$

According to the Paley–Wiener theorem [22, Lemma 3.1], there exists $G(t) \in L^2(0, \infty; \mathcal{H})$ such that

$$R(\lambda, \mathcal{A})W_0 = \int_0^{\infty} e^{-(\lambda+\varepsilon)t}G(t)dt, \quad \text{for all } \mathrm{Re}(\lambda) \geq -\varepsilon. \tag{34}$$

On the other hand, if $W_0 \in D(\mathcal{A}^2)$, by [19, part i) of Th. 1 and Lemma 4.6], we have

$$R(\lambda, \mathcal{A})W_0 = \int_0^{\infty} e^{-\lambda t}W(t)dt \tag{35}$$

for $\mathrm{Re}(\lambda)$ sufficiently large. By the uniqueness of the Fourier transform on $L^2(-\infty, \infty; \mathcal{H})$, (34) and (35) imply that

$$e^{\varepsilon t}W(t) = G(t) \in L^2(0, \infty; \mathcal{H}). \tag{36}$$

Let ω be a constant satisfying $0 < \omega < \varepsilon$. Taking the derivative with respect to λ on both sides of (34), we have

$$R^2(\lambda, \mathcal{A})W_0 = \int_0^{\infty} e^{-\lambda t}tW(t)dt, \quad \text{for all } \mathrm{Re}(\lambda) \geq -\omega.$$

Since it can be verified that $e^{-\lambda t}tW(t) \in L^1(0, \infty; \mathcal{H})$ by (36), the above integration exists in the sense of the usual Bochner integral. Now considering a special case where $\lambda = -\omega + i\tau$, we have

$$R^2(-\omega + i\tau, \mathcal{A})W_0 = \int_0^{\infty} e^{-i\tau t}te^{\omega t}W(t)dt.$$

By the inverse Fourier transform on $L^2(-\infty, \infty; \mathcal{H})$, it is easy to see that

$$\begin{aligned} te^{\omega t}W(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\tau t}R^2(-\omega + i\tau, \mathcal{A})W_0 d\tau \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\tau t}R(-\omega + i\tau, \mathcal{A}) \\ &\quad \times \left[\frac{1}{\delta - \omega + i\tau}W_0 - \frac{1}{\delta - \omega + i\tau}\right. \\ &\quad \left. \times R(-\omega + i\tau, \mathcal{A})(-\delta - \mathcal{A})W_0\right] d\tau \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\tau t}\frac{1}{\delta - \omega + i\tau}R(-\omega + i\tau, \mathcal{A})W_0 d\tau \\ &\quad - \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\tau t}\frac{1}{\delta - \omega + i\tau} \\ &\quad \times R^2(-\omega + i\tau, \mathcal{A})(-\delta - \mathcal{A})W_0 d\tau. \end{aligned}$$

In view of (32), there exists a constant C_0 such that

$$\begin{aligned} \|R(-\omega + i\tau, \mathcal{A})\| &\leq C_0(\omega^2 + \tau^2)^{-1/4} \\ &\leq C_0((\omega - \delta)^2 + \tau^2)^{-1/4}. \end{aligned}$$

Therefore

$$\begin{aligned} \|W(t)\|_{\mathcal{H}} &\leq \frac{C_0}{2\pi t}e^{-\omega t}\int_0^{\infty}((\delta - \omega)^2 + \tau^2)^{-3/4}d\tau\|W_0\|_{\mathcal{H}} \\ &\quad + \frac{C_0^2}{2\pi t}e^{-\omega t}\int_0^{\infty}((\delta - \omega)^2 + \tau^2)^{-1}d\tau \\ &\quad \times [\|\delta W_0\|_{\mathcal{H}} + \|\mathcal{A}W_0\|_{\mathcal{H}}] \\ &\leq Me^{-\omega t}\|W_0\|_{D(\mathcal{A})}, \\ &\qquad \text{for all } t \geq 1, W_0 \in D(\mathcal{A}). \end{aligned} \tag{37}$$

This completes the proof. □

Remark 2: Reference [19, Th. 1–part i), Th. 2, and Th. 4.2] implies that $\mathcal{A}$ generates a one-time integrated semigroup $S(t)$, and the solution to (18) is expressed as

$$W(t) = S(t)\mathcal{A}W_0 + W_0 \in C^1(0, \infty, D(\mathcal{A})) \quad \forall W_0 \in D(\mathcal{A}^2).$$

Moreover, for any $W_0 \in D(\mathcal{A})$

$$S(t)W_0 \in D(\mathcal{A}), \quad \mathcal{A}S(t)W_0 = S(t)\mathcal{A}W_0$$

$$S(t)W_0 = \int_0^t S(s)\mathcal{A}W_0 ds + tW_0.$$

These facts can also be obtained by using the results in [18].

It should be noted that if an operator A generates a strongly continuous semigroup, then A generates a one-time integrated semigroup, but not vice versa, because the generator of a one-time integrated semigroup need not be densely defined. Thus, the one-time (more generally n-times) integrated semigroups can be considered as a generalization of strongly continuous semigroups.

III. Asymptotic Behavior of the Spectrum

In this section, we shall estimate the spectrum with large module based on the characteristic equation $G(\lambda) = 0$ defined in Lemma 3. The following theorem establishes explicitly a relationship between the feedback gain k and the eigenvalues λ_n and indicates the interesting fact that the spectral distributions are totally different for $k = 1$ and $k \neq 1$.

Theorem 3: $G(\lambda)=0$ has solutions $\{\lambda_n\}$ which satisfy

$$\begin{cases}\lambda_n=-2\big(n\pi+\frac{\pi}{2}\big)^2, \text{ if } k=1\\ |\lambda_n|=(n\pi)^2+\frac{1}{4}\big(\log\big|\frac{k-1}{k+1}\big|\big)^2+\mathcal{O}(e^{-\gamma_0|n|})\\ \mathrm{Re}(\lambda_n)=|n|\pi\log\big|\frac{k-1}{k+1}\big|+\mathcal{O}(|n|^{-1}), \quad \text{if } k>1\\ |\lambda_n|=\big(n\pi+\frac{\pi}{2}\big)^2+\frac{1}{4}\big(\log\big|\frac{k-1}{k+1}\big|\big)^2+\mathcal{O}(e^{-\gamma_0|n|})\\ \mathrm{Re}(\lambda_n)=\big|n+\frac{1}{2}\big|\pi\log\big|\frac{k-1}{k+1}\big|+\mathcal{O}(|n|^{-1}), \text{ if } k<1\end{cases}$$

for some $\gamma_0>0$.

