A New Projective Invariant Associated to the Special Parabolic Points of Surfaces and to Swallowtails

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Abstract. We show some generic (robust) properties of smooth surfaces immersed in the real 3-space (Euclidean, affine or projective), in the neighbourhood of a *godron*: an isolated parabolic point at which the (unique) asymptotic direction is tangent to the parabolic curve. With the help of these properties and a projective invariant that we associate to each godron we present all possible local configurations of the flecnodal curve at a generic swallowtail in $\mathbb{R}^3$. We present some global results, for instance: *In a hyperbolic disc of a generic smooth surface, the flecnodal curve has an odd number of transverse self-intersections.*

1 Introduction

A generic smooth surface in $\mathbb{R}^3$ has three (possibly empty) parts: an open *hyperbolic domain* at which the Gaussian curvature K is negative, an open *elliptic domain* at which K is positive and a *parabolic curve* at which K vanishes. A *godron* is a parabolic point at which the (unique) asymptotic direction is tangent to the parabolic curve. We present various robust geometric properties of generic surfaces, associated to the godrons. For example (Theorem 2):

Any smooth curve of a surface of $\mathbb{R}^3$ *tangent to the parabolic curve at a godron* g *has at least* 4*-point contact with the tangent plane of the surface at* g.

The line formed by the inflection points of the asymptotic curves in the hyperbolic domain is called *flecnodal curve*. The next theorem is well known.

Theorem 1. ([14, 10, 13, 7, 9]) *The godrons of a generic smooth surface are the points of the simplest tangency of the flecnodal curve with the parabolic curve.*

For a generic smooth surface we have the following stricking global result (Proposition 5 and Theorem 6):

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A closed parabolic curve bounding a hyperbolic disc has a positive even number of godrons, and the flecnodal curve lying in that disc has an odd number of transverse self-intersections.

The *conodal curve* of a surface S is the closure of the locus of points of contact of S with its *bitangent planes* (planes which are tangent to S at least at two distinct points). It is well known ([14, 10]) that:

A godron of a generic smooth surface is a point of the simplest tangency of the conodal curve with the parabolic curve.

So the parabolic, flecnodal and conodal curves of a surface are mutually tangent at the godrons. At each godron, these three tangent curves determine a projective invariant ρ cross-ratio (see the cr-invariant below). We show all possible configurations of these curves at a godron, according to the value of ρ (Theorem 4). There are six generic configurations, see Fig. 3.

The invariant ρ and the geometric properties of the godrons presented here are useful for the study of the local affine (projective) differential properties of swallowtails. So, for example, we present all generic configurations of the flecnodal curve in the neighbourhood of a swallowtail point of a surface of $\mathbb{R}^3$ in general position (see Theorem 8 and Fig. 6).

The paper is organised as follows. In section 2, we recall the classification of points of a generic smooth surface in terms of the order of contact of the surface with its tangent lines. In section 3, we give some definitions and present our results. Finally, in section 4, we give the proofs of the theorems.

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2 Projective properties of smooth surfaces

The points of a generic smooth surface in the real 3-space (projective, affine or Euclidean) are classified in terms of the contact of the surface with its tangent lines. In this section, we recall this classification and some terminology.

A generic smooth surface S is divided in three (possibly empty) parts:
(E) An open *domain of elliptic points*: there is no real tangent line exceeding 2-point contact with S;
(H) An open *domain of hyperbolic points*: there are two such lines, called *asymptotic lines* (their directions at the point of tangency are called *asymptotic directions*); and
(P) A smooth *curve of parabolic points*: a unique, but double, asymptotic line.

The *parabolic curve*, divides S into the *elliptic* and *hyperbolic domains*.

In the closure of the hyperbolic domain there is:
(F) A smooth immersed *flecnodal curve*: it is formed by the points at which an asymptotic tangent line exceeds 3-point contact with S.

One may also encounter isolated points of the following four types: (g) A *godron* is a parabolic point at which the (unique) asymptotic direction is tangent to the parabolic curve; (sh) A *special hyperbolic point* is a point of the simplest self-intersection of the flecnodal curve; (b) A *biinflection point* is a point of the flecnodal curve at which one asymptotic tangent exceeds 4-point contact with S; (se) A *special elliptic point* is a real point in the elliptic domain of the simplest self-intersection of the complex conjugate flecnodal curves associated to the complex conjugate asymptotic lines. In Fig. 1 the hyperbolic domain is represented in gray colour and the elliptic one in white. The flecnodal curve has a left branch F_l (white) and a right branch F_r (black). These brances will be defined in the next section.

