Two-Dimensional Affine Generalized Fractional Fourier Transform

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Abstract—As the one-dimensional (1-D) Fourier transform can be extended into the 1-D fractional Fourier transform (FRFT), we can also generalize the two-dimensional (2-D) Fourier transform. Sahin et al. have generalized the 2-D Fourier transform into the 2-D separable FRFT (which replaces each variable 1-D Fourier transform by the 1-D FRFT, respectively) and 2-D separable canonical transform (further replaces FRFT by canonical transform). Sahin et al., in another paper, have also generalized it into the 2-D unseparable FRFT with four parameters. In this paper, we will introduce the 2-D affine generalized fractional Fourier transform (AGFFT). It has even further extended the 2-D transforms described above. It is unseparable, and has, in total, ten degrees of freedom. We will show that the 2-D AGFFT has many wonderful properties, such as the relations with the Wigner distribution, shifting-modulation operation, and the differentiation-multiplication operation. Although the 2-D AGFFT form seems very complex, in fact, the complexity of the implementation will not be more than the implementation of the 2-D separable FRFT. Besides, we will also show that the 2-D AGFFT extends many of the applications for the 1-D FRFT, such as the filter design, optical system analysis, image processing, and pattern recognition and will be a very useful tool for 2-D signal processing.

Index Terms—Canonical transform, Fourier transform, fractional Fourier transform, two-dimensional fractional Fourier transform.

I. INTRODUCTION

The fractional Fourier transform (FRFT) [3], [4], which is the generalization of the 1-D Fourier transform, is defined as

$$F_\alpha(f(t)) = O_\alpha^2(f(t)) = \sqrt{\frac{1}{2\pi}} \cdot e^{j\frac{\alpha}{2} \cdot \frac{t}{\alpha} \cdot \cot \alpha \cdot \frac{\alpha}{2}} \cdot \int_{-\infty}^{\infty} e^{j\frac{\alpha}{2} \cdot \frac{t}{\alpha} \cdot \cot \alpha \cdot \frac{\alpha}{2}} \cdot f(t) \cdot dt. \quad (1)$$

It has the following additivity property:

$$O_{\alpha_1}^2 O_{\alpha_2}^2 = O_{\alpha_1 + \alpha_2}^2. \quad (2)$$

It has been used in many applications such as optical system analysis, filter design, solving differential equations, phase retrieval, and pattern recognition, etc.

In fact, the FRFT is the special case of the canonical transform [5] (which is also called the special affine Fourier transform (SAFT) [6]). The canonical transform is defined as

$$F_{(a,b,c,d)}(f(t)) = O_{(a,b,c,d)}^2(f(t)) = \sqrt{\frac{1}{2\pi b}} \cdot e^{j\frac{\alpha}{2} \cdot \frac{t}{\alpha} \cdot \cot \alpha \cdot \frac{\alpha}{2}} \cdot \int_{-\infty}^{\infty} e^{j\frac{\alpha}{2} \cdot \frac{t}{\alpha} \cdot \cot \alpha \cdot \frac{\alpha}{2}} \cdot f(t) \cdot dt \quad (3)$$

and the constraint that $ad - bc = 1$ must be satisfied. The FRFT is just the special case of SAFT with $\{a,b,c,d\} = \{\cos \alpha, \sin \alpha, -\sin \alpha, \cos \alpha\}$

$$F_\alpha(u) = \sqrt{e^{j\alpha}} \cdot O_{(\cos \alpha, \sin \alpha, -\sin \alpha, \cos \alpha)}(f(t)). \quad (5)$$

The canonical transform also has the following the additivity property:

$$O_{(a_1,b_1,c_1,d_1)}^2 O_{(a_2,b_2,c_2,d_2)}^2 = O_{(a_1+a_2,b_1+b_2,c_1+c_2,d_1+d_2)}^2. \quad (6)$$

where

$$\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \cdot \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}. \quad (7)$$

The canonical transform has extended the utilities of FRFT in some applications and is a useful tool for the optical system analysis.

The FRFT and SAFT (canonical transform) defined above in (1) and (3) are all one-dimensional (1-D) transforms. In [1], they have generalized them from 1-D into the 2-D cases. The 2-D
canonical transform they introduce is equivalent to the following equation:

\[
O_F^{(a,b,c,d)}(g(x,y)) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K_{(a,b,c,d)}(f,x) \cdot K_{(a_1,b_1,c_1,d_1)}(h,y) \cdot g(x,y) \, dx \, dy
\]  

where

\[
K_{(a,b,c,d)}(f,x) = \sqrt{\frac{j}{2\pi}} \cdot e^{(j/2)\cdot(b \cdot f \cdot x)} \cdot e^{-(j/2)\cdot(a \cdot f \cdot x^2)} \quad \text{when} \quad b \neq 0
\]

\[
K_{(a,b,c,d)}(f,x) = \sqrt{a} \cdot e^{(j/2)\cdot\alpha \cdot f \cdot x} \cdot \delta(x - d \cdot f) \quad \text{when} \quad b = 0
\]

and \(K_{(a_1,b_1,c_1,d_1)}(h,y)\) is of the same form as \(K_{(a,b,c,d)}(f,x)\), and \(ad - bc = 1\). That is, the 2-D canonical transform defined as (8) can be viewed as the combination of two independent 1-D canonical transforms. The 2-D FRFT they introduce is the special case of the 2-D canonical transform defined above with \(a_1, b_1, c_1, d_1\) = \{cos \alpha, sin \alpha, -sin \alpha, cos \alpha\} and \(a_2, b_2, c_2, d_2\) = \{cos \beta, sin \beta, -sin \beta, cos \beta\}. Although the 2-D canonical transform introduced by [1] has generalized the 2-D Fourier transform, it is not general enough because it treats two variables independently. In this paper, we will call the 2-D fractional Fourier/canonical transforms introduced by [1] the 2-D separable fractional Fourier/canonical transform.

Recently, in [2], the 2-D unseparable FRFT is introduced. This transform is the same as the 2-D separable FRFT for

\[
f \left( \frac{\cos \theta_1 x + \sin \theta_1 y}{\cos \theta_1 - \theta_2}, \frac{-\sin \theta_2 x + \cos \theta_2 y}{\cos \theta_1 - \theta_2} \right)
\]

and there are, in total, four parameters (\(\theta_1, \theta_2, \) and the order of the FRFT for each dimension). It treats the two variables unseparably and generalizes the 2-D separable FRFT. In fact, the 2-D unseparable FRFT [2] can be further generalized. In this paper, we will introduce a new type of generalized 2-D FRFT, which will be much more general than the transforms introduced in [1] and [2].

In [7], an \(N\)-D operator (the operation for the \(N\) dimensional functions) has been introduced and defined as

\[
O^{(A,B,C,D)}(f(x)) = (\det(B))^{-1/2} \cdot \int \exp \left( -B^{-1} \cdot \mathbf{x}^T \cdot B^{-1} \right) \cdot \exp \left( -B^{-1} \cdot \mathbf{x}^T \cdot A \cdot \mathbf{x} \right) \cdot f(\mathbf{x}) \cdot d\mathbf{x}
\]

where

\[
\mathbf{x} = (x_1, x_2, \ldots, x_N) \quad \mathbf{w} = (w_1, w_2, \ldots, w_N).
\]

In (12), \(A, B, C, D\) are all \(N \times N\) matrices and satisfy the following constraints:

\[
A^H C = C^H A, \quad B^H D = D^H B, \quad A^H D - C^H B = I
\]

or equivalently

\[
AB^H = BA^H, \quad CD^H = DC^H, \quad AD^H - BC^H = I
\]

We use \(H\) to denote the Hermitian operation (conjugation and transpose). The operators defined in (12) can be represented by the following \(2N \times 2N\) matrix (in [7]), it is called the metaplectic representation:

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\]

and the operation defined in (12) has the additivity as in

\[
O^{(A', B', C', D')} (O^{(A, B, C, D)} (g(x,y))) = O^{(P, Q, R, S)} (g(x,y))
\]

where

\[
\begin{bmatrix}
P & Q \\
R & S
\end{bmatrix} = \begin{bmatrix}
A & B \\
C & D
\end{bmatrix} \cdot \begin{bmatrix}
A' & B' \\
C' & D'
\end{bmatrix}.
\]

In this paper, we will discuss a special case of (12) with \(N = 2\) (in two dimensions). We call it the 2-D affine generalized fractional Fourier/canonical transform (2-D AGFFT). It is unseparable, totally has ten degrees of freedom, and is more general than the transforms defined in [1] and [2]. We will discuss it in detail, especially for its properties and implementation. We will also show that AGFFT can do many things that cannot be done for the 2-D separable fractional Fourier/canonical transform introduced by [1] and the 2-D unseparable FRFT introduced by [2]. In Section II, we will give the definition of the 2-D affine generalized FFT (AGFFT), some special cases of it, and the 2-D affine generalized fractional convolution and correlation. Then, we will discuss the properties of AGFFT in Section III. In Section IV, we will discuss some efficient ways to calculate and implement this 2-D transform and some simplified form of 2-D AGFFT. In Section V, we will discuss some applications of AGFFT, such as the 2-D filter design, and optical system analysis. Finally, in Section VI, we make some conclusions.

II. 2-D AFFINE GENERALIZED FFT

A. Definition of 2-D AGFFT

The 2-D affine generalized fractional Fourier transform (2-D AGFFT) we define here is the special case of (12) with dimension 2

\[
AGFFT: O^{(A,B,C,D)}_F (g(x,y)) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K_{(A,B,C,D)}(f,h,x,y) \cdot g(x,y) \, dx \, dy
\]

where

\[
A = \begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix}, \quad B = \begin{bmatrix}
b_11 & b_12 \\
b_21 & b_22
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
c_{11} & c_{12} \\
c_{21} & c_{22}
\end{bmatrix}, \quad D = \begin{bmatrix}
d_11 & d_12 \\
d_21 & d_22
\end{bmatrix}
\]

represents the 16 parameters of 2-D AGFFT (here, we restrict all the parameters to be real), and the kernel is

\[
K_{(A,B,C,D)}(f,h,x,y) = \frac{1}{2\pi \sqrt{-\det(B)}} \cdot e^{(j/2)\cdot\det(B)\cdot(b_11 f^2 + b_12 f h + b_22 h^2)} \cdot e^{(j/2)\cdot\det(B)\cdot(c_{11} x^2 + c_{12} x y + c_{22} y^2)}
\]

\[
\cdot e^{(j/2)\cdot\det(B)\cdot(d_{11} x f + d_{12} x y + d_{22} y f)}
\]
where
\[ k_1 = d_{11} b_{22} - d_{21} b_{21} \]
\[ k_2 = 2 \left( -d_{11} b_{12} + d_{12} b_{11} \right) = 2 \left( d_{21} b_{22} - d_{22} b_{21} \right) \]
\[ k_3 = -d_{21} b_{12} + d_{22} b_{11} \]
\[ p_1 = a_{11} b_{22} - a_{21} b_{21} \]
\[ p_2 = 2 \left( a_{11} b_{12} - a_{22} b_{12} \right) = 2 \left( -a_{11} b_{21} + a_{21} b_{11} \right) \]
\[ p_3 = -a_{22} b_{21} + a_{21} b_{21} \].

The constraints of (13) will become the following six constraints:
\[ a_{11} c_{22} + a_{21} c_{21} = a_{12} c_{11} + a_{22} c_{21} \]
\[ b_{11} d_{22} + b_{21} d_{22} = b_{12} d_{11} + b_{22} d_{21} \]
\[ a_{11} d_{11} + a_{21} d_{21} = \left( a_{12} d_{11} + a_{22} d_{21} \right) = 1 \]
\[ a_{12} d_{12} + a_{22} d_{22} = \left( a_{11} d_{12} + a_{21} d_{22} \right) = 1 \]
\[ a_{11} d_{12} + a_{22} d_{21} = a_{11} d_{12} + a_{22} d_{21} \]
\[ a_{12} d_{11} + a_{21} d_{22} = a_{12} d_{11} + a_{21} d_{22} \].

(22)

or equivalently, from (14)
\[ a_{11} b_{21} + a_{12} b_{22} = a_{21} b_{11} + a_{22} b_{12} \]
\[ c_{11} d_{21} + c_{12} d_{22} = c_{11} d_{21} + c_{22} d_{22} \]
\[ a_{11} d_{11} + a_{12} d_{12} = \left( a_{11} d_{11} + a_{12} d_{12} \right) = 1 \]
\[ a_{21} d_{21} + a_{22} d_{22} = \left( a_{21} d_{21} + a_{22} d_{22} \right) = 1 \]
\[ a_{11} d_{12} + a_{22} d_{21} = a_{11} d_{21} + a_{22} d_{22} \]
\[ a_{21} d_{11} + a_{22} d_{22} = a_{21} d_{11} + a_{22} d_{22} \].

(23)

Specially, we find that when \( A \) or \( D \) is an identical matrix \( I \), then \( b_{12} = b_{21} \), and \( c_{12} = c_{21} \). When \( B \) or \( C \) is \( I \), then \( a_{12} = a_{21} \), and \( d_{12} = d_{21} \). Because there are 16 parameters and six constraints, the free dimension of the 2-D AGFFT is 10. In contrast, the free dimension for the 1-D canonical transform is 3, and for the 2-D separable canonical transform defined as (8), it is 6 (eight parameters with two constraints).