Proof: For $k=1$, $G(\lambda)=0$ reduces to

$$2+e^{\sqrt{2\lambda}}+e^{-\sqrt{2\lambda}}=0$$

the solutions of which are given by those λ_n satisfying $\sqrt{2\lambda_n}=(2n\pi+\pi)i$ for integer n, i.e.,

$$\lambda_n=-2\Big(n\pi+\frac{\pi}{2}\Big)^2.$$

For the case $k\neq 1$, write $\lambda_n=|\lambda_n|e^{i\theta}$. θ must satisfy $\pi/2<\theta<3\pi/2$, since λ_n lies on the left-half plane as shown in Lemma 5.

Let $\delta>0$ be a sufficiently small constant. Consider first the case $\frac{\pi}{2}<\theta\le\pi-\delta$. Then

$$\sqrt{\lambda_n}=|\lambda_n|^{1/2}e^{i\theta/2}=|\lambda_n|^{1/2}\left[\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right]$$

and

$$e^{-\sqrt{2\lambda_n}}=\mathcal{O}\big(e^{-\gamma\sqrt{|\lambda_n|}}\big), e^{i\sqrt{2\lambda_n}}=\mathcal{O}\big(e^{-\gamma\sqrt{|\lambda_n|}}\big)$$

for some $\gamma>0$. From equation $G(\lambda_n)=0$, we have

$$e^{\sqrt{2\lambda_n}+i\sqrt{2\lambda_n}}=\frac{k-1}{k+1}+\mathcal{O}\big(e^{-\gamma\sqrt{|\lambda_n|}}\big) \tag{38}$$

which implies

$$e^{\sqrt{2|\lambda_n|}\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)} =\frac{k-1}{k+1}e^{-i\sqrt{2|\lambda_n|}\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)}+\mathcal{O}\big(e^{-\gamma\sqrt{|\lambda_n|}}\big). \tag{39}$$

Since the left-hand side of (39) is a positive real number, we have

$$e^{\sqrt{2|\lambda_n|}\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)}=\left|\frac{k-1}{k+1}\right|+\mathcal{O}\big(e^{-\gamma\sqrt{|\lambda_n|}}\big) \tag{40}$$

$$e^{-i\sqrt{2|\lambda_n|}\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)}=\mathrm{sign}(k-1)+\mathcal{O}\big(e^{-\gamma\sqrt{|\lambda_n|}}\big). \tag{41}$$

In view of (40), we see that

$$\theta\to\frac{\pi}{2}, \quad \text{as } |\lambda_n|\to\infty$$

since otherwise the left-hand side of (40) would go to zero. Furthermore, from (40)

$$\sqrt{2|\lambda_n|}\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)=\log\left(\left|\frac{k-1}{k+1}\right|+\mathcal{O}\big(e^{-\gamma\sqrt{|\lambda_n|}}\big)\right) =\log\left|\frac{k-1}{k+1}\right|+\mathcal{O}\big(e^{-\gamma\sqrt{|\lambda_n|}}\big).$$

Thus

$$2|\lambda_n|(1-\sin\theta)=\left(\log\left|\frac{k-1}{k+1}\right|\right)^2+\mathcal{O}\big(e^{-\gamma\sqrt{|\lambda_n|}}\big)$$

or

$$\sin\theta=1-\frac{1}{2|\lambda_n|}\left(\log\left|\frac{k-1}{k+1}\right|\right)^2+\mathcal{O}\big(e^{-\gamma\sqrt{|\lambda_n|}}\big). \tag{42}$$

Hence

$$\mathrm{Re}(\lambda_n)=|\lambda_n|\cos\theta=-|\lambda_n|\sqrt{1-\sin^2\theta} =-\sqrt{|\lambda_n|}\log\left|\frac{k-1}{k+1}\right|+\mathcal{O}\big(|\lambda_n|^{-1/2}\big). \tag{43}$$

On the other hand, from (41) we get (z), shown at the bottom of the page, where n is an integer. Thus $|\lambda_n|=\mathcal{O}(n^2)$ and

$$\begin{cases}2|\lambda_n|(1+\sin\theta)=(2n\pi)^2+\mathcal{O}(|n|e^{-|n|}), \quad \text{if } k>1\\ 2|\lambda_n|(1+\sin\theta)=[(2n+1)\pi]^2+\mathcal{O}(|n|e^{-|n|}), \quad \text{if } k<1.\end{cases}$$

In view of (42), we have

$$2|\lambda_n|(1+\sin\theta)=4|\lambda_n|-\left(\log\left|\frac{k-1}{k+1}\right|\right)^2+\mathcal{O}\big(e^{-\gamma\sqrt{|\lambda_n|}}\big).$$

Hence we have (44), shown at the bottom of the page. Substituting (44) into (43) yields

$$\begin{cases}\mathrm{Re}(\lambda_n)=|n|\pi\log\big|\frac{k-1}{k+1}\big|+\mathcal{O}(|n|^{-1}), \quad \text{if } k>1\\ \mathrm{Re}(\lambda_n)=\big|n+\frac{1}{2}\big|\pi\log\big|\frac{k-1}{k+1}\big|+\mathcal{O}(|n|^{-1}), \quad \text{if } k<1.\end{cases} \tag{45}$$

We now show that, in the area $\pi-\delta<\theta\le\pi$, there exist no zeros of $G(\lambda)$ for large $|\lambda|$. In this case, $\frac{\pi}{2}-\frac{\delta}{2}<\frac{\theta}{2}\le\frac{\pi}{2}$ and

$$e^{-\sqrt{2\lambda}}=\mathcal{O}(1), \quad e^{i\sqrt{2\lambda}}=\mathcal{O}(1).$$

If there exists a λ such that $G(\lambda)=0$, then

$$e^{\sqrt{2\lambda}+i\sqrt{2\lambda}}=\mathcal{O}(1)$$

or

$$(1+k)e^{|2\lambda|^{1/2}\left[\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right]} =(1-k)e^{|2\lambda|^{1/2}\left[\sin\frac{\theta}{2}-i\cos\frac{\theta}{2}\right]}+\mathcal{O}(1). \tag{46}$$

$$\begin{cases}-\sqrt{2|\lambda_n|}\big(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\big)=2n\pi+\mathcal{O}\big(e^{-\gamma\sqrt{|\lambda_n|}}\big), \quad \text{if } k>1\\ -\sqrt{2|\lambda_n|}\big(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\big)=(2n+1)\pi+\mathcal{O}\big(e^{-\gamma\sqrt{|\lambda_n|}}\big), \quad \text{if } k<1\end{cases} \tag{z}$$

$$\begin{cases}|\lambda_n|=(n\pi)^2+\frac{1}{4}\big(\log\big|\frac{k-1}{k+1}\big|\big)^2+\mathcal{O}(|n|e^{-\gamma|n|}), \quad \text{if } k>1\\ |\lambda_n|=(n\pi+\frac{\pi}{2})^2+\frac{1}{4}\big(\log\big|\frac{k-1}{k+1}\big|\big)^2+\mathcal{O}\big(|n|e^{-\gamma|n|}\big), \quad \text{if } k<1.\end{cases} \tag{44}$$