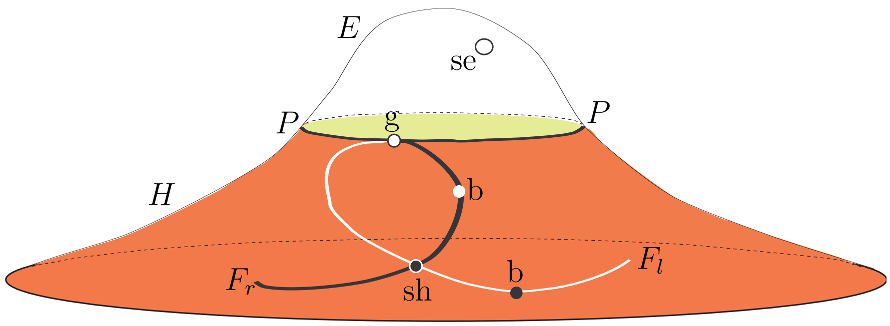

Figure 1: The 8 tangential singularities of a generic smooth surface.

The term "godron" is due to R. Thom [9]. In other papers one can find the terms "special parabolic point" or "cusp of the Gauss map". We keep Thom's terminology since it is shorter. Here we will study the local projective differential properties of the godrons.

The above 8 classes of tangential singularities, Theorem 1 and all the theorems presented in this paper are projectively invariant and are robust features of a smooth surface, that is, they are stable in the sense that under a sufficiently small perturbation (taking derivatives into account) they do not vanish but only deform slightly. Seven of these classes were known at the end of the 19th century in the context of the enumerative geometry of complex algebraic surfaces, with prominent works of Cayley, Zeuthen and Salmon, see [14]. For these seven classes, the normal forms of surfaces at such points up to the 5-jet, under the group of projective transformations, were independently found by E.E. Landis ([7]) and O.A. Platonova ([13]). The special elliptic points were found by D. Panov ([8]).

For surfaces in $\mathbb{R}^3$, these tangential singularities depend only on the affine structure of $\mathbb{R}^3$ (because they depend only on the contact with lines), that is, they are independent of any Euclidean structure defined on $\mathbb{R}^3$ and of the Gaussian curvature of the surface which could be induced by such a Euclidean structure.

3 Statement of results

Consider the pair of fields of asymptotic directions in the hyperbolic domain. An *asymptotic curve* is an integral curve of a field of asymptotic directions.

Left and right asymptotic and flecnodal curves. Fix an orientation in the 3-space $\mathbb{R}P^3$ (or in $\mathbb{R}^3$). The two asymptotic curves passing through a point of the hyperbolic domain of a generic smooth surface can be distinguished in a natural geometric way: One twists like a left screw and the other like a right screw. More precisely, a regularly parametrised smooth curve is said to be a *left (right) curve* if its first three derivatives at each point form a negative (resp. a positive) frame.

Proposition 1. *At a hyperbolic point of a surface one asymptotic curve is left and the other is right.*

A proof is given (for generic surfaces) in Euclidean Remark below.

The hyperbolic domain is therefore foliated by a family of left asymptotic curves and by a family of right asymptotic curves. The corresponding asymptotic tangent lines are called respectively *left* and *right asymptotic lines*.

Definition 1. The *left* (*right*) *flecnodal curve* F_l (resp. F_r) of a surface S consists of the points of the flecnodal curve of S whose asymptotic line, having higher order of contact with S, is a left (resp. right) asymptotic line.

The following statement (complement to Theorem 1) is used and implicitly proved (almost explicitly) in [16, 18]. A proof is given in section 4, see Fig. 2:

Proposition 2. *A godron separates locally the flecnodal curve into its right and left branches.*

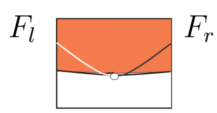

Figure 2: The flecnodal curve near a godron.

Definition 2. A *flattening* of a generic curve is a point at which the first three derivatives are linearly dependent. Equivalently, a flattening is a point at which the curve has at least 4-point contact with its osculating plane.

The flattenings of a generic curve are isolated points separating the right and left intervals of that curve.