Since the 2-D AGFFT has too many parameters, in this paper, we will usually use the matrix or vector notations instead of the explicit notations. We will usually use
\[ \vec{x} = (x, y), \quad \vec{w} = (f, h) \]
(24)
to denote the variables in time and frequency domains and use \( A, B, C, \) and \( D \) defined as (19) to denote the 16 parameters. Besides, in this paper, we will use \( G_{(A, B, C, D)}(f, h) \) to denote the result of the 2-D AGFFT of \( g(x, y) \)
\[ G_{(A, B, C, D)}(\vec{w}) = G_{(A, B, C, D)}(f, h) = \frac{1}{|\det(D)|} \cdot \exp \left( j \cdot \vec{w} \cdot \vec{D} T \cdot \vec{w} / 2 \right) \cdot \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left( j \cdot \vec{p} \cdot \vec{D} T \cdot \vec{w} / 2 \right) \cdot g(\vec{x}) \cdot d\vec{x} \cdot d\vec{y} \cdot d\vec{p} \cdot d\vec{q} \]
(25)
The additivity property for the 2-D AGFFT is also the same as (16) and (17), and the reversible property of the 2-D AGFFT is
\[ \left( G_{(A, B, C, D)} \right)^{-1} = G_{(D^{T}, -B^{T}, -C^{T}, A^{T})}. \]

(26)

We note that when \( a_{12} = a_{21} = b_{21} = b_{21} = c_{12} = c_{21} = d_{21} = d_{21} = 0 \), the 2-D AGFFT becomes the 2-D separable canonical transform defined as (8). More specially, when \( A = D = 0 \) and \( B = -C = I \), the 2-D AGFFT becomes the 2-D forward Fourier transform, and when
\[ A = \begin{bmatrix} \cos \alpha \cdot \cos \beta & -\cos \alpha \cdot \sin \beta \\ \cos \beta \cdot \sin \beta & \cos \alpha \cdot \cos \beta \end{bmatrix} \]
\[ B = \begin{bmatrix} \sin \alpha \cdot \cos \beta & -\sin \alpha \cdot \sin \beta \\ \sin \beta \cdot \sin \beta & \sin \alpha \cdot \cos \beta \end{bmatrix} \]
\[ C = \begin{bmatrix} -\sin \alpha \cdot \cos \beta & \sin \alpha \cdot \sin \beta \\ -\sin \beta \cdot \sin \beta & -\sin \beta \cdot \cos \beta \end{bmatrix} \]
\[ D = \begin{bmatrix} \cos \alpha \cdot \cos \beta & -\cos \alpha \cdot \sin \beta \\ \cos \beta \cdot \sin \beta & \cos \alpha \cdot \cos \beta \end{bmatrix} \]
(27)
the 2-D AGFFT becomes the 2-D unseparable FRFT introduced by Sahin et al. [2].

We show the relations between the AGFFT and its special cases in Fig. 1. The number in the ( ) shows the degree of freedom of the corresponding transform.

B. Definition of 2-D AGFFT When \( \det(B) = 0 \)

We note, in (18), that if \( \det(B) = 0 \), then we cannot apply this equation directly. In these cases, we must convert the 2-D AGFFT defined as (18)–(21) into another form. We discuss each of these cases as follows:

(1) \( B = 0 \).

We note that because \( A^T D - C^T B = I \), when \( B = 0 \), \( A^T D = I \). That is, the equality relation \( A = (D^T)^{-1} \) must be satisfied in the case that \( B = 0 \), and since
\[ \begin{bmatrix} \vec{D}^{-1} & 0 \\ 0 & \vec{D} \end{bmatrix} = \begin{bmatrix} 0 & -\vec{D}^{-1} \\ \vec{D} & -\vec{C} \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -\vec{I} & 0 \end{bmatrix} \]
(28)
in the case that \( B = 0 \), the 2-D AGFFT can be defined as
\[ G_{(D^T, 0, C, D)}(\vec{w}) = \frac{1}{|\det(D)|} \cdot \exp \left( j \cdot \vec{w} \cdot \vec{D} T \cdot \vec{w} / 2 \right) \cdot \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left( j \cdot \vec{p} \cdot \vec{D} T \cdot \vec{w} / 2 \right) \cdot g(\vec{x}) \cdot d\vec{x} \cdot d\vec{y} \cdot d\vec{p} \cdot d\vec{q} \]
(29)
where \( \vec{p} = (p_r, q) \), and \( \vec{x}, \vec{w} \) are defined as (24). After some calculating, we obtain (30), shown at the bottom of the page. This is the formula for 2-D AGFFT when \( B = 0 \). We find that
\[ G_{(D^T, 0, C, D)}(f, h) = \frac{1}{|\det(D)|} \cdot \exp \left( j \cdot \vec{w} \cdot \vec{D} T / 2 \right) \cdot \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left( j \cdot \vec{p} \cdot \vec{D} T \cdot \vec{w} / 2 \right) \cdot g(d_{11} f + d_{21} h, d_{12} f + d_{22} h). \]
(30)
when \( C = 0 \), it is just the geometric twisting operation, and when \( D = I \), it is just the chirp multiplication operation.

\[(2')\] \[
\{A, B, C, D\} = \{I, B, 0, I\} \quad \text{det}(B) = 0, \quad B \neq 0.
\]

Before discussing the other cases, we first discuss the formula of the 2-D AGFFT with the parameters \( \{A, B, C, D\} \) = \( \{I, B, 0, I\} \). In the case that \( \text{det}(B) \neq 0 \), we can apply (18)–(21) directly, but in the case that \( \text{det}(B) = 0 \), we must use other ways to find the formula. Because

\[
\begin{bmatrix}
I & B \\
0 & I
\end{bmatrix} = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \cdot \begin{bmatrix} I & 0 \\ -B & I \end{bmatrix} \cdot \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}
\]

(31)

therefore

\[
O_F^{(I,B,0,I)}(g(x,y)) = \text{IFT} \left( O_F^{(0,0,-B,0)}(\text{FT}(g(x,y))) \right)
\]

\[
= \text{IFT} \left( e^{-j/2(0,1,1,0) \cdot (\cdot)^2} \cdot (x,y) \right).
\]

(32)

We have used (30) and the multiplication theory of the original Fourier transform. In this paper, the definition of Fourier transform, inverse Fourier transform, and the convolution we used are as follows:

\[
\text{FT}(g(x,y)) = O_F^{(0,0,-I,0)}(g(x,y))
\]

\[
= \frac{1}{\sqrt{-1} \cdot 2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-j (x+h) y} g(x,y) \cdot dx \cdot dy
\]

\[
\text{IFT}(g(x,y)) = O_F^{(0,0,-I,0)}(g(x,y))
\]

\[
= \frac{1}{\sqrt{-1} \cdot 2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{j (x+h) y} g(x,y) \cdot dx \cdot dy
\]

(33)

(34)

\[
(2') \quad \text{det}(B) = 0, \quad B \neq 0, \quad \text{det}(A) \neq 0.
\]

We will generalize (2') and discuss all the cases where \( \text{det}(B) = 0, B \neq 0 \), and \( \text{det}(A) \neq 0 \). In this case, because

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix} = \begin{bmatrix} I & 0 \\
0 & D - CA^{-1} B
\end{bmatrix}
\]

\[
= \begin{bmatrix} I & 0 \\
0 & D - CA^{-1} B
\end{bmatrix}
\]

\[
\cdot \begin{bmatrix} A & 0 \\
0 & A^{-1}
\end{bmatrix}
\]

(42)
and from (14) (the Hermitian operations can be replaced with the transpose operations in this equation since all the parameters are real)

\[ \begin{align*}
\text{BA}^T &= \text{AB}^T \\
\text{DA}^T - \text{CA}^{-1}\text{BA}^T &= \text{DA}^T - \text{CA}^{-1}\text{AB}^T \\
&= \text{DA}^T - \text{CB}^T \\
&= (\text{AD}^T - \text{BC}^T)^T = I
\end{align*} \]

(43)

we obtain

\[ \begin{bmatrix}
\text{A} & \text{B} \\
\text{C} & \text{D}
\end{bmatrix} = \begin{bmatrix}
\text{I} & 0 \\
\text{CA}^{-1} & \text{I}
\end{bmatrix} \cdot \begin{bmatrix}
\text{I} & \text{AB}^T \\
0 & \text{I}
\end{bmatrix} \cdot \begin{bmatrix}
\text{A} & 0 \\
0 & \text{AT}^{-1}
\end{bmatrix}
\]

\((\det(A) \neq 0).
\]

(44)

Then, together with (30) and (32), we obtain

\[ G_{(A,B,C,D)}(f,h) = \frac{1}{2\pi \sqrt{-\det(A)}} \exp \left( j \left( \hat{f} \cdot \text{CA}^{-1} \cdot \hat{f}^T / 2 \right) \right) \]

\[ \cdot \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} r(f - x, h - y) g(\hat{x} \cdot \text{AT}^{-1}) \cdot dx \cdot dy \]

(45)

where

\[ r(x,y) = \text{IFT} \left( \exp \left( -j \left( \hat{f} \cdot \text{AB}^T \cdot \hat{f}^T / 2 \right) \right) \right) \]

(46)

and the explicit formula for \( r(x,y) \) can be calculated from (36), (38), or (40).

(3)

\[ \begin{align*}
\det(B) &= 0, \quad B \neq 0, \\
\det(A) &= 0
\end{align*} \]

\( \det(A) \neq 0, \quad \det(D) \neq 0. \)

Since

\[ \begin{bmatrix}
\text{A} & \text{B} \\
\text{C} & \text{D}
\end{bmatrix} = \begin{bmatrix}
\text{AD}^T & \text{BD}^{-1} \\
\text{CD}^T & \text{I}
\end{bmatrix} \cdot \begin{bmatrix}
\text{D}^{-1} & 0 \\
0 & \text{D}
\end{bmatrix} \\
&= \begin{bmatrix}
\text{I} & \text{BD}^{-1} \\
0 & \text{I}
\end{bmatrix} \cdot \begin{bmatrix}
\text{AD}^T - \text{BD}^{-1} \text{CD}^T & 0 \\
0 & \text{CD}^T \text{I}
\end{bmatrix} \cdot \begin{bmatrix}
\text{D}^{-1} & 0 \\
0 & \text{D}
\end{bmatrix}
\]

from (13) and (14)

\[ \text{BD}^{-1} \text{CD}^T = \text{BD}^{-1} \text{DC}^T = I \\
\text{AD}^T - \text{BD}^{-1} \text{CD}^T = \text{AD}^T - \text{BC}^T = I
\]

(47)

so that

\[ \begin{bmatrix}
\text{A} & \text{B} \\
\text{C} & \text{D}
\end{bmatrix} = \begin{bmatrix}
\text{I} & \text{BD}^{-1} \\
0 & \text{I}
\end{bmatrix} \cdot \begin{bmatrix}
\text{I} & 0 \\
\text{CD}^T & \text{I}
\end{bmatrix} \cdot \begin{bmatrix}
\text{D}^{-1} & 0 \\
0 & \text{D}
\end{bmatrix}
\]

(48)

Thus, in this case

\[ G_{(A,B,C,D)}(f,h) = \frac{1}{2\pi \sqrt{-\det(D)}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} r(f - x, h - y) \exp \left( j \left( \hat{x} \cdot \text{CD}^T \cdot \hat{x}^T / 2 \right) \right) g(\hat{x}) \cdot dx \cdot dy \]

(49)

where

\[ r(x,y) = \text{IFT} \left( \exp \left( -j \left( \hat{f} \cdot \text{BD}^{-1} \cdot \hat{f}^T / 2 \right) \right) \right). \]

(50)

The explicit formula of \( r(x,y) \) can be calculated from (36), (38), or (40).

(4)

\[ \begin{align*}
\det(B) &= 0, \quad B \neq 0, \\
\det(A) &= 0
\end{align*} \]

\( \det(A) \neq 0, \quad \det(D) = 0, \quad D \neq 0. \)

Since

\[ O_{F_{A,B,C,D}}^{(A,B,C,D)}(f,h) = \text{IFT} \left( O_{F_{C,D,-A,-B}}^{(C,D,-A,-B)} \right) \]

(51)

from (45), we obtain, in this case

\[ G_{(A,B,C,D)}(f,h) = \frac{1}{4\pi^2 \sqrt{-\det(C)}} \cdot \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left( j \left( \hat{f} \cdot \text{CD}^T \cdot \hat{f}^T / 2 \right) \right) \]

\[ \cdot \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} r(w - x, v - y) \cdot g(\hat{w}) \cdot dw \cdot dv \]

(52)

where

\[ r(x,y) = \text{IFT} \left( \exp \left( -j \left( \hat{f} \cdot \text{CD}^T \cdot \hat{f}^T / 2 \right) \right) \right) \]

\( \hat{w} = (u, v). \)

(53)

(5)

\[ \begin{align*}
\det(B) &= 0, \quad B \neq 0, \\
\det(A) &= \det(D) = \det(C) = 0
\end{align*} \]

\( \det(A) \neq 0, D \neq 0 \) must be satisfied.