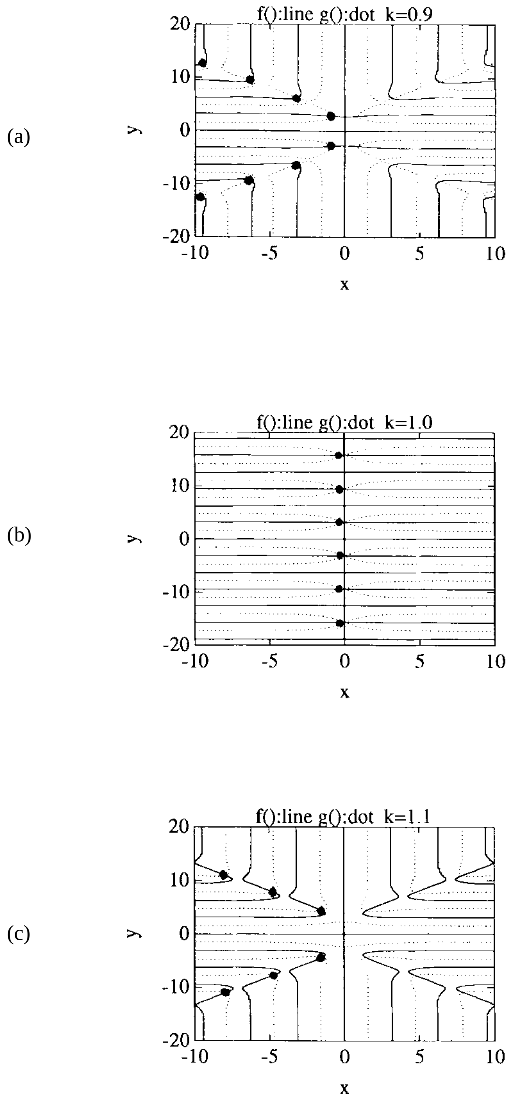

Fig. 2. Spectral distribution for system (17) with different feedback gain k: (a) $k = 0.9$, (b) $k = 1.0$, and (c) $k = 1.1$.

When $\theta \to \pi$, the modulus of the right-hand side of (46) is much larger than that of the left-hand side. Therefore, there exists no solution of $G(\lambda) = 0$ for $|\lambda|$ sufficiently large.

Remaining to be shown is the existence of the zeros which satisfy the relation in this theorem. We only consider the case of $k > 1$ since the treatment for $k < 1$ is very similar. Returning to (38), we see that the solution of the equation

$$e^{(1+i)\mu} = \frac{k-1}{k+1}$$

can be expressed as

$$\mu_n = \frac{1}{2}\log\left|\frac{k-1}{k+1}\right| + n\pi + i\left[n\pi - \frac{1}{2}\log\left|\frac{k-1}{k+1}\right|\right]$$

with $|\mu_n| = \mathcal{O}(|n|)$. Let O_n be the circle with center at μ_n and radius $|\mu_n|^{-2}$.

$$O_n\colon \lambda = \mu_n + |\mu_n|^{-2}e^{i\theta}, \quad 0 \le \theta < 2\pi.$$

Then for all λ located on the circumference of O_n, we see that

$$\begin{aligned}
&\left|e^{(1+i)\lambda} - \frac{k-1}{k+1}\right| \\
&\quad = \left|e^{(1+i)\lambda} - e^{(1+i)\mu_n}\right| \\
&\quad = \left|\frac{k-1}{k+1}\right|\left|e^{(1+i)(\lambda-\mu_n)} - 1\right| \\
&\quad = \left|\frac{k-1}{k+1}\right|\left|e^{(1+i)|\mu_n|^{-2}e^{i\theta}} - 1\right| \\
&\quad = \left|\frac{k-1}{k+1}\right|\left|(1+i)|\mu_n|^{-2}e^{i\theta} + \mathcal{O}\big(|\mu_n|^{-4}\big)\right| \\
&\quad = \left|\frac{k-1}{k+1}\right|\left||1+i||\mu_n|^{-2} + \mathcal{O}\big(|\mu_n|^{-4}\big)\right| \\
&\quad = \left|\frac{k-1}{k+1}\right|\left||1+i||\lambda|^{-2} + \mathcal{O}\big(|\lambda|^{-4}\big) > \mathcal{O}\big(e^{-\gamma\sqrt{|\lambda|}}\big)\right|
\end{aligned}$$

holds for all sufficiently large $|n|$. By Rouche's theorem, there exists, inside O_n, a unique solution σ_n to equation

$$e^{(1+i)\lambda} = \frac{k-1}{k+1} + \mathcal{O}\big(e^{-\gamma\sqrt{|\lambda|}}\big)$$

such that

$$|\sigma_n - \mu_n| \le |\mu_n|^{-2}.$$

Hence $|\sigma_n^2 - \mu_n^2| = |\sigma_n - \mu_n||\sigma_n + \mu_n| \le 2|\mu_n|^{-1} + |\mu_n|^{-4}$. In view of (44), we see that $\lambda_n = \frac{1}{2}\sigma_n^2$ is the unique solution of (38) such that

$$\left|\lambda_n - \frac{1}{2}\mu_n^2\right| \le |\mu_n|^{-1} + \frac{1}{2}|\mu_n|^{-4}.$$

Since

$$\frac{1}{2}\mu_n^2 = n\pi\log\left|\frac{k-1}{k+1}\right| + i\left[(n\pi)^2 - \frac{1}{4}\left(\log\left|\frac{k-1}{k+1}\right|\right)^2\right]$$

and taking n to be a postive integer, we have the desired result. □

To see how different the eigenvalues are for the cases $k = 1$ and $k \ne 1$, we performed some computer simulations. Let

$$\sqrt{2\lambda} = x + iy \tag{47}$$

with x and y being real. Substituting this into $G(\lambda) = 0$ in Lemma 3, we get

$$f(x, y) + ig(x, y) = 0 \tag{48}$$

where

$$\begin{aligned}
f(x, y) &= 4 + (1+k)(e^x + e^{-x}) \\
&\qquad \times \cos(y) + (1-k)(e^{-y} + e^y)\cos(x) \\
g(x, y) &= (1+k)(e^x - e^{-x}) \\
&\qquad \times \sin(y) + (1-k)(e^{-y} - e^y)\sin(x).
\end{aligned}$$