Euclidean Remark. If we fix an arbitrary Euclidean structure in the affine oriented space $\mathbb{R}^3$, then the lengths of the vectors and the angles between vectors are defined. Therefore, for such Euclidean structure, the torsion τ of a curve and the Gaussian curvature K of a surface are defined. In this case *a point of a curve is right, left or flattening if the torsion at that point satisfies* $\tau > 0$, $\tau < 0$ *or* $\tau = 0$, *respectively*. The Gaussian curvature K on the hyperbolic domain of a smooth surface is negative. The Beltrami-Enepper Theorem states that *the values of the torsion of the two asymptotic curves passing through a hyperbolic point with Gaussian curvature* K *are given by* $\tau = \pm\sqrt{-K}$. This proves Proposition 1.

Definition 3. An *inflection* of a (regularly parametrised) smooth curve is a point at which the first two derivatives are linearly dependent. Equivalently, an inflection is a point at which the curve has at least 3-point contact with its tangent line.

A generic curve in the affine space $\mathbb{R}^3$ has no inflection. However, a generic 1-parameter family of curves can have isolated parameter values for which the corresponding curve has one isolated inflection.

Theorem 2. *Let* S *be a generic smooth surface. All smooth curves of* S *which are tangent to the parabolic curve at a godron* g *have either a flattening or an inflection at* g, *and their osculating plane is the tangent plane of* S *at* g.

The proof of Theorem 2 is given in section 4.

Corollary 1. *The godrons of a generic smooth surface are flattenings of its parabolic and flecnodal curves.*

3.1 The cr-invariant and classification of godrons

The conodal curve. Let S be a smooth generic surface. A *bitangent plane* of S is a plane which is tangent to S at least at two distinct points. The *conodal curve* D of a surface S is the closure of the locus of points of contact of S with its bitangent planes.

At a godron of S, the curve D is simply tangent to the curves P (parabolic) and F (flecnodal). This fact will be clear from our calculation of D for Platonova's normal forms of godrons.

The projective invariant. At any godron g, there are three tangent smooth curves F, P and D, to which we will associate a projective invariant:

Consider the Legendrian curves L_F, L_P, L_D and L_g (of the 3-manifold of contact elements of S, PT^*S) consisting of the contact elements of S tangent to F, P, D and to the point g, respectively (the contact elements of S tangent to a point are just the contact elements of S at that point). These four Legendrian curves are tangent to the same contact plane Π of PT^*S. The tangent directions of these curves determine four lines l_F, l_P, l_D and l_g, through the origin of Π.

Definition 4. The *cr-invariant* $\rho(g)$ of a godron g is defined as the cross-ratio of the lines l_F, l_P, l_D and l_g of Π:

$$\rho(g) = (l_F, l_P, l_D, l_g).$$

Platonova's normal form. According to Platonova's Theorem [13], in the neighbourhood of a godron, a surface can be sent by projective transformations to the normal form

$$z = \frac{y^2}{2} - x^2y + lx^4 + \varphi(x,y), \quad \text{(for some } l \neq \tfrac{1}{2}\text{)}, \tag{G1}$$

where φ is the sum of homogeneous polynomials in x and y of degree greater than 4 and (possibly) of flat functions.

Theorem 3. *Let g be a godron, with cr-invariant value ρ, of a generic smooth surface S. Put S (after projective transformations) in Platonova's normal form* $(G1)$*. Then the coefficient l equals $\rho/2$.*

It turns out that among the 2-jets of the curves in S, tangent to P at a godron, there is a special 2-jet at which "something happens". We introduce it in the following lemma.

Tangential Map and Separating 2-jet. Let g be a godron of a generic smooth surface S. The *tangential map* of S, $\tau_S : S \to (\mathbb{R}P^3)^\vee$, associates to each point of S its tangent plane at that point. The image $S^\vee$ of τ_S is called the *dual surface of S*.

Write $J^2(g)$ for the set of all 2-jets of curves of S tangent to P at g. By *the image of a 2-jet γ in $J^2(g)$ under the tangential map τ_S* we mean the image, under τ_S, of any curve of S whose 2-jet is γ. By Theorem 2, all the 2-jets of $J^2(g)$ are plane. In suitable affine coordinates, the elements of $J^2(g)$ can be identified with the curves $t \mapsto (t, ct^2, 0)$, $c \in \mathbb{R}$.