This case seldom occurs. One example is the 1-D Fourier transform in the \( y \) axis \((a_{11} = b_{22} = -c_{22} = d_{11} = 1, \) others \( = 0)\). We can show in this case that all the 2-D AGFFTs can be decomposed as

\[ O_{F_{A,B,C,D}}^{(A^{-1},0,0,\Lambda)}(g(x,y)) = O_{F_{A',B',C',D'}}^{(A',0,0,\Lambda)} \left( O_{F_{A,B',C',D'}}^{(A',0,0,\Lambda)}(g(x,y)) \right) \]

(54)

where

\[ \begin{align*}
A' &= \begin{bmatrix}
1 & a \\
0 & 0
\end{bmatrix}, \\
B' &= \begin{bmatrix}
0 & 0 \\
-\alpha & 1
\end{bmatrix} \\
C' &= \begin{bmatrix}
0 & 0 \\
0 & -1
\end{bmatrix}, \\
D' &= \begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix}
\]

\[ O_{F_{A,B',C',D'}}^{(A',B',C',D')}(g(x,y)) = \text{FT}_y(g(x - a_y, y)) \]

(56)

and \( \text{FT}_y \) means the 1-D Fourier transform along the \( y \) axis. Thus, when \( \det(A) = \det(B) = \det(C) = \det(D) = 0 \) and \( A \neq 0, B \neq 0, D \neq 0 \), the 2-D AGFFT can be decomposed as the combination of a 1-D Fourier transform for the variable \( y \) on \( g(x - a_y, y) \), which is a multiplication operation with the quadratic phase function \([i.e., \exp(j(s_1 x^2 + s_2 x y + s_3 y^2))])\) and a geometric twisting operation.
C. Basic Operations

We will discuss the components of 2-D AGFFT. We find that there are at most ten independent basic operations for the 2-D AGFFT, and they correspond to the ten free dimensions of 2-D AGFFT. We list one example of the independent basic operations set in the following and denote them by \(O_{F(1)}, O_{F(2)}, \ldots, O_{F(10)}\).

1) Chirp multiplication for the \(x\)-axis \( (a_{11} = a_{22} = d_{11} = d_{22} = 1, \; \tau_x = 0) \):
\[
O_{F(1)}(g(x,y)) = e^{j\tau_xx^2/2} \cdot g(x,y).
\] (57)

2) Chirp convolution for the \(x\)-axis \( (a_{11} = a_{22} = d_{11} = d_{22} = 1, \; \eta_x = 0) \):
\[
O_{F(2)}^{\eta_x}(g(x,y)) = e^{j\eta_x y^2/2} \cdot g(x,y).
\] (58)

3) Scaling along the \(x\)-axis \( (a_{11} = 1/d_{11} = \rho_x, \; a_{22} = d_{22} = 1, \; \text{others} = 0) \):
\[
O_{F(3)}^{\rho_x}(g(x,y)) = g(x/\rho_x, y)/\sqrt{\rho_x}.
\] (59)

4) Chirp multiplication for the \(y\)-axis \( (a_{11} = a_{22} = d_{11} = d_{22} = 1, \; \tau_y = 0) \):
\[
O_{F(4)}(g(x,y)) = e^{j\tau_y y^2} \cdot g(x,y).
\] (60)

5) Chirp convolution for the \(y\)-axis \( (a_{11} = a_{22} = d_{11} = d_{22} = 1, \; \eta_y = 0) \):
\[
O_{F(5)}^{\eta_y}(g(x,y)) = g(x,y)/\sqrt{\eta_y}.
\] (61)

6) Scaling along the \(y\)-axis \( (a_{22} = 1/d_{22} = \rho_y, \; a_{11} = d_{11} = 1, \; \text{others} = 0) \):
\[
O_{F(6)}^{\rho_y}(g(x,y)) = g(x, y/\rho_y)/\sqrt{\rho_y}.
\] (62)

7) Multiplication of the \(x\)-axis \( (a_{11} = a_{22} = d_{11} = d_{22} = 1, \; \tau = 0) \):
\[
O_{F(7)}(g(x,y)) = e^{j\tau xy} \cdot g(x,y).
\] (63)

8) Convolution of the \(x\)-axis \( (a_{11} = a_{22} = d_{11} = d_{22} = 1, \; \eta = 0) \):
\[
O_{F(8)}^{\eta}(g(x,y)) = g(x, y)/\sqrt{\eta}.
\] (64)

9) Shearing along the \(x\)-axis \( (a_{11} = a_{22} = d_{11} = d_{22} = 1, \; -a_{21} = d_{21} = p, \; \text{others} = 0) \):
\[
O_{F(9)}^{p}(g(x,y)) = g(x + py, y).
\] (65)

10) Shearing along the \(y\)-axis \( (a_{11} = a_{22} = d_{11} = d_{22} = 1, \; -a_{21} = d_{21} = q, \; \text{others} = 0) \):
\[
O_{F(10)}^{q}(g(x,y)) = g(x, qx + y).
\] (66)

Each of the above basic operations has only one free parameter and, hence, has the free dimension of 1. They all have the additive property relates to their parameter, that is
\[
O_{F(11)}^{p+q}(g(x,y)) = O_{F(11)}^{p}(g(x,y)) \cdot O_{F(11)}^{q}(g(x,y)).
\] (67)

Some of these basic operations are exchangeable. That is
\[
O_{F(12)}^{p}(g(x,y)) = O_{F(12)}^{q}(g(x,y))
\] when \( p, q \in \{1, 4, 7\} \), when \( p, q \in \{2, 5, 8\} \), or when \( p, q \in \{3, 6\} \).

All the 2-D AGFFTs can be decomposed as the combination of the above ten basic operations (this is why the free dimension of the 2-D AGFFT is 10). For example, when \( \det(A) \neq 0 \) and \( a_{22} \neq 0 \), we can decompose the 2-D AGFFT as
\[
O_{F(A,B,C,D)}^{(A,B,C,D)} = O_{F(11)}^{(A,B,C,D)} \cdot O_{F(12)}^{(A,B,C,D)} \cdot O_{F(13)}^{(A,B,C,D)} \cdot O_{F(14)}^{(A,B,C,D)} \cdot O_{F(15)}^{(A,B,C,D)} \cdot O_{F(16)}^{(A,B,C,D)} \cdot O_{F(17)}^{(A,B,C,D)} \cdot O_{F(18)}^{(A,B,C,D)} \cdot O_{F(19)}^{(A,B,C,D)} \cdot O_{F(20)}^{(A,B,C,D)}.
\] (69)

For example
11) 1-D FRFT on the \(x\)-axis: \( a_{11} = d_{11} = \cos \alpha, b_{11} = -\alpha = \sin \alpha, a_{22} = d_{22} = 1, \; \text{others} = 0 \):
\[
O_{F(11)}^{(A)}(g(x,y)) = \frac{1}{\sqrt{2\pi}} \cdot e^{j\alpha/2 \cdot \alpha} \int_{-\infty}^{\infty} e^{-j\alpha \cdot \cos \theta \cdot x} \cdot e^{j\alpha \cdot \cos \theta \cdot y} \cdot dx.
\] (70)

12) 1-D FRFT on the \(y\)-axis: \( a_{22} = d_{22} = \cos \alpha, b_{22} = -\alpha = \sin \alpha, a_{11} = d_{11} = 1, \; \text{others} = 0 \):
\[
O_{F(12)}^{(A)}(g(x,y)) = \frac{1}{\sqrt{2\pi}} \cdot e^{j\alpha/2 \cdot \alpha} \int_{-\infty}^{\infty} e^{-j\alpha \cdot \cos \theta \cdot y} \cdot e^{j\alpha \cdot \cos \theta \cdot x} \cdot dy.
\] (71)

13) Clockwise rotation of the \(x\)-axis with angle \( \theta_1 \):
\[
O_{F(13)}^{(A)}(g(x,y)) = g(f + \tan \theta_1 \cdot h, \sec \theta_1 \cdot h).
\] (72)

14) Counterclockwise rotation of the \(x\)-axis with angle \( \theta_2 \):
\[
O_{F(14)}^{(A)}(g(x,y)) = g(\tan \theta_2 \cdot f, -\tan \theta_2 \cdot h + \theta_2 \cdot h).
\] (73)

or
\[
g(f, h) = O_{F(14)}^{(A)}(g(x,y)) = g(f + \tan \theta_2 \cdot h, \sec \theta_2 \cdot h).
\] (74)

or
\[
g(f, h) = O_{F(13)}^{(A)}(g(x,y)) = g(f, h + \tan \theta_1 \cdot h, \sec \theta_1 \cdot h).
\] (75)
TABLE I

<table>
<thead>
<tr>
<th>Basic ops.</th>
<th>(A)</th>
<th>(B)</th>
<th>(C)</th>
<th>(D)</th>
<th>(E)</th>
<th>(F)</th>
<th>(G)</th>
<th>2-D AGFFT</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>×</td>
<td>×</td>
<td>O</td>
<td>×</td>
<td>O</td>
<td>×</td>
<td>×</td>
<td>O</td>
</tr>
<tr>
<td>(2)</td>
<td>×</td>
<td>O</td>
<td>×</td>
<td>O</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>O</td>
</tr>
<tr>
<td>(3)</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>O</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>O</td>
</tr>
<tr>
<td>(4)</td>
<td>×</td>
<td>×</td>
<td>O</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>O</td>
</tr>
<tr>
<td>(5)</td>
<td>×</td>
<td>O</td>
<td>×</td>
<td>O</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>O</td>
</tr>
<tr>
<td>(6)</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>O</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>O</td>
</tr>
<tr>
<td>(7)</td>
<td>×</td>
<td>×</td>
<td>O</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>O</td>
</tr>
<tr>
<td>(8)</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>O</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>O</td>
</tr>
<tr>
<td>(9)</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>O</td>
<td>O</td>
</tr>
<tr>
<td>(10)</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>O</td>
<td>O</td>
</tr>
<tr>
<td>(11)</td>
<td>O</td>
<td>×</td>
<td>×</td>
<td>O</td>
<td>×</td>
<td>×</td>
<td>O</td>
<td>O</td>
</tr>
<tr>
<td>(12)</td>
<td>O</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>O</td>
<td>×</td>
<td>O</td>
<td>O</td>
</tr>
<tr>
<td>(13)</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>O</td>
<td>O</td>
<td>O</td>
</tr>
<tr>
<td>(14)</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>O</td>
<td>O</td>
<td>O</td>
</tr>
</tbody>
</table>

| Degree of freedom | 2 | 2 | 3 | 3 | 6 | 4 | 10 |

In fact, these four basic operations can all be decomposed as the combination of the former ten basic operations so that they will not increase the free dimension of the 2-D AGFFT:

\[
\begin{align*}
O_{F(11)}^{a} & = O_{F(2)}^{sc \alpha \cos \theta} O_{F(1)}^{\sin \alpha} O_{F(3)}^{sc \alpha \cos \theta} \\
O_{F(12)}^{a} & = O_{F(2)}^{sc \alpha \cos \theta} O_{F(4)}^{\sin \alpha} O_{F(3)}^{sc \alpha \cos \theta} \\
O_{F(13)}^{b} & = O_{F(6)}^{sc \alpha \cos \theta} \\
O_{F(14)}^{b} & = O_{F(3)}^{sc \alpha \cos \theta} O_{F(10)}^{\sin \alpha}.
\end{align*}
\]

From the basic operations, we can see the structure of the 2-D AGFFT and realize how the 2-D AGFFT generalizes the 2-D separable FRFT and other transforms listed in Fig. 1. We list Table I to show whether the basic operations described above exist for each of the special cases of the 2-D AGFFT. We use “O” and “×” to indicate whether the basic operations exist for each of the transforms and use A)–G) to indicate the following:

A) 2-D separable FRFT;
B) 2-D separable Fresnel transform;
C) multiplication of \( \exp \left( j(\eta_1 x^2 + \eta_2 y^2 + \eta_3 y^2) \right) \);
D) convolution with \( \exp \left( j(\eta_1 x^2 + \eta_2 y^2 + \eta_3 y^2) \right) \);
E) 2-D separable canonical transform;
F) Sahin’s 2-D unseparable FRFT;
G) Geometric twisting operation.

The degree of freedom for each transform in Table I can be calculated from the total number of “O’s” in the corresponding column, and in each case as below, the degree of freedom must be decreased by 1:

a) The first, second, and 11th items are all “O’s.”
b) The fourth, fifth, and 12th items are all “O’s.”
c) The sixth, ninth, and 13th items are all “O’s.”
d) The third, 10th, and 14th items are all “O’s.”