Obviously, λ satisfies $G(\lambda) = 0$ if and only if x and y satisfy

$$f(x, y) = 0, \quad g(x, y) = 0.$$

The graphs of $f(x, y) = 0$ and $g(x, y) = 0$ are plotted in Figs. 2(a)–(c) where the solid line denotes the graph of

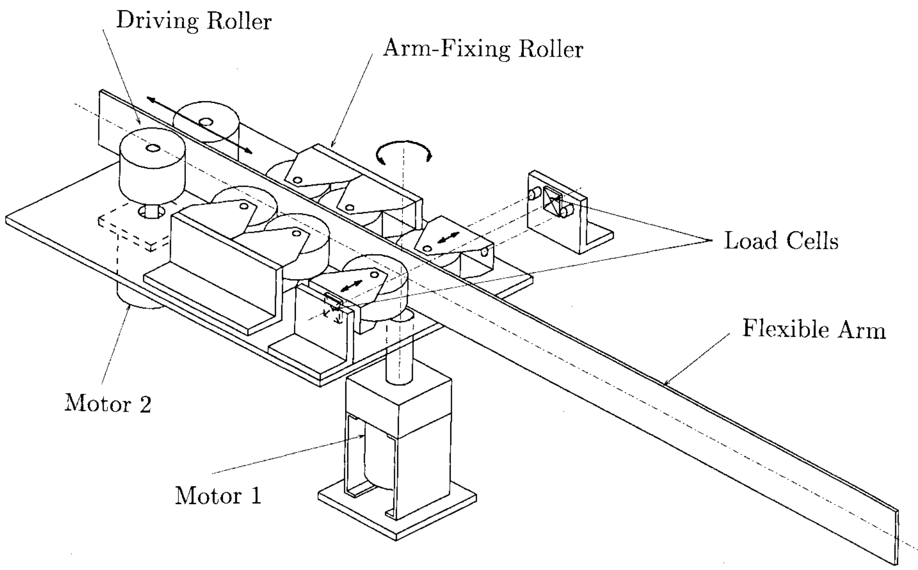

Fig. 3. Schematic of the flexible polar robot used in experiments.

$f(x,y)=0$ and the dotted line denotes the graph of $g(x,y)=0$. From the intersection points (the black points) of these two graphs we can recover the zero points of $G(\lambda)$ by using the relation

$$\lambda = \frac{1}{2}(x^2 - y^2) + ixy.$$

It can be seen that the case for $k=1$ is very different from the cases for $k=0.9$ and $k=1.1$.

IV. Control Experiments

To demonstrate the applicability of the proposed feedback law, a polar robot, as shown in Fig. 3, was constructed in our laboratory. The robot has a revolute joint and a prismatic joint. The tip arm driven by the prismatic joint motor is a flexible one, with its length variable. The dynamics of this kind of robot is discussed in [23], but so far no effective method to control the vibration in the flexible arm has been reported. The difficulty lies in the fact that the strain feedback control law cannot be used because it is very difficult, if not impossible, to measure the bending strain at the root end of the flexible arm, which is not fixed, and that the well-known linear control theory cannot be used because the dynamic model is nonlinear and time-varying.

The revolute joint motor (motor 1, DC, 60 watt) is coupled to its load via a gear with the gear ratio being equal to 50, while the prismatic joint motor (motor 2, DC, 40 watt) is directly coupled to the driving roller. Commercial motor drivers of speed reference type are selected to provide power for the two control motors.

To measure shear force, two load cells are fixed at a place as indicated in Fig. 3. Each sensor can detect the force acting on it. A thin aluminum beam with total length $\ell = 1.0$ [m] is driven by motor 2 and can slide between the two load cells. At the stationary state, there is an equal constant force acting on the two sensors. When there is vibration, however, there will be differences between the outputs of these two sensors. From these differences the shear force, $y'''(t,0)$ at the root end of the flexible arm, can be detected.

Potentiometers and tachometers are used to measure the angular position $\theta(t)$, arm length $\ell(t)$, angular velocity $\dot{\theta}(t)$, and the arm length variation speed $\dot{\ell}(t)$. These signals together with the shear force signal $y'''(t,0)$ are sent to the controller (NEC9801 personal computer with 80 386 CPU). The control program is written in C language, and the sampling period is set at 8 [ms].

As a first step, let us consider the case where only the revolute joint is driven, while the prismatic joint remains inactive. In this case, the device can be thought of as a one-link flexible robot with a revolute joint, as has appeared in the literature [3], [10], for which it has been shown that the bending strain at the root end can damp out vibration in a flexible arm. Our purpose here is to show whether shear force (or shear strain) feedback can control the vibration in a flexible arm. Since the shear force $y'''(t,0)$ is already available, the control voltage $V_{\text{ref}}(t)$ to the motor driver of motor 1 is set to be

$$V_{\text{ref}}(t) = -ky'''(t,0) \tag{49}$$

where $k > 0$ is a feedback constant. Since the reference voltage is approximately proportional to the angular velocity for a motor with a motor driver of speed reference type, i.e.,

$$V_{\text{ref}}(t) = k_f\dot{\theta}(t) \tag{50}$$

we get

$$\dot{\theta}(t) = -ky'''(t,0) \tag{51}$$

by combining (49) and (50) and by letting $k_f = 1$ without loss of generality. Taking the time derivative of both sides of (51) yields (15). In engineering applications, it may be required that both the vibration and angular position of motor 1 be controlled simultaneously. In this case, the motor voltage

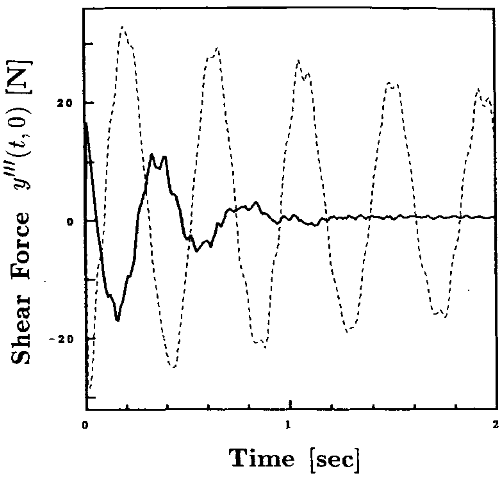

Fig. 4. Experimental results of shear force feedback control when the arm length is a constant. The real line represents sensor output with shear force feedback, and the dashed line represents sensor output without shear force feedback.