Separating 2-jet Lemma. *There exists a unique 2-jet σ in $J^2(g)$* (that we call *separating 2-jet at g*) *satisfying the following properties:*

(a) *The images, under τ_S, of all elements of $J^2(g)$ different from σ are cusps of $S^\vee$ sharing the same tangent line l_g, at $\tau_S(g)$.*

(b) *The image of σ under τ_S is a singular curve of $S^\vee$ whose tangent line at $\tau_S(g)$ is different from $l_g^\vee$.*

(c) (separating property)*: The images under τ_S of any two elements of $J^2(g)$, separated by σ, are cusps pointing to opposite directions.*

Remark 1. Once a godron with cr-invariant ρ of a smooth surface is sent (by projective transformations) to the normal form $z = y^2/2 - x^2y + \rho x^4/2 + \varphi(x,y)$, the separating 2-jet is independent of ρ: It is given by the equation $y = x^2$, in the (x,y)-plane.

For generic values of ρ the curves F, P and D are simply tangent one to the others. However, for isolated values of ρ two of these curves may have higher order of tangency and then some bifurcation occurs. We will look for the values of ρ at which 'something happens'.

Theorem 4. *Let g be a godron of a generic smooth surface S. There are six possible generic configurations of the curves F, P and D with respect to the separating 2-jet and to the asymptotic line at g. They are represented in* Fig. 3*. The actual configuration at g depends on which of the following six open intervals the cr-invariant $\rho(g)$ belongs to, respectively:* $(1, \infty)$, $(\frac{2}{3}, 1)$, $(\frac{1}{2}, \frac{2}{3})$, $(0, \frac{1}{2})$, $(-\frac{1}{2}, 0)$ *or* $(-\infty, -\frac{1}{2})$.

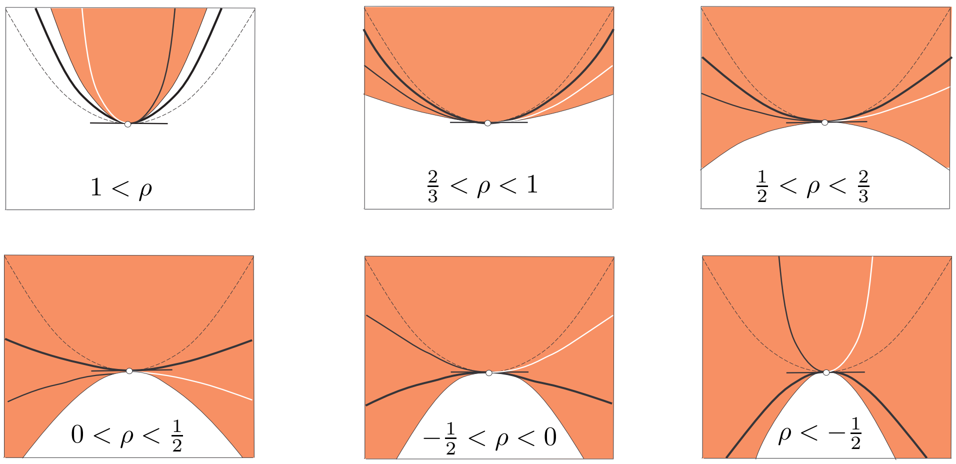

Figure 3: The configurations of the curves F (half-white half-black curves), P (boundary between white and gray domains), D (thick curves), the separating 2-jet (broken curves) and the asymptotic line (horizontal segments) at generic godrons.

3.2 The index of a godron

Definition 5. A godron is said to be *positive* or *of index* $+1$ (resp. *negative* or *of index* -1) if at the neighbouring parabolic points the half-asymptotic lines, directed to the hyperbolic domain, point towards (resp. away from) the godron. See Fig. 4.

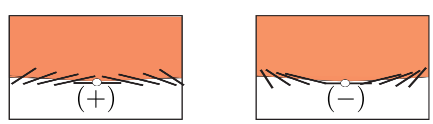

Figure 4: A positive godron and a negative godron.