The basic operations 1–6 exist for the 2-D separable canonical transform, but the basic operations 7–10 do not exist for this transform. This is because for the former six basic operations, the \( x \)- and \( y \)-axes are independent, but for the latter four operations, the \( x \)- and \( y \)-axes are dependent. Because, for the remaining four basic operations, the seventh and eighth basic operations are the multiplication and convolution of \( \exp \left( j(\eta_1 x^2 + \eta_2 y^2) \right) \), the ninth and tenth basic operations exist for the geometric twisting operations; therefore, we can say that the 2-D AGFFT is the combination of

1) 2-D separable canonical transform E) [basic ops. (1)–(6)]
2) multiplication or convolution of \( \exp \left( j(\eta_1 x^2 + \eta_2 y^2) \right) \), C), D) [basic ops. (7), (8)]
3) geometric twisting operation G) [basic ops. (3), (6), (9), (10)].

In addition, from [8], we find that all the canonical transforms can be decomposed as the combination of chirp multiplication, Fourier transform, and scaling operation (one kind of geometric twisting operation). Thus, we can also view the 2-D AGFFT as the combination of the

1) 2-D Fourier transform (2-D FT);
2) geometric twisting operation G) [basic ops. (3), (6), (9), (10)];
3) multiplication of quadratic phase function \( [i.e., \exp \left( j(\eta_1 x^2 + \eta_2 y^2 + \eta_3 y^2) \right) \) C) [basic ops. (1), (4), (7)];
4) convolution with the quadratic phase function D) [basic ops. (2), (5), (8)].

Since the 2-D FT can be further decomposed as the combination of the chirp multiplication and chirp convolution and can be absorbed into the third and the fourth components, we can thus just say that the 2-D AGFFT is the combination of the latter 3 components described above.

D. 2-D Affine Generalized Fractional Convolution and Correlation

Analogous to the conventional 2-D convolution defined as (35), we can define the 2-D affine generalized fractional con-
TABLE II

<table>
<thead>
<tr>
<th>(1) Projection property</th>
<th>(G_{(A,B,C,D)}(x,y)) = (\sum_{f} \int_{u} W_{f} \left( \frac{D}{B} \cdot C \right) \cdot \left( \frac{D}{B} \cdot A \right) \cdot df\cdot dh ), (\vec{v} = [x, y, f, h] ).</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2) Recovery property</td>
<td>(G_{(A,B,C,D)}(x,y)) = (\sum_{f} \int_{u} W_{f} \left( \frac{D}{B} \cdot C \right) \cdot \left( \frac{D}{B} \cdot A \right) \cdot e^{i(f\cdot x + h\cdot y)} \cdot df\cdot dh ) / (G_{(A,B,C,D)}(0,0)).</td>
</tr>
</tbody>
</table>
| (3) Relations with generalized fractional convolution | If \(z(x,y)\) is the 2-D affine generalized fractional convolution of \(f(x,y)\) and \(g(x,y)\), then \(W_{f}(\vec{x}, \vec{w}) = \sum_{f} \int_{u} W_{f} \left( \frac{D}{B} \cdot C \right) \cdot \left( \frac{D}{B} \cdot A \right) \cdot \left( \vec{x} A + \vec{w} B^{T} D - \vec{p} B - \vec{x} A^{T} C - \vec{w} B^{T} C + \vec{p} A \right) \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \·

A. Relations with the 2-D Wigner Distribution Function (WDF)

The 2-D Wigner distribution function (2-D WDF) is the extension of 1-D Wigner distribution. Its formula is

\[
W_{g}(x,y,f,h) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-j\pi f \cdot x - j\pi h \cdot y} \cdot (x + \frac{\tau}{2}, y + \frac{\eta}{2}) \cdot g(x,y,f,h) \cdot d\tau \cdot d\eta
\]

The 2-D WDF has close relations with the 2-D AGFFT. If \(W_{G}(A,B,C,D)(x,y,f,h)\)

\[
W_{G}(A,B,C,D) \cdot \left( (x - \frac{\tau}{2}, y - \frac{\eta}{2}) \right) \cdot d\tau \cdot d\eta
\]

then we can prove

\[
W_{g}(x,y,f,h) = W_{G}(A,B,C,D) \cdot \left( a_{11}x + a_{12}y + b_{11}f + b_{12}h \cdot a_{21}x + a_{22}y + b_{21}f + b_{22}h \cdot c_{11}x + c_{12}y + d_{11}f + d_{12}h \cdot c_{21}x + c_{22}y + d_{21}f + d_{22}h \right)
\]

That is, if \(\vec{v} = [x, y, f, h]\), then

\[
W_{g}(\vec{v}) = W_{G}(A,B,C,D) \cdot \left( \vec{v} \cdot \left[ \begin{array}{c} A \\ B \\ C \\ D \end{array} \right] \right)
\]

Equation (85) can also be written as

\[
W_{G}(A,B,C,D) \cdot \left( \vec{v} \cdot \left[ \begin{array}{c} A^{T} \\ B^{T} \\ C^{T} \\ D^{T} \end{array} \right] \right)
\]

Except for (84)–(86), there are also some important relations between the 2-D AGFFT and 2-D WDF. We list them in Table II. The proofs of these relations are listed in the Appendix.

B. Relations with the Shifting-Modulation Operation and the Differentiation-Multiplication Operation

Except for the 2-D Wigner distribution function, the 2-D AGFFT also has close relations with the shifting-modulation
TABLE III
SHIFTING, MODULATION, DIFFERENTIATION, AND MULTIPLICATION PROPERTIES FOR 2-D AGFFT

<table>
<thead>
<tr>
<th>Property</th>
<th>Input</th>
<th>Transform results</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) Space shifting</td>
<td>$g(x \mp x_s)$, where $x_s = (x_s, y_s)$</td>
<td>$\exp[i(\varphi + 2\pi xc_1)]G_{\text{A,B,C,D}}(w \mp x_s A^T)$. where $\varphi = -x_s c_1 A^T x_s / 2$.</td>
</tr>
<tr>
<td>(2) Modulation</td>
<td>$\exp[i(\varphi + 2\pi wc_2)]G_{\text{A,B,C,D}}(w \mp y_c B^T)$, where $\varphi = -y_c B^T y_c / 2$.</td>
<td></td>
</tr>
<tr>
<td>(3) Multiplied by $x$</td>
<td>$x \cdot g(x,y)$</td>
<td>$j b_1 \frac{\partial}{\partial f} G_{\text{A,B,C,D}}(f,h) + j b_2 \frac{\partial}{\partial h} G_{\text{A,B,C,D}}(f,h) + d_1 f \cdot G_{\text{A,B,C,D}}(f,h) + d_2 h \cdot G_{\text{A,B,C,D}}(f,h)$</td>
</tr>
<tr>
<td>(4) Multiplied by $y$</td>
<td>$y \cdot g(x,y)$</td>
<td>$j b_1 \frac{\partial}{\partial f} G_{\text{A,B,C,D}}(f,h) + j b_2 \frac{\partial}{\partial h} G_{\text{A,B,C,D}}(f,h) + d_1 f \cdot G_{\text{A,B,C,D}}(f,h) + d_2 h \cdot G_{\text{A,B,C,D}}(f,h)$</td>
</tr>
<tr>
<td>(5) Differentiation for $x$-axis</td>
<td>$\frac{\partial}{\partial x} g(x,y)$</td>
<td>$a_{11} \frac{\partial}{\partial f} G_{\text{A,B,C,D}}(f,h) + a_{12} \frac{\partial}{\partial h} G_{\text{A,B,C,D}}(f,h) - f c_1 g_{\text{A,B,C,D}}(f,h) - f c_2 h \cdot G_{\text{A,B,C,D}}(f,h)$</td>
</tr>
<tr>
<td>(6) Differentiation for $y$-axis</td>
<td>$\frac{\partial}{\partial y} g(x,y)$</td>
<td>$a_{13} \frac{\partial}{\partial f} G_{\text{A,B,C,D}}(f,h) + a_{14} \frac{\partial}{\partial h} G_{\text{A,B,C,D}}(f,h) - h c_1 g_{\text{A,B,C,D}}(f,h) - h c_2 f \cdot G_{\text{A,B,C,D}}(f,h)$</td>
</tr>
</tbody>
</table>

From the space-shifting property, we find, after the 2-D AGFFT, that the space shifting will partially become the modulation operation (due to $c_1$ or $c_2$) and partially remain the geometric shifting operation (due to $b_1$ or $b_2$). Thus, unlike the 2-D Fourier transform, the 2-D AGFFT is not space invariant.

Combining the space shifting and modulation properties, we have

$$
O_F^{(A,B,C,D)} = e^{i\varphi} e^{i2\pi f_1 x_1} e^{i2\pi f_2 x_2} G_{\text{A,B,C,D}}(f_r, h_r + h_s) (87)
$$

where

$$
[r_1 \ r_2]
= [A \ B] \cdot 
[1 \ m_1]

[m_2 \ n_1]
$$

and $\varphi$ is some constant phase. Thus, the 2-D AGFFT has a closed relation with the shifting-modulation operations pair.

From the above discussion, we find that the 2-D AGFFT defined as (18)–(21) can be further generalized to include the space shift term and modulation term. That is

$$
G_{\text{A,B,C,D,M,N}}(f,h) = e^{i(\varphi f + \eta h)} G_{\text{A,B,C,D}}(f - m_f, h - m_h) (89)
$$

where $M = [m_f, m_h]^T$ represents the space shifting, and $N = [\eta_f, \eta_h]^T$ represents the frequency shifting. We will call this the 2-D AGFFT with space-shifting and frequency modulation (2-D TFAGFFT). Together with the ten free dimensions of the original 2-D AGFFT, there are totally 14 free dimensions for the 2-D TFAGFFT. The 2-D TFAGFFT can be represented as

$$
\begin{bmatrix}
A & B & C & D
\end{bmatrix} \cdot 
[\mathbf{x}^T]
= 
M
= 
N
= 
\begin{bmatrix}
[A] \cdot [B]
[1 \ h_1]
[h_2]
[h_3]
[h_4]
\end{bmatrix} (90)
$$

We can prove that the following additivity property will be satisfied for the 2-D TFAGFFT:

$$
O_F^{(A',B',C',D',M',N')} = O_F^{(A,B,C,D,M,N)} \cdot [A' \ B' \ C' \ D'] \cdot [M' \ N'] (91)
$$

where

$$
\begin{bmatrix}
\hat{A} \ \hat{B} \\
\hat{C} \ \hat{D}
\end{bmatrix} = 
[\mathbf{x}^T] \cdot 
\begin{bmatrix}
[A] & [B] \\
[C] & [D]
\end{bmatrix} \cdot 
\begin{bmatrix}
\mathbf{x}^T \\
\mathbf{w}^T
\end{bmatrix}
= 
\begin{bmatrix}
\hat{M} \\
\hat{N}
\end{bmatrix} (92)
$$

The additivity relation in (91) and (92) can also be expressed as

$$
\begin{bmatrix}
\hat{A} \ \hat{B} \\
\hat{C} \ \hat{D}
\end{bmatrix} \cdot [\mathbf{x}^T] \cdot [\mathbf{w}^T] = 
\begin{bmatrix}
\hat{M} \\
\hat{N}
\end{bmatrix} (93)
$$

The reversible property of the 2-D TFAGFFT is

$$
O_F^{(A,B,C,D,M,N)} = O_F^{(A',B',C',D',M',N')} (94)
$$

The 2-D TFAGFFT will be very useful for the filter design. In addition, from the multiplication differentiation properties in Table III, we find

$$
O_F^{(A,B,C,D)} \left[ h_1 \frac{\partial g(x,y)}{\partial x} + h_2 \frac{\partial g(x,y)}{\partial y} - h_3 \cdot j x \cdot g(x,y) - h_4 \cdot j y \cdot g(x,y) \right] = k_1 \frac{\partial G_{\text{A,B,C,D}}(f,h)}{\partial f} + k_2 \frac{\partial G_{\text{A,B,C,D}}(f,h)}{\partial h} - k_3 j f \cdot G_{\text{A,B,C,D}}(f,h) - k_4 j h \cdot G_{\text{A,B,C,D}}(f,h) (95)
$$

where the coefficients $k_1, k_2, k_3,$ and $k_4$ can be calculated from

$$
\begin{bmatrix}
k_1 \\
k_2 \\
k_3 \\
k_4
\end{bmatrix} = 
[A] \cdot 
\begin{bmatrix}
h_1 \\
h_2 \\
h_3 \\
h_4
\end{bmatrix} (96)
$$

Thus, the differentiation-multiplication operations pair also has closed relations with the 2-D AGFFT. The 2-D Wigner transform, shifting-modulation operations pair, and the differentiation-multiplication operation pair are the three operations closed to the 2-D AGFFT.