$V_{\text{ref}}(t)$ can be set to

$$V_{\text{ref}}(t) = -ky'''(t,0) - k_p(\theta(t) - \theta_r) - k_i \int_0^t (\theta(\tau) - \theta_r)d\tau \quad (52)$$

where θ_r is the desired position of motor 1 and k_p and k_i are the PI feedback gains. Note that when this feedback law is implemented, the closed-loop equation is a coupled partial differential equation on $y(t,x)$ and an ordinary differential equation on $\theta(t) - \theta_r$ which is called a hybrid system. The stability analysis can be similarly done as in [24], based on the results presented here. And conditions on feedback gains k, k_p, and k_i can be derived.

To keep consistency with the theoretical discussions, we use (49), instead of (52), as the control law to control the vibration only, leaving the motor angular position uncontrolled. The experimental results are displayed in Fig. 4, where the dotted line denotes the difference of the two sensor outputs when the arm is undergoing free vibration after an external impact force is applied [which corresponds to the case where $k = 0$ in (49)]. It is observed that it takes a long time for the vibration in the arm to die down. When shear force feedback is considered [k in (49) is set to be a suitable value], we get the sensor output response as indicated in Fig. 4 with the solid line. The damping effect of shear force feedback is obvious. Note that k can be determined by several times of trial and error, starting from a very small value and gradually increasing it until a sufficient damping rate is observed.

V. Conclusion and Remarks

This paper has presented a shear force feedback control method for flexible robot arms with revolute joints. Emphasis was placed on the investigation of stability of the closed-loop feedback system. It has been shown that there exists a unique classical solution to the closed-loop equation, and moreover the solution is exponentially stable. Based on these results, it is clarified that the system operator associated with the closed-loop equation generates a one-time integrated semigroup. Whether this operator generates a strongly continuous semigroup remains for further research.

Appendix

Proof of Lemma 6: For any $(f,g) \in \mathcal{H}$, we need to solve the following boundary value problem:

$$(\mathcal{A} - \lambda)\begin{bmatrix}\phi(x)\\ \psi(x)\end{bmatrix} = \begin{bmatrix}\psi(x) + kx\phi'''(0) - \lambda\phi(x)\\ -\phi''''(x) - \lambda\psi(x)\end{bmatrix} = \begin{bmatrix}f(x)\\ g(x)\end{bmatrix}, \quad \begin{bmatrix}\phi(x)\\ \psi(x)\end{bmatrix} \in D(\mathcal{A})$$

from which we have

$$\psi(x) = \lambda\phi(x) - kx\phi'''(0) + f(x) \quad (53)$$

with $\phi(x)$ satisfying

$$\begin{cases}\phi''''(x) - k\lambda x\phi'''(0) + \lambda^2\phi(x) = -[\lambda f(x) + g(x)]\\ \phi(0) = \phi'(0) = \phi''(1) = \phi'''(1) = 0.\end{cases} \quad (54)$$

Since we only consider those λ with $\text{Re}(\lambda) \geq 0$, we are able to write λ as

$$\lambda = |\lambda|e^{i\theta}, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}.$$

We consider only the case where $0 \leq \theta \leq \frac{\pi}{2}$. The estimates for $\frac{-\pi}{2} \leq \theta \leq 0$ are similar. In this case, $\frac{\pi}{4} \leq \frac{\theta}{2} + \frac{\pi}{4} \leq \frac{\pi}{2}$ and the following facts hold:

$$\begin{aligned}\lambda i &= |\lambda|e^{i(\theta+\frac{\pi}{2})}, \quad \sqrt{\lambda i} = \sqrt{|\lambda|}e^{i(\frac{\theta}{2}+\frac{\pi}{4})}\\ e^{-\sqrt{\lambda i}} &= e^{\sqrt{|\lambda|}[-\cos(\frac{\theta}{2}+\frac{\pi}{4}) - i\sin(\frac{\theta}{2}+\frac{\pi}{4})]} = \mathcal{O}(1)\\ e^{i\sqrt{\lambda i}} &= e^{\sqrt{|\lambda|}[-\sin(\frac{\theta}{2}+\frac{\pi}{4}) - i\cos(\frac{\theta}{2}+\frac{\pi}{4})]} = \mathcal{O}(1)\\ e^{i\sqrt{\lambda i}}e^{\sqrt{\lambda i}} &= \mathcal{O}(1).\end{aligned} \quad (55)$$

These relations will be frequently used in the sequel. It can be checked that the solution $\phi(x)$ to (54) equals the sum of $\tilde{\phi}(x)$ and $\phi_p(x)$, i.e., $\phi(x) = \tilde{\phi}(x) + \phi_p(x)$, where $\tilde{\phi}(x)$ satisfies

$$\begin{cases}\tilde{\phi}''''(x) - k\lambda x\tilde{\phi}'''(0) + \lambda^2\tilde{\phi}(x) = 0\\ \tilde{\phi}(0) = \tilde{\phi}'(0) = 0\\ \tilde{\phi}''(1) = -\phi_p''(1)\\ \tilde{\phi}'''(1) = -\phi_p'''(1)\end{cases} \quad (56)$$

and $\phi_p(x)$ satisfies

$$\begin{cases}\phi_p''''(x) + \lambda^2\phi_p(x) = -[\lambda f(x) + g(x)]\\ \phi_p(0) = \phi_p'(0) = \phi_p'''(0) = 0.\end{cases} \quad (57)$$

A particular solution of (57) is given by

$$\phi_p(x) = -\frac{1}{2}(\lambda i)^{-3/2}\int_0^x [\sinh\sqrt{\lambda i}(x-\xi) - \sin\sqrt{\lambda i}(x-\xi)] \times [\lambda f(\xi) + g(\xi)]d\xi. \quad (58)$$

Clearly, $\|R(\lambda, \mathcal{A})\| = \mathcal{O}(|\lambda|^{-1/2})$ holds if we can find a constant M, independent of λ, such that for all $\text{Re}(\lambda) \geq 0$

$$\left(\int_0^1 [|\phi''(x)|^2 + |\psi(x)|^2]dx\right)^{1/2} \leq M|\lambda|^{-1/2}\left(\int_0^1 [|f''(x)|^2 + |g(x)|^2]dx\right)^{1/2}. \quad (59)$$

The estimation of (59) is divided into four steps.