C. Other Properties of the 2-D AGFFT

Except for the relations with the 2-D WDF, shifting-modulation operation, and differentiation-multiplication operation, there are also some important properties for the 2-D AGFFT. We use Tables IV and V to summarize these properties. We prove Property 3 in Table IV and Property 4 in Table V in the Appendix.
TABLE IV

PROPERTIES OF THE 2-D AGFFT AND ITS KERNEL (A)

<table>
<thead>
<tr>
<th>Property</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) Orthogonality property</td>
<td>$\sum_{x,y} K_{(A,B,C,D)}(f,h,x,y) \cdot \overline{K_{(A,B,C,D)}(f',h',x,y)} = \delta(f - f', h - h')$</td>
</tr>
<tr>
<td>(2) Energy conservation property</td>
<td>$\sum_{x,y}</td>
</tr>
<tr>
<td>(3) Generalized energy conservation property</td>
<td>$\sum_{x,y} \mathcal{S}(x,y) \cdot g(x,y) \cdot \overline{g(x,y)} \cdot dx , dy = \sum_{x,y} \mathcal{S}<em>{(A,B,C,D)}(f,h) \cdot S</em>{(A,B,C,D)}(f,h) \cdot df , dh$ Also called as the generalized Parseval’s theorem</td>
</tr>
<tr>
<td>(4) Conjugation property</td>
<td>$K_{(A,B,C,D)}^*(f,h,x,y) = K_{(A,B,C,D)}(f,h,x,y)$</td>
</tr>
<tr>
<td>(5) Symmetry property</td>
<td>$K_{(A,B,C,D)}(f,h,x,y) = K_{(A,B,C,D)}(f,h',x,y)$</td>
</tr>
<tr>
<td>(6) Another form of the inverse 2-D AGFFT</td>
<td>$g(x,y) = \sum_{x,y} K_{(A,B,C,D)}^*(f,h,x,y) \cdot G_{(A,B,C,D)}(f,h) \cdot df , dh$</td>
</tr>
</tbody>
</table>

D. Transform Results for Some Special Signals

1) We have $\delta(x,y)$, as shown in (97) at the bottom of the page.

2) We have generalized transform results for $\exp[j(\eta_1 x^2 + \eta_2 xy + \eta_3 y^2) + j(\tau_1 x + \tau_2 y)]$.

We note that $\exp[j(\eta_1 x^2 + \eta_2 xy + \eta_3 y^2)]$ can be obtained from the unity equation [i.e., $g(x,y) = 1$] by the 2-D AGFFT with parameter $[\{0,0,0\}, \Phi]$.

$$e^{j(\eta_1 x^2 + \eta_2 xy + \eta_3 y^2)} = O_F^{(L_0,B_0,1)}(\Phi)$$ (98)

Where $\Phi = \left[ \begin{array}{cc} 2\eta_1 & \eta_2 \\ \eta_2 & 2\eta_3 \end{array} \right]$.

The function $\exp[j(\eta_1 x^2 + \eta_2 xy + \eta_3 y^2) + j(\tau_1 x + \tau_2 y)]$ is the modulation of $\exp[j(\eta_1 x^2 + \eta_2 xy + \eta_3 y^2)]$ by $\exp[j(\tau_1 x + \tau_2 y)]$.

Then, because the 2-D Fourier transform of $(-1)^{\frac{1}{2}2\pi \delta(x,y)}$ is 1

$$e^{j(\eta_1 x^2 + \eta_2 xy + \eta_3 y^2)} = O_F^{(L_0,-I,\Phi)}(\sqrt{-1}) \cdot 2\pi \delta(x,y) \cdot \Phi$$ (99)

From (99) and the modulation property listed in Table III, we obtain

$$O_F^{(A,B,C,D)}(g(x,y)) = \exp(j(\phi + \bar{\omega} \bar{D} \bar{W}_0^T)) R_{(A,B,C,D)}(\bar{\omega} - \bar{W}_0B^T)$$ (100)

Where $\Phi = -\bar{W}_0B^T \bar{D} \bar{W}_0^T/2, \bar{W}_0 = (\tau_1, \tau_2)$, and

$$R_{(A,B,C,D)}(f,h) = O_F^{(-B,A,1,-D,C,1)}(\bar{W} - \bar{W}_0B) \cdot (\sqrt{-1} \cdot 2\pi \delta(x,y)) \cdot \Phi$$ (101)

Then, applying (97), we obtain

$$O_F^{(A,B,C,D)}(e^{j(\eta_1 x^2 + \eta_2 xy + \eta_3 y^2 + \tau_1 x + \tau_2 y)}) = \frac{1}{\sqrt{\det(\Phi)}}$$

$$\cdot \exp\left(\frac{-j}{2} \bar{W}_0B^T \bar{D} \bar{W}_0^T + j \bar{D} \bar{W}_0^T + \frac{j}{2}(\bar{W} - \bar{W}_0B^T) \cdot (C + D\Phi)(A + B\Phi)^{-1}(\bar{W}^T - \bar{W}_0B^T)\right)$$ (102)

IV. CALCULATION AND DIGITAL IMPLEMENTATION

Although the 2-D AGFFT form seems to be very complex, we can in fact implement it in very simple way. From Section II-C, we have discussed that all the 2-D AGFFT can be decomposed as the combinations of the ten basic operations. Among these basic operations, we find quadratic phase term convolution (i.e., the second, fifth, and eighth basic operations) can be done by the combination of Fourier transform and quadratic phase term multiplication. Thus, all the 2-D AGFFT can be decomposed as the combinations of the quadratic phase term multiplication (i.e., the first, fourth, and seventh basic operations), geometric twisting operations (i.e., the third, sixth, ninth, and tenth basic operations), and the Fourier transform. This will be a great help in calculating the 2-D AGFFT because these operations are all easier to calculate than the integral operation in (18). We will discuss how to decompose the 2-D AGFFT for each case in Section IV-A.

A. Calculation and Digital Implementation for the 2-D AGFFT

1) $\det(B) \neq 0$:

If $\det(B) \neq 0$, then the 2-D AGFFT can be decomposed as

$$O_F^{(A,B,C,D)}(g(x,y)) = \frac{1}{\sqrt{\det(B)}} \cdot e^{j(\eta_1 x^2 + \eta_2 xy + \eta_3 y^2 + \tau_1 x + \tau_2 y)} \cdot \Phi$$ (103)

That is, in the case that $\det(B) \neq 0$, the 2-D AGFFT can be rewritten as

$$O_F^{(A,B,C,D)}(g(x,y)) = \sqrt{\det(B)} \cdot e^{j(\eta_1 x^2 + \eta_2 xy + \eta_3 y^2)} \cdot \Phi \cdot FT\left(e^{j(\eta_1 x^2 + \eta_2 xy + \eta_3 y^2)} \cdot g(b_{11}x + b_{21}y, b_{12}x + b_{22}y)\right)$$ (105)
TABLE V

<table>
<thead>
<tr>
<th>Property</th>
<th>Input</th>
<th>Transform Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) Space reversion</td>
<td>(g(-x,-y))</td>
<td>(G(A,B,C,D)(-f,-h))</td>
</tr>
<tr>
<td>(2) Mirror property</td>
<td>(g(-x,y), g(x,-y))</td>
<td>(G(A,B,C,D)(-f,h), G(A,B,C,D)(f,-h)) when (a_{12} = a_{21} = b_{12} = b_{21} = c_{12} = c_{21} = d_{12} = d_{21} = 0), i.e., for the case of the separable 2-D canonical transform.</td>
</tr>
<tr>
<td>(3) Twisting preservation</td>
<td>(g(px+qy-qx+py), ) (p^2 + q^2 = 1)</td>
<td>(G(A,B,C,D)(pf + qg, -qf + ph)) when (a_{12} = a_{21} = b_{12} = b_{21} = c_{12} = c_{21} = d_{12} = d_{21} = 0).</td>
</tr>
<tr>
<td>(4) Division by (x)</td>
<td>(g(x,y)/x)</td>
<td>(-j \exp\left(j \omega \cdot DB^{-1} \cdot \hat{w}^T / 2\right) \cdot \sum_{p} \exp\left(-j \hat{s} \cdot B T \cdot \hat{s}^T / 2\right) \cdot G(A,B,C,D)(\hat{s}B^T) \cdot dp), where ( \hat{w} = [f,h], \hat{s} = [p,q], ), (z = (fb_{12} - hb_{21})/\det(B)), and (q = -(fb_{21} + hb_{12})/\det(B)).</td>
</tr>
<tr>
<td>(5) Division by (y)</td>
<td>(g(x,y)/y)</td>
<td>(-j \exp\left(j \omega \cdot DB^{-1} \cdot \hat{w}^T / 2\right) \cdot \sum_{q} \exp\left(-j \hat{s} \cdot B T \cdot \hat{s}^T / 2\right) \cdot G(A,B,C,D)(\hat{s}B^T) \cdot dq), where ( \hat{w} = [f,h], \hat{s} = [p,q], ), (z = (-fb_{21} + hb_{12})/\det(B)), and (p = (fb_{12} - hb_{21})/\det(B)).</td>
</tr>
</tbody>
</table>

where
\[
\begin{bmatrix}
2v_1 & q_2 \\
q_2 & 2v_3
\end{bmatrix} = DB^{-1}, \quad \begin{bmatrix}
v_2 \\
v_3
\end{bmatrix} = AB^T. \tag{106}
\]

Therefore, the 2-D AGFFT can be calculated by the following four steps:
1) scaling;
2) multiplication of a quadratic phase function;
3) 2-D Fourier transform;
4) multiplication of a quadratic phase function.

For the digital implementation of 2-D AGFFT, we can also apply (105). The sampling will not be done directly to the input function. Instead, it will be done after the quadratic phase multiplication and geometric twisting (i.e., before the 2-D FFT), as follows:

\[
G(A,B,C,D)(r \Delta f, s \Delta h) = e^{j(q_1^2 \Delta f^2 + q_2^2 \Delta y^2 + v_2^2 \Delta f^2)} \cdot \text{DFT}\left(e^{j v_3 m^2 \Delta f^2 + v_2 n^2 \Delta y^2} \cdot \sqrt{\det(B)} \cdot g(b_{11} m \Delta x + b_{21} n \Delta y, b_{12} m \Delta x + b_{22} n \Delta y)\right). \tag{107}
\]

where \(q_1, q_2, q_3, v_1, v_2,\) and \(v_3\) are also the same as (106), and the sampling interval \(\Delta x, \Delta y, \Delta f,\) and \(\Delta h\) must satisfy
\[
\Delta x \cdot \Delta f = 2\pi / M, \quad \Delta y \cdot \Delta h = 2\pi / N. \tag{108}
\]

where \(M\) and \(N\) are the number of sampling points in the \(x\)-axis and \(y\)-axis for \(g(b_{11} x + b_{21} y, b_{12} x + b_{22} y)\). If in (107) we set the dc point at the middle, then (107) can be rewritten as

\[
G(A,B,C,D)(r \Delta f, s \Delta h) = \Delta x \Delta y \sqrt{-\det(B)} \cdot e^{j q_1^2 \Delta f^2 + q_2^2 \Delta y^2 + v_2^2 \Delta f^2} \cdot \frac{1}{2\pi} \cdot e^{j v_3 m^2 \Delta f^2 + v_2 n^2 \Delta y^2} \cdot \sqrt{\det(B)} \cdot g(b_{11} m \Delta x + b_{21} n \Delta y, b_{12} m \Delta x + b_{22} n \Delta y). \tag{109}
\]

The digital implementation of inverse 2-D AGFFT is

\[
g(b_{11} m \Delta x + b_{21} n \Delta y, b_{12} m \Delta x + b_{22} n \Delta y) = e^{j v_3 m^2 \Delta f^2 + v_2 n^2 \Delta y^2} \cdot \frac{1}{2\pi} \cdot e^{j q_1^2 \Delta f^2 + q_2^2 \Delta y^2 + v_2^2 \Delta f^2} \cdot \sqrt{\det(B)} \cdot g(b_{11} m \Delta x + b_{21} n \Delta y, b_{12} m \Delta x + b_{22} n \Delta y). \tag{110}
\]

We note, in (109), that the sampling points array will no longer align with the \(x\)- and \(y\)-axes for the original function \(g(x,y)\). The comparison between the sampling points of \(g(b_{11} m \Delta x + b_{21} n \Delta y, b_{12} m \Delta x + b_{22} n \Delta y)\) and the Cartesian grid sampling points is shown in Fig. 2.
Fig. 2. Cartesian grid sampling points and the twisted grid sampling points.

Since the available input datum are usually Cartesian grid sampling points
\[ g(p_{\Delta x_1}, q_{\Delta y_1}) \quad p = \ldots, -1, 0, 1, 2, \ldots \]
\[ q = \ldots, -1, 0, 1, 2, \ldots \]  
(111)

Therefore, in (109), if we want to obtain the value of
\[ g(b_{11}m_{\Delta x} + b_{21}n_{\Delta y}, b_{12}m_{\Delta x} + b_{22}n_{\Delta y}) \],
we can calculate it by the bilinear interpolation
\[
g(b_{11}m_{\Delta x} + b_{21}n_{\Delta y}, b_{12}m_{\Delta x} + b_{22}n_{\Delta y})
= (1 - h)(1 - s)g(p_{\Delta x_1}, q_{\Delta y_1})
+ h(1 - s)g((p + 1)_{\Delta x_1}, q_{\Delta y_1})
+ (1 - h)s \cdot g(p_{\Delta x_1}, (q + 1)_{\Delta y_1})
+ hs \cdot g((p + 1)_{\Delta x_1}, (q + 1)_{\Delta y_1})
\]
(112)

where \( h = (b_{11}m_{\Delta x} + b_{21}n_{\Delta y})/\Delta x \), and \( s = (b_{12}m_{\Delta x} + b_{22}n_{\Delta y})/\Delta y \). We use Fig. 3 to illustrate the concept of bilinear interpolation. Equation (112) can also be rewritten as
\[
g(b_{11}m_{\Delta x} + b_{21}n_{\Delta y}, b_{12}m_{\Delta x} + b_{22}n_{\Delta y})
= (1 - h)[g(p_{\Delta x_1}, (q + 1)_{\Delta y_1}) - g(p_{\Delta x_1}, q_{\Delta y_1})]
+ hs[g((p + 1)_{\Delta x_1}, (q + 1)_{\Delta y_1}) - g(p_{\Delta x_1}, q_{\Delta y_1})]
+ (1 - h)s \cdot g(p_{\Delta x_1}, (q + 1)_{\Delta y_1})
+ hs \cdot g((p + 1)_{\Delta x_1}, (q + 1)_{\Delta y_1})
\]
\[
\]  
(113)

Therefore, each interpolation requires three multiplications. If there are \( M \) different values of \( m \) and \( N \) different values of \( n \), then the total number of multiplications required for the resampling is
\[ S = MN \cdot f(4HK), \]  
(116)

Since it is unnecessary to use all the values of \( g(m_{\Delta x}, n_{\Delta y}) \) to calculate one resampling, \( H \), and hence, \( f(4HK) \) is independent of the number of resampling points \( MN \).

The complexity for the resampling usually can be ignored, except for the case in which the resampling algorithm is too complicated and \( f(4HK) \) is too large. When \( f(4HK) \) is too large, the complexity of resampling must be considered when discussing the complexity of 2-D AGFFT, but since \( f(4HK) \) is always independent of \( MN \), and \( S = MN \cdot f(4HK) \), even when \( f(4HK) \) is large, the amount of multiplications required for resampling is still proportional to \( MN \), and the complexity of 2-D AGFFT is still dominated by the 2-D separable FRFT.

From (109), the number of the multiplications for the digital implementation of 2-D AGFFT when \( \det(B) \neq 0 \) is approximated to
\[ 2MN + \frac{(MN/2)\log_2(MN)}{\text{(required for 2-D separable FRFT)}} \]
\[ + S\text{(required for resampling)} \]
\[ = (2 + f(4HK))MN + \frac{(MN/2)\log_2(MN)}{\text{.}} \]  
(117)
Therefore, the complexity of the digital implementation of 2-D AGFFT by the method of (109) is dependent on the resampling algorithm we use. When we use the interpolation algorithm in (112) for the resampling, then $f(AH_K) = 3, S = 3MN$, and the total number of the multiplication operations required for digital implementation of 2-D AGFFT is

$$5MN + (MN/2) \log_2 (MN) \approx (MN/2) \log_2 (MN) \quad \text{when } M, N \text{ are large.} \quad (118)$$

We note that this is similar to the complexity of the digital implementation of 2-D separable fractional Fourier transform [1]. Thus, the 2-D AGFFT has more parameters, but it is, in fact, as simple as the 2-D separable FFT and even almost as simple as the 2-D separable Fourier transform.

2) $\mathbf{B} = \mathbf{0}$:

In this case, we just use (30). This is much simple, and for the digital implementation, the complexity is proportion to $M \times N$.

3) $\det(\mathbf{B}) = \mathbf{0}, \mathbf{B} \neq \mathbf{0}$, and $\det(\mathbf{A}) \neq \mathbf{0}$:

In this case, we use (45) to implement the 2-D AGFFT as

$$G_{(A,B,C,D)}(f,h) = \frac{1}{\sqrt{\det(\mathbf{A})}} e^{j\phi_p \cdot f^p + p_2 \cdot f^h + p_3 \cdot h^2} \cdot \text{IFT}_{v \rightarrow f} \cdot \text{IFT}_{h \rightarrow h}$$

$$\cdot \left[ e^{j(\phi_2 \cdot v^2 + q_2 \cdot v^h + q_3 \cdot h^2)} \cdot \text{FT}_{u \rightarrow v} \right]$$

$$\cdot \left( g \left( \frac{a_2 x - a_1 y}{\det(\mathbf{A})}, -\frac{a_2 y + a_1 x}{\det(\mathbf{A})} \right) \right) \quad (119)$$

where

$$\begin{bmatrix} 2p_1 & -p_2 \\ -p_2 & 2p_3 \end{bmatrix} = CA^{-1}, \quad \begin{bmatrix} 2q_1 & -q_2 \\ -q_2 & 2q_3 \end{bmatrix} = -AB^T. \quad (120)$$

For the digital implementation, (119) becomes

$$G_{(A,B,C,D)}(r \Delta f, s \Delta h) = \frac{1}{\sqrt{\det(\mathbf{A})}} e^{j(\phi_p \cdot r^2 + p_2 \cdot r^h + p_3 \cdot h^2 \cdot \Delta^2)}$$

$$\cdot \sum_{t = (-M+1)/2}^{(M-1)/2} \sum_{k = (-N+1)/2}^{(N-1)/2} e^{j2\pi \left( \frac{r \cdot t}{M} + \frac{s \cdot k}{N} \right)}$$

$$\cdot e^{j(\phi_2 \cdot r^2 + q_2 \cdot r^h + q_3 \cdot h^2 \cdot \Delta^2)}$$

$$\cdot \sum_{m = (-M+1)/2}^{(M-1)/2} \sum_{n = (-N+1)/2}^{(N-1)/2} \exp \left( -j2\pi \left( \frac{t \cdot m}{M} + \frac{k \cdot n}{N} \right) \right)$$

$$\cdot g \left( \frac{a_2 x - a_1 y}{\det(\mathbf{A})}, -\frac{a_2 y + a_1 x}{\det(\mathbf{A})} \right) \quad (121)$$

where

$$\Delta_2 \Delta_2 = 2\pi / M, \quad \Delta_2 \Delta_2 = 2\pi / N$$

$$\Delta_x = \Delta_f, \quad \Delta_y = \Delta_h. \quad (122)$$

The complexity of (121) is

$$2MN + MN \cdot \log_2 (MN) + S \quad (123)$$

where $S$ is the number of the multiplication operations required for the resampling. When $M$ and $N$ are large and $S$ is proportional to $MN$, then (123) is approximated to $MN \log_2 MN$. It will be twice the complexity of the case where $\det(\mathbf{B}) \neq 0$ due to there being two DFTs required.

4) $\det(\mathbf{B}) = \mathbf{0}, \mathbf{B} \neq \mathbf{0}$, and $\det(\mathbf{A}) = \mathbf{0}, \det(\mathbf{D}) \neq \mathbf{0}$.

In this case, we use (49), and

$$G_{(A,B,C,D)}(f,h) = \sqrt{\det(\mathbf{D})} \cdot \text{IFT}_{v \rightarrow f} \cdot \text{IFT}_{h \rightarrow h}$$

$$\cdot \left[ e^{j(\phi_2 \cdot r^2 + q_2 \cdot r^h + q_3 \cdot h^2 \cdot \Delta^2)} \cdot g(d_{11} x + d_{21} y, d_{12} x + d_{22} y) \right] \quad (124)$$

where

$$\begin{bmatrix} 2p_1 & -p_2 \\ -p_2 & 2p_3 \end{bmatrix} = -BD^{-1}, \quad \begin{bmatrix} 2q_1 & -q_2 \\ -q_2 & 2q_3 \end{bmatrix} = CD^T. \quad (125)$$

For the digital implementation

$$G_{(A,B,C,D)}(r \Delta f, s \Delta h) = \frac{1}{\sqrt{\det(\mathbf{D})}} e^{j(\phi_2 \cdot r^2 + q_2 \cdot r^h + q_3 \cdot h^2 \cdot \Delta^2)}$$

$$\cdot \sum_{t = (-M+1)/2}^{(M-1)/2} \sum_{k = (-N+1)/2}^{(N-1)/2} \exp \left( \frac{j2\pi \left( \frac{r \cdot t}{M} + \frac{s \cdot k}{N} \right)}{\frac{M}{N}} \right)$$

$$\cdot \sum_{m = (-M+1)/2}^{(M-1)/2} \sum_{n = (-N+1)/2}^{(N-1)/2} \exp \left( -j2\pi \left( \frac{t \cdot m}{M} + \frac{k \cdot n}{N} \right) \right)$$

$$\cdot e^{j(\phi_2 \cdot r^2 + q_2 \cdot r^h + q_3 \cdot h^2 \cdot \Delta^2)}$$

$$\cdot g(\Delta_2 x, \Delta_2 y, d_{11} \Delta_2 x + d_{21} \Delta_2 y, d_{12} \Delta_2 x + d_{22} \Delta_2 y) \quad (126)$$

where

$$\Delta_2 \Delta_2 = 2\pi / M, \quad \Delta_2 \Delta_2 = 2\pi / N$$

$$\Delta_x = \Delta_f, \quad \Delta_y = \Delta_h. \quad (127)$$

must also be satisfied. The complexity of (126) is the same as (123), i.e., it is approximated to twice of the complexity in the case where $\det(\mathbf{B}) \neq 0$.

5) $\det(\mathbf{B}) = \mathbf{0}, \mathbf{B} \neq \mathbf{0}$, $\det(\mathbf{A}) = \mathbf{0}$, $\det(\mathbf{D}) = \mathbf{0}$. $\mathbf{D} \neq \mathbf{0}$. $\det(\mathbf{C}) \neq \mathbf{0}$. 


In this case, we can use (51), and together with (121), we obtain the formula of digital implementation as

\[
G_{(A,B,C,D)}(\tau \Delta f, \kappa \Delta \eta) = \sum_{\tau = -(M+1)/2}^{(M-1)/2} \sum_{\kappa = -(N+1)/2}^{(N-1)/2} \frac{\Delta \nu \Delta \sigma}{2 \sqrt{N M \det(C)}} \cdot \exp \left( j 2 \pi \left( \frac{\tau \cdot \tau}{M} + \frac{\kappa \cdot \kappa}{N} \right) \right) \cdot \exp \left( j \frac{\eta \cdot \sigma}{N} \right) \cdot g \left( \frac{\nu \Delta x + \sigma \Delta y}{\det(C)} \right)
\]

where

\[
\begin{bmatrix}
2p_1 & 2p_2 \\
-2p_2 & 2p_3
\end{bmatrix} = -AC^{-1}, \quad \begin{bmatrix}
2q_1 & -q_2 \\
-q_2 & 2q_3
\end{bmatrix} = -CD^T
\]

\[
\Delta \nu \Delta \sigma = 2 \pi / M, \quad \Delta \kappa \Delta \eta = 2 \pi / N, \quad \Delta \tau = \Delta \eta, \quad \Delta \kappa = \Delta \sigma.
\]

The complexity of (128) is

\[
2MN + (3MN/2) \cdot \log_2(MN) + S
\]

where \(S\) is the number of the multiplication operations required for resampling. When \(M\) and \(N\) are large and \(S\) is proportional to \(MN\), then (130) is approximated to \((3MN/2) \cdot \log_2 MN\). It is three times the complexity of the case where \(\det(B) \neq 0\) because three DFTs are required.

6) \(\det(B) = 0\), \(B \neq 0\), and \(\det(A) = \det(D) = \det(C) = 0\):

We can use (54)–(56) to implement the 2-D AGFFT for this case.

### B. Simplified Form of the 2-D AGFFT

The 2-D AGFFT defined as (18)–(21) has, in total, ten free parameters, ten exponential terms, and two integration operations. It is very complex, and for the practical applications, we will use it many times for design because there are ten parameters to be adjusted. Therefore, it would be better to use the simplified form of the 2-D AGFFT without losing its utilities.

From Section II-C, we have discussed that the 2-D AGFFT is the combination of the 2-D Fourier transform, the geometric twisting operation, and the multiplication and convolution of the quadratic phase term \([i.e., \exp(j(\eta \Delta f^2 + \eta \Delta \eta \Delta f + \eta \Delta \eta^2))]\). We conclude that the convolution of the quadratic phase term will not increase the ability of the 2-D AGFFT since it can be replaced by the combination of the multiplication \(f\) the quadratic phase term, and the 2-D Fourier transform. Therefore, we just want the simplified 2-D AGFFT to contain the 2-D Fourier transform, the geometric twisting operation, and the multiplication of the quadratic phase term.