First Step—Estimation of $\phi_p''(x)$:

$$
\begin{aligned}
\phi_p''(x) &= -\frac{1}{2}(\lambda i)^{-1/2}\int_0^x [\sinh\sqrt{\lambda i}(x-\xi)+\sin\sqrt{\lambda i}(x-\xi)] \\
&\quad\times[\lambda f(\xi)+g(\xi)]d\xi \\
&= -\frac{1}{2}(\lambda i)^{-1/2}\int_0^x [\sinh\sqrt{\lambda i}(x-\xi)+\sin\sqrt{\lambda i}(x-\xi)] \\
&\quad\times g(\xi)d\xi \\
&\quad-\frac{1}{2}(\lambda i)^{-3/2}\int_0^x [\sinh\sqrt{\lambda i}(x-\xi)-\sin\sqrt{\lambda i}(x-\xi)] \\
&\quad\times\lambda f''(\xi)d\xi \\
&= -\frac{1}{4}(\lambda i)^{-1/2}\int_0^x \left[e^{\sqrt{\lambda i}(x-\xi)}+ie^{-i\sqrt{\lambda i}(x-\xi)}\right]g(\xi)d\xi \\
&\quad+\frac{1}{4}i(\lambda i)^{-1/2}\int_0^x \left[e^{\sqrt{\lambda i}(x-\xi)}-ie^{-i\sqrt{\lambda i}(x-\xi)}\right] \\
&\quad\times f''(\xi)d\xi+\mathcal{O}(|\lambda|^{-1/2}[\|f''\|+\|g\|]). \\
&= \frac{1}{4}(\lambda i)^{-1/2}e^{\sqrt{\lambda i}x}\int_0^x e^{-\sqrt{\lambda i}\xi}[if''(\xi)-g(\xi)]d\xi \\
&\quad-i\frac{1}{4}(\lambda i)^{-1/2}e^{-i\sqrt{\lambda i}x}\int_0^x e^{i\sqrt{\lambda i}\xi}[if''(\xi)+g(\xi)]d\xi \\
&\quad+\mathcal{O}(|\lambda|^{-1/2}[\|f''\|+\|g\|]).
\end{aligned}
\tag{60}
$$

Second Step—Estimations of $-\phi_p''(1)$ and $-\phi_p'''(x)$: From the result in (60), we see that

$$
\begin{aligned}
-\phi_p''(1) &= -\frac{1}{4}(\lambda i)^{-1/2}\int_0^1 e^{\sqrt{\lambda i}(1-\xi)}[if''(\xi)-g(\xi)]d\xi \\
&\quad+i\frac{1}{4}(\lambda i)^{-1/2}\int_0^1 e^{-i\sqrt{\lambda i}(1-\xi)}[if''(\xi)+g(\xi)]d\xi \\
&\quad+\mathcal{O}(|\lambda|^{-1/2}[\|f''\|+\|g\|]).
\end{aligned}
\tag{61}
$$

By straightforward calculation, we obtain

$$
\begin{aligned}
\phi_p'''(x) &= -\frac{1}{2}\int_0^x [\cosh\sqrt{\lambda i}(x-\xi)+\cos\sqrt{\lambda i}(x-\xi)] \\
&\quad\times g(\xi)d\xi \\
&\quad+i\frac{1}{2}\int_0^x [\cosh\sqrt{\lambda i}(x-\xi)-\cos\sqrt{\lambda i}(x-\xi)] \\
&\quad\times f''(\xi)d\xi \\
&= -\frac{1}{4}\int_0^x \left[e^{\sqrt{\lambda i}(x-\xi)}+e^{-i\sqrt{\lambda i}(x-\xi)}\right]g(\xi)d\xi \\
&\quad+i\frac{1}{4}\int_0^x \left[e^{\sqrt{\lambda i}(x-\xi)}-e^{-i\sqrt{\lambda i}(x-\xi)}\right]f''(\xi)d\xi \\
&\quad+\mathcal{O}(\|f''\|+\|g\|). \\
&= \frac{1}{4}\int_0^x e^{\sqrt{\lambda i}(x-\xi)}[if''(\xi)-g(\xi)]d\xi \\
&\quad-\frac{1}{4}\int_0^x e^{-i\sqrt{\lambda i}(x-\xi)}[if''(\xi)+g(\xi)]d\xi \\
&\quad+\mathcal{O}(\|f''\|+\|g\|)
\end{aligned}
$$

from which we have

$$
\begin{aligned}
-\phi_p'''(1) &= -\frac{1}{4}\int_0^1 e^{\sqrt{\lambda i}(1-\xi)}[if''(\xi)-g(\xi)]d\xi \\
&\quad+\frac{1}{4}\int_0^1 e^{-i\sqrt{\lambda i}(1-\xi)}[if''(\xi)+g(\xi)]d\xi \\
&\quad+\mathcal{O}(\|f''\|+\|g\|).
\end{aligned}
\tag{62}
$$

Third Step—Estimation of $\tilde{\phi}''(x)$: Let $h(x) = \tilde{\phi}''(x)$, where $\tilde{\phi}(x)$ is the solution of (56). Then, $h(x)$ should satisfy

$$
\begin{cases}
h''''(x)+\lambda^2 h(x)=0 \\
h(1)=-\phi_p''(1), h'(1)=-\phi_p'''(1) \\
h''(0)=0 \\
h'''(0)=k\lambda h'(0)
\end{cases}
\tag{63}
$$

the solution of which is of the form

$$
\begin{aligned}
h(x) &= A_1 e^{\sqrt{\lambda i}x}+A_2 e^{-\sqrt{\lambda i}x} \\
&\quad+A_3 e^{i\sqrt{\lambda i}x}+A_4 e^{-i\sqrt{\lambda i}x}.
\end{aligned}
\tag{64}
$$

Here $A_i, i=1,2,3,4$ are constants to be determined. By the boundary conditions $h''(0)=h'''(0)-k\lambda h'(0)=0$, we get

$$
\begin{cases}
A_1 = \frac{(1-i)(1+k)}{2(k-i)}A_3+\frac{(1+i)(k-1)}{2(k-i)}A_4 \\
A_2 = \frac{(1+i)(k-1)}{2(k-i)}A_3+\frac{(1-i)(k+1)}{2(k-i)}A_4.
\end{cases}
\tag{65}
$$