From the above discussion, we conclude that the 2-D AGFFT can be simplified by extracting the outside quadratic exponent term by setting \(D = 0\) in (18) and (20)

\[
\begin{aligned}
O_{(A,B,C,D)}(g(x,y)) &= \frac{1}{2 \sqrt{-\det(B)}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdot \exp \left( \frac{j}{\det(B)} \left( (b_{22} f + b_{21} h)x + (b_{21} f - b_{11} h)y \right) \right) \cdot \exp \left( \frac{j}{2 \det(B)} (p_1 x^2 + p_2 x y + p_3 y^2) \right) \cdot g(x,y) \cdot dx dy
\end{aligned}
\]

(131)

where \(p_1, p_2,\) and \(p_3\) are defined as (21). We find, in (131), that the multiplication of the quadratic phase term has been preserved in the terms of \(\exp(j(p_1 x^2 + p_2 x y + p_3 y^2)/2\det(B))\), and the 2-D Fourier transform together with the geometric twisting operation have been preserved in terms of \(\exp(j((b_{22} f + b_{21} h)x + (b_{21} f - b_{11} h)y)/\det(B))\), and all three parts of AGFFT still exist. This simplified 2-D AGFFT has only seven free dimensions because of the 12 parameters \((D = 0)\) and five constraints (the second constraint is naturally satisfied). The seven free dimensions correspond to the seven exponent terms in the (131). The inverse of (131) is

\[
\begin{aligned}
O_{(A,B,C,D)}^{-1}(f,h) &= \frac{1}{2 \sqrt{-\det(B)}} \cdot \exp \left( \frac{j}{\det(B)} \left( (b_{22} x - b_{21} y)f + (b_{12} x + b_{11} y)h \right) \right) \cdot \exp \left( \frac{j}{2 \det(B)} (p_1 x^2 + p_2 x y + p_3 y^2) \right) \cdot G_{(A,B,C,D)}(f,h) \cdot df dh.
\end{aligned}
\]

(132)

Although this simplified form of 2-D AGFFT only has the free dimension of 7, it is sufficient for most of the applications of the 2-D AGFFT defined in (18)–(21), except for the optical system analysis. We note it has only one more free dimension than the 2-D separable canonical transform defined as (8), but it can do many things that cannot be done by the 2-D separable canonical transform.

We note, for the simplified 2-D AGFFT defined in (131) that \(\det(B) \neq 0\) must be satisfied (due to \(D = 0\)). Then, from Section IV-A, the complexity of the digital implementation is approximated to

\[
MN + (MN/2) \cdot \log_2 MN + S.
\]

(133)

Because \(D = 0\), one chirp multiplication operation is saved.

For the application of filter design (see Section IV-A), it would be better to further simplify (131) by setting \(B = I\).
In this case there are only three parameters, and all six constraints are naturally satisfied. Therefore, the free dimension is only 3. This simplified transform is sufficient to filter out all the quadratic type noise easily. Because there are only three parameters, the design of the optimal filter is easy. Its inverse is

\[ g(x, y) = \frac{1}{2\pi} e^{-j\alpha_1 x^2 + 2\alpha_2 xy + \alpha_3 y^2}/2 \cdot \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{jfx} \cdot G(A)(f, h) \cdot df dh. \]

(134)

There are also another choices for the simplified 2-D AGFFT. For example, we can use the following equation:

\[ g_{\text{modified}}(x, y) = \frac{1}{2\pi \sqrt{b_2 b_1 b_2}} \int_{-\infty}^{\infty} e^{-j\lambda x^2 + 2\alpha_2 xy + \alpha_3 y^2}/2 \cdot g(x + p, y + q) \cdot dx dy. \]

(136)

This is the combination of 2-D separable canonical transform (with the parameters \( \{a_1, b_1, -1/b_2, 0\} \) for the \( x \)-axis and \( \{a_2, b_2, -1/b_2, 0\} \) for the \( y \)-axis) and the geometric twisting operation. It is the special case of 2-D AGFFT defined in (18)–(21) has in total the free dimension of 10, it is very i.e., the 2-D affine generalized convolution of the signal \( s(x, y) \) and the filter \( Z(f, h) \). There are at least two ways to design the filter with 2-D AGFFT:

1) optimal filter;
2) pass-stopband filter.

We first discuss the optimal filter. Because the 2-D AGFFT is also an orthonormal transform, the formula of the optimal filter for the 1-D FRFT [11] can also be applied here with a little modification. We assume that a) the correlation between the input signal \( g(x, y) \) and the received signal \( s(x, y) \) [denoted by \( R_{gs}(x, y, \sigma, \tau) \)] and that b) the auto-correlation of the received signal [denoted by \( R_{ss}(x, y, \sigma, \tau) \)] have been known in advance. Then, the optimal filter can be calculated from

\[ z_{opt}(f, h) = \frac{R_{G(A,B,C,D)}S_{A,B,C,D}(f, h, f, h)}{R_{S(A,B,C,D)}S_{A,B,C,D}(f, h, f, h)}. \]

(139)

In addition, \( R_{G(A,B,C,D)}S_{A,B,C,D}(f, h, f, h), R_{S(A,B,C,D)}S_{A,B,C,D}(f, h, f, h) \) can be calculated from

\[ R_{G(A,B,C,D)}S_{A,B,C,D}(f, h, f, h) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K_{A,B,C,D}(f, h, x, y) \cdot R_{gs}(x, y, \sigma, \tau) \cdot dx dy d\sigma d\tau \]

(140)

\[ R_{S(A,B,C,D)}S_{A,B,C,D}(f, h, f, h) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K_{A,B,C,D}(f, h, x, y) \cdot R_{ss}(x, y, \sigma, \tau) \cdot dx dy d\sigma d\tau. \]

(141)

To calculate the mean square error (MSE), suppose that we also know that c) the auto-correlation of the input signal [denoted by \( R_{gg}(x, y, \sigma, \tau) \)]. Then, we can calculate \( R_{G(A,B,C,D)G(A,B,C,D)}(f, h, f, h) \) from

\[ R_{G(A,B,C,D)G(A,B,C,D)}(f, h, f, h) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K_{A,B,C,D}(f, h, x, y) \cdot R_{gg}(x, y, \sigma, \tau) \cdot dx dy d\sigma d\tau \]

(142)

and the MSE for the optimal filter can be calculated as

\[ \text{MSE} = \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdot R_{G(A,B,C,D)G(A,B,C,D)}(f, h, f, h) - 2 \text{Re} \right] z_{opt}(f, h) \cdot R_{S(A,B,C,D)S(A,B,C,D)}(f, h, f, h) \]

\[ + |z_{opt}(f, h)|^2 \cdot R_{S(A,B,C,D)S(A,B,C,D)}(f, h, f, h) \cdot df dh. \]

(143)

We can use (139) to determine the optimal filter for each choice of \( \{A, B, C, D\} \) and then use (143) iteratively to determine the optimal choice of \( \{A, B, C, D\} \). Because the 2-D AGFFT defined in (18)–(21) has in total the free dimension of 10, it is very
difficult to determine the optimal choice of \{A, B, C, D\}, but if we use the modified 2-D AGFFT defined as (134), then there are only three free parameters, and the optimal choice of the parameters is much easier to determine.

We then discuss the pass-stopband filter. That is, \( Z(f,h) = 1 \) in the passband, and \( Z(f,h) = 0 \) in the stopband. This type of filter is especially useful when the noise has the form

\[
n(x, y) = P_0 \exp(j(p_1 x^2 + p_2 xy + p_3 y^2 + p_4 x + p_5 y))
\]

(144)

or in general

\[
n(x, y) = \sum_{s=0}^{\infty} P_{s0} \exp(j(p_{s1} x^2 + p_{s2} xy + p_{s3} y^2 + p_{s4} x + p_{s5} y)),
\]

(145)

If, in (145), \( p_{s2} = 0 \) for all \( s \), then this type of noise can just be removed by the 2-D separable canonical transform, but if \( p_{s2} \neq 0 \) for some \( s \), then we must use the 2-D AGFFT introduced in this paper to remove the noise. To remove the noise as the form of (144), we can choose the parameters \{A, B, C, D\} of the 2-D AGFFT as

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix} = \begin{bmatrix}
-\Psi & I \\
-I & 0
\end{bmatrix}, \quad \text{where} \quad \Psi = \begin{bmatrix}
2p_1 & p_2 \\
p_2 & 2p_3
\end{bmatrix}.
\]

(146)

This is because

\[
O_F(\begin{bmatrix}
\Psi & -I \\
I & 0
\end{bmatrix}) (P_0 e^{j(p_1 x^2 + p_2 xy + p_3 y^2 + p_4 x + p_5 y)})
= \text{FT} \left( O_F(\begin{bmatrix}
\Psi & -I \\
I & 0
\end{bmatrix}) (P_0 e^{j(p_1 x^2 + p_2 xy + p_3 y^2 + p_4 x + p_5 y)}) \right)
= \text{FT} \left( P_0 e^{j(p_1 x + p_2 y)} \right)
= 2\pi \sqrt{-1} \cdot P_0 \cdot \delta(f-p_1, h-p_5).
\]

(147)

Therefore, the noise in (144) will become the impulse function after the 2-D AGFFT with the parameters in (146). To remove the noise in (145), we can repeat the filter operation several times, and each time, we choose the parameters of the 2-D AGFFT according to (146) (\( \{p_1, p_2, p_3\} \) is changed as \( \{p_{s1}, p_{s2}, p_{s3}\} \)) to remove one of the quadratic phase components in (145).

However, when we do not know the components of the noise, we cannot apply the method described above. In this case, we must resort to other ways to search the parameters \{A, B, C, D\} of the 2-D AGFFT. For example, we can calculate the 2-D Wigner distribution function (2-D WDF, which is described in Section III-A) for \( s(x, y) + n(x, y) \), where \( s(x, y) \) is the signal and \( n(x, y) \) is the noise, and use the WDF to conclude that we must choose a set of parameters. Since it would require much effort to discuss this method in detail, we will not discuss it in this paper.

We will give an example below. Here, the signal is the fruit image with the size 256 \times 256 shown in Fig. 4 [the location (0, 0) is at the center], and the noise as

\[
n(x, y) = \exp(j \cdot 0.001(2x^2 - 10xy + 1.5y^2)),
\]

(148)

In Fig. 5, we plotted the fruit plus the noise. This noise cannot be moved by the separable 2-D canonical transforms because of the existence of the unseparable term \( \exp(j \cdot 0.001(2x^2 - 10xy + 1.5y^2)) \). However, we can remove it by the 2-D AGFFT defined as (18)–(21), and its simplified form of (134). We choose the parameters as \( B = -C = I, D = 0, a_{11} = -0.004, a_{12} = 0.01, a_{22} = -0.003 \). In Fig. 6, we show the result of the 2-D AGFFT of Fig. 5 with the parameters described above. After the 2-D AGFFT, we use the filter as

\[
Z(f,h) = 1 - \delta(f,h).
\]

(149)
Fig. 7. Recovered signal after filtering in the AGFFT domain.

Fig. 8. Optical system with a tilted cylinder lens.

Then, we do the inverse 2-D AGFFT and obtain the recovered signal as Fig. 7. We find that the noise has been completely removed.

**B. Optical Implementation for the 2-D AGFFT**

The 2-D AGFFT can also be applied to the optical system analysis for the monochromatic light. The 2-D separable canonical transform has been used for this application [1]. It can analyze the cylinder lens with the width-variation direction of the x- or y-axis. However, for the tilted cylinder lens, the 2-D separable canonical transform will fail to analyze it, and we must use the 2-D AGFFT defined as (18)–(21). For example, for the optical system in Fig. 8, the transform function of the first lens is

\[
t_1(x, y) = e^{-j\pi (y \cos \alpha - x \sin \alpha)^2 / 2\lambda f_1}
\]

and it corresponds to the 2-D AGFFT with the parameters

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix} =
\begin{bmatrix}
1 & 0 \\
\Phi & 1
\end{bmatrix}
\]

where

\[
\Phi =
\begin{bmatrix}
-k \sin^2 \alpha & k \sin \alpha \cos \alpha \\
-k \sin \alpha \cos \alpha & -k \cos^2 \alpha
\end{bmatrix}, \quad k = 2\pi / \lambda
\]

(151)

Then, the overall system in Fig. 8 will correspond to the 2-D AGFFT with parameters

\[
\begin{bmatrix}
A' & B' \\
C' & D'
\end{bmatrix} =
\begin{bmatrix}
1 & 0 \\
\Omega & 1
\end{bmatrix} \cdot
\begin{bmatrix}
I & \Psi \\
0 & I
\end{bmatrix} \cdot
\begin{bmatrix}
I & 0 \\
\Phi & 1
\end{bmatrix}
\]

(152)

where

\[
\Omega =
\begin{bmatrix}
0 & 0 \\
0 & -k / f_2
\end{bmatrix}, \quad \Psi =
\begin{bmatrix}
d / k & 0 \\
0 & d / k
\end{bmatrix}
\]

(153)

Besides, the 2-D AGFFT can also be applied to describe how the monochromatic light propagates in the gradient index medium. In [12], if the monochromatic light propagates in the ellipsoidal gradient index (ellipsoidal GRIN) medium with the refractive index as

\[
n^2(x, y) = n_0^2 \left[ 1 - \frac{n_z}{n_0} x^2 - \frac{n_y}{n_0} y^2 \right]
\]

(154)

and the propagation length is L, then

\[
T(g(x, y)) = \exp(-j2\pi n_0 L / \lambda) \cdot O_{F_x}^2 \cdot O_{F_y}^3 (g(x, y))
\]

(155)

where \( O_{F_x}^2 \) is the 1-D SAFT along the x-axis with parameters

\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix} =
\begin{bmatrix}
\cos \alpha & \sqrt{\frac{n_0 \sin \alpha}{n_x}} / k \\
\frac{-n_x}{n_0} k \sin \alpha & \cos \alpha
\end{bmatrix}
\]

(156)

and \( O_{F_y}^3 \) is the 1-D SAFT along the y-axis with parameters that are similar as above, except that \( \alpha \) is changed as \( \beta \) and \( n_x \) is changed as \( n_y \).