By the boundary conditions $h(1) = -\phi_p''(1)$ and $h'(1) = -\phi_p'''(1)$, we have

$$
E\begin{bmatrix}A_3\\A_4\end{bmatrix} = \begin{bmatrix}-\phi_p''(1)\\-\phi_p'''(1)/\sqrt{\lambda i}\end{bmatrix}
$$

where E is a 2×2 matrix with elements

$$
\begin{aligned}
e_{11} &= \frac{(1-i)(1+k)}{2(k-i)}e^{\sqrt{\lambda i}} \\
&\quad+\frac{(1+i)(k-1)}{2(k-i)}e^{-\sqrt{\lambda i}}+e^{i\sqrt{\lambda i}} \\
e_{12} &= \frac{(1+i)(k-1)}{2(k-i)}e^{\sqrt{\lambda i}} \\
&\quad+\frac{(1-i)(k+1)}{2(k-i)}e^{-\sqrt{\lambda i}}+e^{-i\sqrt{\lambda i}} \\
e_{21} &= \frac{(1-i)(1+k)}{2(k-i)}e^{\sqrt{\lambda i}} \\
&\quad-\frac{(1+i)(k-1)}{2(k-i)}e^{-\sqrt{\lambda i}}+ie^{i\sqrt{\lambda i}} \\
e_{22} &= \frac{(1+i)(k-1)}{2(k-i)}e^{\sqrt{\lambda i}} \\
&\quad-\frac{(1-i)(k+1)}{2(k-i)}e^{-\sqrt{\lambda i}}-ie^{-i\sqrt{\lambda i}}.
\end{aligned}
$$

The determinant of E is given by

$$
\begin{aligned}
\Delta &= \det(E) \\
&= -\frac{1}{k-i}\left[4+(1+k)\left(e^{\sqrt{2\lambda}}+e^{-\sqrt{2\lambda}}\right)\right. \\
&\qquad\left.+(1-k)\left(e^{i\sqrt{2\lambda}}+e^{-i\sqrt{2\lambda}}\right)\right] \\
&= -\frac{1}{k-i}\left\{4+(1+k)\left[e^{(1-i)\sqrt{\lambda i}}+e^{(i-1)\sqrt{\lambda i}}\right]\right. \\
&\quad\left.+(1-k)\left[e^{(1+i)\sqrt{\lambda i}}+e^{-(1+i)\sqrt{\lambda i}}\right]\right\} \\
&= -\frac{1}{k-i}e^{-i\sqrt{\lambda i}}\left\{\left[(1+k)e^{\sqrt{\lambda i}}+(1-k)e^{-\sqrt{\lambda i}}\right]\right. \\
&\quad\left.+4e^{i\sqrt{\lambda i}}+(1+k)e^{(2i-1)\sqrt{\lambda i}}+(1-k)e^{(1+2i)\sqrt{\lambda i}}\right\} \\
&= -\frac{1}{k-i}e^{-i\sqrt{\lambda i}}\left[(1+k)e^{\sqrt{\lambda i}}+(1-k)e^{-\sqrt{\lambda i}}\right] \\
&\quad\times\left[1+\mathcal{O}\left(\left|e^{i\sqrt{\lambda i}}\right|\right)\right]
\end{aligned}
$$

$$= -\frac{1+k}{k-i}e^{(1-i)\sqrt{\lambda i}}\left[1+\frac{1-k}{1+k}e^{-2\sqrt{\lambda i}}+\mathcal{O}\left(\left|e^{i\sqrt{\lambda i}}\right|\right)\right]. \tag{66}$$

Using this, it is easy to check that

$$\begin{aligned}\Delta\sqrt{\lambda i}A_3 &= -e_{22}\sqrt{\lambda i}\phi_p''(1)+e_{12}\phi_p'''(1)\\ &= (1+i)e^{-i\sqrt{\lambda i}}\frac{1}{4}\int_0^1 e^{\sqrt{\lambda i}(1-\xi)}[if''(\xi)-g(\xi)]d\xi\\ &\quad+\frac{1-k}{k-i}e^{\sqrt{\lambda i}}\frac{1}{4}\int_0^1 e^{-i\sqrt{\lambda i}(1-\xi)}\\ &\quad\times[if''(\xi)+g(\xi)]d\xi+\mathcal{O}\left(\left|e^{-i\sqrt{\lambda i}}\right|[\|f''\|+\|g\|]\right)\\ \Delta\sqrt{\lambda i}A_4 &= e_{21}\sqrt{\lambda i}\phi_p''(1)-e_{11}\phi_p'''(1)\\ &= -i\left[\frac{1+k}{k-i}e^{\sqrt{\lambda i}}+\frac{1-k}{k-i}e^{-\sqrt{\lambda i}}\right]\\ &\quad\times\frac{1}{4}\int_0^1 e^{-i\sqrt{\lambda i}(1-\xi)}[if''(\xi)+g(\xi)]d\xi\\ &\quad+\mathcal{O}\left(\left|e^{\sqrt{\lambda i}}\right|[\|f''\|+\|g\|]\right).\end{aligned}$$

By virtue of (66), we get

$$\begin{cases}\sqrt{\lambda i}A_3 = -\frac{1+i}{4}\frac{k-i}{1+k}\int_0^1 e^{-\sqrt{\lambda i}\xi}[if''(\xi)-g(\xi)]d\xi\\ \qquad -\frac{1}{4}\frac{1-k}{1+k}\int_0^1 e^{i\sqrt{\lambda i}\xi}[if''(\xi)+g(\xi)]d\xi\\ \qquad +\mathcal{O}(|e^{-\sqrt{\lambda i}}|[\|f''\|+\|g\|]),\\ \sqrt{\lambda i}A_4 = \frac{i}{4}\int_0^1 e^{i\sqrt{\lambda i}\xi}[if''(\xi)+g(\xi)]d\xi\\ \qquad +\mathcal{O}(|e^{i\sqrt{\lambda i}}|[\|f''\|+\|g\|]).\end{cases} \tag{67}$$

Consequently

$$A_k = \mathcal{O}(|\lambda|^{-1/2}[\|f''\|+\|g\|]), \quad k=1,\cdots,4. \tag{68}$$

Also, it can be verified that

$$\begin{aligned}\Delta\sqrt{\lambda i}A_1 &= \frac{1+k}{k-i}e^{-i\sqrt{\lambda i}}\\ &\quad\times\frac{1}{4}\int_0^1 e^{\sqrt{\lambda i}(1-\xi)}[if''(\xi)-g(\xi)]d\xi\\ &\quad+\mathcal{O}\left(\left|e^{-i\sqrt{\lambda i}}\right|[\|f''\|+\|g\|]\right).\end{aligned}$$

Thus

$$\begin{aligned}\sqrt{\lambda i}A_1 &= -\frac{1}{4}\int_0^1 e^{-\sqrt{\lambda i}\xi}[if''(\xi)-g(\xi)]d\xi\\ &\quad+\mathcal{O}\left(\left|e^{-\sqrt{\lambda i}}\right|[\|f''\|+\|g\|]\right).\end{aligned} \tag{69}$$