Thus, we can conclude that if the monochromatic light propagates in the ellipsoidal GRIN medium with the refractive index as

\[
n^2(x, y) = n_0^2 \left[ 1 - \frac{n_1}{n_0} (x + py)^2 - \frac{n_2}{n_0} (qx + y)^2 \right]
\]

(157)

and the propagation length is L, then the result can be represented by the 2-D AGFFT

\[
T(g(x, y)) = \exp(-j2\pi n_0 L / \lambda) \cdot O_{F_x}^{(A, B, C, D)} (g(x, y))
\]

(158)

where

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix} =
\begin{bmatrix}
\frac{1}{\Delta} & -p / \Delta & 0 & 0 \\
p / \Delta & \frac{1}{\Delta} & 0 & 0 \\
0 & 0 & 1 & q \\
0 & 0 & p & 1
\end{bmatrix}
\]

\[
\cdot
\begin{bmatrix}
\cos \alpha & 0 & \frac{\sin \alpha}{m_1 k} & 0 \\
0 & \cos \beta & 0 & \frac{\sin \beta}{m_2 k} \\
\frac{-m_1 k \sin \alpha}{m_1 k} & 0 & \cos \alpha & 0 \\
0 & -m_2 k \sin \beta & 0 & \cos \beta
\end{bmatrix}
\]

\[
\cdot
\begin{bmatrix}
1 & p & 0 & 0 \\
q & 1 & 0 & 0 \\
0 & 0 & 1 / \Delta & -q / \Delta \\
0 & 0 & -p / \Delta & 1 / \Delta
\end{bmatrix}
\]

(159)
and
\[ m_1 = \sqrt{\frac{n_1}{n_0}}, \quad m_2 = \sqrt{\frac{n_2}{n_0}}, \quad \alpha = \frac{2L}{\pi}m_1 \]
\[ \beta = \frac{2L}{\pi}m_2, \quad \Delta = 1 - \rho q. \]

(160)

**C. Other Potential Applications**

In the 1-D case, we can use fractional correlation for pattern recognition [13], [14]. Thus, in the 2-D case, we can also use the 2-D affine generalized fractional correlation defined in (81) for pattern recognition. When using the generalized fractional correlation for pattern recognition, objects will be identified only when we have the following.

1) The objects are similar to the reference pattern.
2) The objects are in a certain region (because the 2-D AGFFT is partially space variant).

Besides, the 2-D AGFFT can also be applied to generalize the 2-D Hilbert transform, mask, signal synthesis, beam shaping, 2-D signal compression, and phase retrieval, etc.

**VI. CONCLUSION**

We have introduced the 2-D AGFFT and some important properties, its calculation and digital implementation, and applications for the filter design, optical system analysis, etc.

The 2-D AGFFT generalizes the 2-D separable FRFT and the 2-D canonical transform introduced by [1] and mixed them with the geometric twisting operation. Thus, the 2-D AGFFT not only can extend most of the applications for 1-D fractional Fourier transform but can also be a useful tool for the pattern recognition with some spatial distortion. The main disadvantages for the generalized transform are the number of the parameter increases, and the calculation will become more complex. With the method introduced in Section IV, however, we find that if \( M \) and \( N \) are large and the re-sampling algorithm we use is not very complex, then the complexity of digital implementation for the 2-D AGFFT is still proportional to \((MN/2)\log_2 MN\) for most of the cases, as in the 2-D separable FRFT. Besides using the simplified forms introduced in Section IV-B, we can reduce the number of parameters with very little effect on the utility of 2-D AGFFT. Thus, in fact, the 2-D AGFFT is only a little more complex than the 2-D separable FRFT, but the utilities would be extended a lot. We believe that the 2-D AGFFT will be a popular tool for 2-D signal processing in the future.

**APPENDIX**

**PROOF OF THE PROPERTIES**

**Proof of the Properties 1 and 2 in Table II:** We know that
\[
|g(x, y)|^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_g(x, y, f, h) \cdot df \cdot dh \]
(161)

\[
g(x, y) = \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \frac{x}{2}, \frac{y}{2} \right, f, h) \cdot e^{j(fx+yh)} \cdot df \cdot dh \right] / g(0, 0). \]
(162)

Therefore, if \( W_{G(AB,CD)}(x, y, f, h) \) is the 2-D WDF of \( G(AB,CD)(f, h) \), then
\[
\left| G(AB,CD)(x, y) \right|^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_{G(AB,CD)}(x, y, f, h) \cdot df \cdot dh \]
(163)

\[
G(AB,CD)(x, y) = \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_{G(AB,CD)} \left( \frac{x}{2}, \frac{y}{2}, f, h \right) \cdot e^{j(fx+yh)} \cdot df \cdot dh \right] / G(AB,CD)(0, 0). \]
(164)

Then, together with (86), we can obtain Properties 1 and 2 in Table II.

**Proof of Property 3 in Table II:** If \( s(x, y) = p(x, y)q(x, y) \) and \( W_s(x, y, f, h) \), \( W_p(x, y, f, h) \), and \( W_q(x, y, f, h) \) are the WDF of \( s(x, y), p(x, y), q(x, y) \), then [15]
\[
W_s(\tilde{x}, \tilde{w}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_p(\tilde{x}, \tilde{p}) \cdot W_q(\tilde{w}, \tilde{w} - \tilde{p}) \cdot d\tilde{p} \cdot d\tilde{w} \]
(165)

where
\[
\tilde{x} = [x, y], \quad \tilde{w} = [f, h], \quad \tilde{p} = [\tau, \sigma].
\]

If \( z(x, y) \) is the 2-D affine generalized fractional convolution of \( f(x, y) \) and \( g(x, y) \), as (79), then \( Z_{(AB,CD)}(f, h) = F_{(AB,CD)}(f, h) \cdot G_{(AB,CD)}(f, h) \), and from (165)
\[
W_z(\tilde{x}, \tilde{w}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_{F(AB,CD)}(\tilde{x}, \tilde{p}) \cdot W_{G(AB,CD)}(\tilde{w}, \tilde{w} - \tilde{p}) \cdot d\tilde{p} \cdot d\tilde{w}. \]
(166)

Then, from (86), we can obtain Property 3 in Table II.

\[
W_z(\tilde{x}, \tilde{w}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_f(\tilde{x}D - \tilde{p}B, -\tilde{x}C + \tilde{p}A) \cdot W_g(\tilde{x}D - \tilde{w}B + \tilde{x}C + \tilde{w}A - \tilde{p}A) \cdot d\tilde{p} \cdot d\tilde{w}. \]

Then, from (85)
\[
W_z(\tilde{x}, \tilde{w}) = W_{Z(AB,CD)}(\tilde{x}A^T + \tilde{w}B^T, \tilde{x}C^T + \tilde{w}D^T) \]
\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_f(\tilde{x}A^TD + \tilde{w}B^TD - \tilde{x}A^TC - \tilde{w}B^TC + \tilde{p}A) \cdot W_g(\tilde{x} + \tilde{p}B, \tilde{w} - \tilde{p}A) \cdot d\tilde{p} \cdot d\tilde{w}. \]

(167)

**Proof of Property 3 in Table III:** From (18)–(21), we find
\[
\frac{\partial}{\partial f} G_{(AB,CD)}(f, h) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{j}{\text{det}(B)} \left[ (d_{11}b_{22} - d_{12}b_{21})f + (-d_{11}b_{22} + d_{12}b_{21})h - b_{22}x + b_{21}y \right] \cdot K_{(AB,CD)}(f, h, x, y) \cdot g(x, y) \cdot dz \cdot dy.
\]
Then, we multiply (167) by $b_{11}$, multiply (168) by $b_{21}$, sum them together, and use the fact that $-d_{11}b_{11} + d_{21}b_{11} = d_{21}b_{21} - d_{11}b_{12}$ [the second constraint of 2-D AGFFT in (22)], and we will obtain the Property 3.

**Proof of Property 5 in Table III**: From (26) and (20), we obtain

\[
\frac{\partial}{\partial x} g(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{j}{\text{det}(B)} \left( (-a_{11}b_{22} + a_{21}b_{12})x + (a_{11}b_{22} - a_{21}b_{11})y + b_{21}f - b_{12} h \right) \\
\cdot \tilde{K}(\tilde{x}, \tilde{y}) \cdot G_{(A, B, C, D)}(f, h) \cdot df \cdot dh.
\]

We can then write

\[
\frac{\partial}{\partial x} g(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{j}{\text{det}(B)} \left( (-a_{11}b_{22} + a_{21}b_{12})x + (a_{11}b_{22} - a_{21}b_{11})y + b_{21}f - b_{12} h \right) \\
\cdot \tilde{K}(\tilde{x}, \tilde{y}) \cdot G_{(A, B, C, D)}(f, h) \cdot df \cdot dh
\]

and therefore

\[
\exp\left( -\frac{j}{2} \tilde{w} \cdot B \tilde{D} \tilde{w} \right) \tilde{G}_{(A, B, C, D)}(\tilde{w} B^T) = 2 \pi \sqrt{-\text{det}(B)} \int \exp\left( \frac{j}{2} \left( -\tilde{x} \tilde{w} T + \tilde{x} B^{-1} A \tilde{x}^T \right) \right) g(\tilde{x}) d\tilde{x}.
\]

**Proof of Property 4 in Table V**: Suppose $\tilde{G}_{(A, B, C, D)}(\tilde{w})$ is the 2-D AGFFT of $g(x, y)/x$:

\[
\tilde{G}_{(A, B, C, D)}(\tilde{w}) = \frac{1}{2\pi \sqrt{-\text{det}(B)}} \exp\left( j \tilde{w} \cdot D^{-1} \tilde{w} \right) \\
\cdot \int \exp\left( \frac{j}{2} \left( -\tilde{x} B^{-1} \tilde{w} T + \tilde{x} B^{-1} A \tilde{x}^T \right) \right) g(\tilde{x}) d\tilde{x}
\]

where $\tilde{x} = [x, y]$, $\tilde{w} = [f, h]$. Then

\[
\tilde{G}_{(A, B, C, D)}(\tilde{w} B^T) = \frac{1}{2\pi \sqrt{-\text{det}(B)}} \exp\left( j \tilde{w} \cdot B^T \tilde{w} \right) \\
\cdot \int \exp\left( \frac{j}{2} \left( -\tilde{x} \tilde{w} T + \tilde{x} B^{-1} A \tilde{x}^T \right) \right) g(\tilde{x}) d\tilde{x}
\]

and therefore

\[
\exp\left( -\frac{j}{2} \tilde{w} \cdot B^T \tilde{D} \tilde{w} \right) \tilde{G}_{(A, B, C, D)}(\tilde{w} B^T) = 2 \pi \sqrt{-\text{det}(B)} \int \exp\left( \frac{j}{2} \left( -\tilde{x} \tilde{w} T + \tilde{x} B^{-1} A \tilde{x}^T \right) \right) g(\tilde{x}) d\tilde{x}.
\]

Since

\[
\tilde{G}_{(A, B, C, D)}(\tilde{w} B^T) = \frac{1}{2\pi \sqrt{-\text{det}(B)}} \exp\left( j \tilde{w} \cdot B^T \tilde{D} \tilde{w} \right) \\
\cdot \int \exp\left( \frac{j}{2} \left( -\tilde{x} \tilde{w} T + \tilde{x} B^{-1} A \tilde{x}^T \right) \right) g(\tilde{x}) d\tilde{x}
\]

(171) can be rewritten as

\[
\frac{\partial}{\partial f} \left[ \exp\left( -\frac{j}{2} \tilde{w} \cdot B^T \tilde{D} \tilde{w} \right) \tilde{G}_{(A, B, C, D)}(\tilde{w} B^T) \right] = -j \exp\left( -\frac{j}{2} \tilde{w} \cdot B^T \tilde{D} \tilde{w} \right) \tilde{G}_{(A, B, C, D)}(\tilde{w} B^T) \\
\exp\left( -\frac{j}{2} \tilde{w} \cdot B^T \tilde{D} \tilde{w} \right) \tilde{G}_{(A, B, C, D)}(\tilde{w} B^T) = -j \int_{-\infty}^{\infty} \exp\left( -\frac{j}{2} \tilde{v} \cdot B^T \tilde{D} \tilde{v} \right) \tilde{G}_{(A, B, C, D)}(\tilde{v} B^T) d\tilde{v}
\]

where $\tilde{v} = [p, h]$. Thus, if we replace $\tilde{w}$ by $\tilde{w} B^{-1}$, then

\[
\tilde{G}_{(A, B, C, D)}(\tilde{w}) = -j \exp( j \tilde{w} \cdot B^{-1} \cdot \tilde{w} / 2) \\
\cdot \int_{-\infty}^{\infty} \exp\left( -\frac{j}{2} \tilde{s} \cdot B^T \tilde{D} \cdot \tilde{s} / 2 \right) \tilde{G}_{(A, B, C, D)}(\tilde{s} B^T) d\tilde{p}
\]

where $z = (f \cdot b_{12} - h \cdot b_{22}) / \text{det}(B)$, $\tilde{s} = [p, q]$, $q = (-f \cdot b_{22} + h \cdot b_{12}) / \text{det}(B)$.
References


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