By use of (67)–(69), we have

$$\begin{aligned}\sqrt{\lambda i}h(x) &= \sqrt{\lambda i}A_1e^{\sqrt{\lambda i}x}+\sqrt{\lambda i}A_2e^{-\sqrt{\lambda i}x}\\ &\quad+\sqrt{\lambda i}A_3e^{i\sqrt{\lambda i}x}+\sqrt{\lambda i}A_4e^{-i\sqrt{\lambda i}x}\\ &= \sqrt{\lambda i}A_1e^{\sqrt{\lambda i}x}+\sqrt{\lambda i}A_4e^{-i\sqrt{\lambda i}x}+\mathcal{O}(\|f''\|+\|g\|)\\ &= -\frac{1}{4}e^{\sqrt{\lambda i}x}\int_0^1 e^{-\sqrt{\lambda i}\xi}[if''(\xi)-g(\xi)]d\xi\\ &\quad+i\frac{1}{4}e^{-i\sqrt{\lambda i}x}\int_0^1 e^{i\sqrt{\lambda i}\xi}[if''(\xi)+g(\xi)]d\xi\\ &\quad+\mathcal{O}(\|f''\|+\|g\|)\\ &= -\sqrt{\lambda i}\phi_p''(x)+\mathcal{O}(\|f''\|+\|g\|)\end{aligned}$$

from which we conclude that

$$\begin{aligned}\phi''(x) &= \tilde{\phi}''(x)+\phi_p''(x)\\ &= h(x)+\phi_p''(x)\\ &= \mathcal{O}(|\lambda|^{-1/2}[\|f''\|+\|g\|]).\end{aligned} \tag{70}$$

Final Step—Estimation of $\psi(x)$*:* To complete the proof, we need to show that

$$\begin{aligned}\psi(x) &= \lambda\phi(x)-kx\phi'''(0)+f(x)\\ &= \mathcal{O}(|\lambda|^{-1/2}[\|f''\|+\|g\|]).\end{aligned}$$

By the boundary condition $h'''(0)-k\lambda h'(0)=0$, we see that

$$\begin{aligned}kx\phi'''(0) &= kx\tilde{\phi}'''(0)=kxh'(0)\\ &= k\sqrt{\lambda i}[A_1-A_2+iA_3-iA_4]x\\ &= i\sqrt{\lambda i}[A_1-A_2-iA_3+iA_4]x.\end{aligned} \tag{71}$$

A simple calculation indicates that

$$\begin{aligned}\lambda\phi_p(x) &= -f(x)-\frac{i}{4}(\lambda i)^{-1/2}\\ &\quad\times\int_0^1 e^{\sqrt{\lambda i}(x-\xi)}[if''(\xi)-g(\xi)]d\xi\\ &\quad+\frac{1}{4}(\lambda i)^{-1/2}\int_0^1 e^{-i\sqrt{\lambda i}(x-\xi)}\\ &\quad\times[if''(\xi)+g(\xi)]d\xi+\mathcal{O}(|\lambda|^{-1/2}[\|f''\|+\|g\|]).\end{aligned} \tag{72}$$

Because of the relations $\tilde{\phi}''(x)=h(x)$ and $\tilde{\phi}(0)=\tilde{\phi}'(0)=0$, we see that

$$\begin{aligned}\tilde{\phi}'(x) &= \int_0^x h(\tau)d\tau\\ &= (\lambda i)^{-1/2}\left[A_1e^{\sqrt{\lambda i}x}-A_2e^{-\sqrt{\lambda i}x}\right.\\ &\qquad\left.-iA_3e^{i\sqrt{\lambda i}x}+iA_4e^{-i\sqrt{\lambda i}x}\right]\\ &\quad-(\lambda i)^{-1/2}[A_1-A_2-iA_3+iA_4]\\ \tilde{\phi}(x) &= \int_0^x \tilde{\phi}'(\tau)d\tau\\ &= (\lambda i)^{-1}\left[A_1e^{\sqrt{\lambda i}x}+A_2e^{-\sqrt{\lambda i}x}\right.\\ &\qquad\left.-A_3e^{i\sqrt{\lambda i}x}-A_4e^{-i\sqrt{\lambda i}x}\right]\\ &\quad-(\lambda i)^{-1}[A_1+A_2-A_3-A_4]\\ &\quad-(\lambda i)^{-1/2}[A_1-A_2-iA_3+iA_4]x\\ &= (\lambda i)^{-1}\left[A_1e^{\sqrt{\lambda i}x}+A_2e^{-\sqrt{\lambda i}x}\right.\\ &\qquad\left.-A_3e^{i\sqrt{\lambda i}x}-A_4e^{-i\sqrt{\lambda i}x}\right]\\ &\quad-(\lambda i)^{-1/2}[A_1-A_2-iA_3+iA_4]x\\ &\quad+\mathcal{O}(|\lambda|^{-3/2}[\|f''\|+\|g\|]).\end{aligned}$$

Therefore

$$\begin{aligned}\lambda\tilde{\phi}(x) &= -iA_1e^{\sqrt{\lambda i}x}+iA_4e^{-i\sqrt{\lambda i}x}\\ &\quad+i\sqrt{\lambda i}[A_1-A_2-iA_3+iA_4]x\\ &\quad+\mathcal{O}(|\lambda|^{-1/2}[\|f''\|+\|g\|])\end{aligned}$$

$$\begin{aligned}&= \frac{i}{4}(\lambda i)^{-1/2}\int_0^1 e^{\sqrt{\lambda i}(x-\xi)}[if''(\xi)-g(\xi)]d\xi \\ &\quad - \frac{1}{4}(\lambda i)^{-1/2}\int_0^1 e^{-i\sqrt{\lambda i}(x-\xi)} \\ &\quad \times [if''(\xi)+g(\xi)]d\xi + \mathcal{O}(|\lambda|^{-1/2}[\|f''\|+\|g\|]) \\ &\quad + i\sqrt{\lambda i}[A_1 - A_2 - iA_3 + iA_4]x \\ &= -\lambda\phi_p(x) - f(x) + k\phi'''(0)x \\ &\quad + \mathcal{O}(|\lambda|^{-1/2}[\|f''\|+\|g\|])\end{aligned}$$

from which we conclude that

$$\begin{aligned}\psi(x) &= \lambda\phi(x) - k\phi'''(0)x + f(x) \\ &= \lambda\tilde{\phi}(x) + \lambda\phi_p(x) - k\phi'''(0)x + f(x) \\ &= \mathcal{O}(|\lambda|^{-1/2}[\|f''\|+\|g\|]). \qquad (73)\end{aligned}$$

Combining (70) and (73), we have the desired estimate (59).

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