The asymptotic double of the hyperbolic domain. A godron g can be positive or negative, depending on the index of the direction field, which is naturally associated to g, on the *asymptotic double* $\tilde{H}$ of the closure of the hyperbolic domain H: The *asymptotic double* is the surface in the manifold of contact elements of S, PT^*S, consisting of the field of asymptotic directions. It doubly covers the hyperbolic domain, and its projection to S has a fold singularity over the parabolic curve. For generic S the lifted direction field on $\tilde{H}$ is smooth and vanishes only over the godrons. If g is a positive godron, then the index of this direction field at its singular point equals $+1$, the point being a node or a focus; if g is negative, the index equals -1 and the point is a saddle. See Fig. 5.

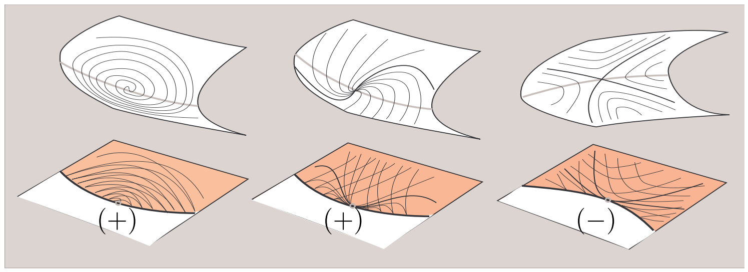

Figure 5: The double of the hyperbolic domain near a godron.

Proposition 3. *A godron g is positive (negative) if and only if the value of its cr-invariant ρ satisfies: $\rho(g) > 1$ (resp. $\rho(g) < 1$).*

Corollary of Proposition 3 and Theorem 4.
(a) *In the neighbourhood of a positive godron the hyperbolic domain is locally convex.*

(b) *There exist negative godrons for which the neighbouring hyperbolic domain is locally convex.*

(c) *In case of item (b), the flecnodal curve lies locally between P and D* (see Fig. 3)*. Moreover, we have: $\frac{2}{3} < \rho < 1$.*

Items (a) and (b) of this Corollary are due to F. Aicardi [1].

Corollary 2. *All godrons of a cubic surface in $\mathbb{R}P^3$ are negative.*

Proof. By the definitions of asymptotic curve and of flecnodal curve, any line contained in a smooth surface is both an asymptotic curve and a connected component of the flecnodal curve of that surface.

Let S be a generic algebraic surface of degree 3. At a point of the flecnodal curve, an asymptotic line has at least 4-point contact with S. Since S is a cubic surface, this line must lie completely in S. So the flecnodal curve of S consists of straight lines.

At a godron g of S, the tangent line to the parabolic curve (that is, the flecnodal curve) lies in the hyperbolic domain. Thus the neighbouring elliptic domain is locally convex. Therefore, by the above corollary, g is negative. □

3.3 Locating the left and right branches of F

Remark on the co-orientation of the elliptic domain. Each connected component of the elliptic domain is 'naturally' co-oriented: At each elliptic point the surface lies locally on one of the two half-spaces determined by its tangent plane at that point. This half-space will be called *positive half-space*. This gives a co-orientation on each connected component of the elliptic domain. By continuity, at a parabolic point a positive half-space is determined by the neighbouring elliptic points.

This simple observation has strong topological consequences. For example:

Proposition 4. *The elliptic domain of any smooth surface in the* 3*-space (Euclidean, affine or projective) can not contain a Möbius strip.*

In the neighbourhood of a godron g of a smooth surface S, we can distinguish explicitly which branch of the flecnodal curve is the right branch and which is the left one. For this, we need only to know the index of g and the co-orientation of S given by its tangent planes in the neighbouring elliptic points:

Let g be a godron of a generic smooth surface S. Take an affine coordinate system x, y, z such that the (x, y)-plane be tangent to S at g, and the x-axis be tangent to the parabolic curve at g (thus also tangent to F at g). Direct the positive z-axis to the positive half-space of the neighbouring elliptic points. Direct the positive y-axis towards the neighbouring hyperbolic domain. Finally, direct the positive x-axis in such way that any basis (e_x, e_y, e_z) of x, y, z form a positive frame for the fixed orientation of $\mathbb{R}^3$ (or of $\mathbb{R}P^3$).

So one can locally parametrise the flecnodal curve at g by projecting it to the x-axis.

Theorem 5. *Under the above parametrisation, the left and right branches of the flecnodal curve at g correspond locally to the negative and positive semi-axes of the x-axis, respectively, if and only if g is a positive godron. The opposite correspondence holds for a negative godron.*

In other words, if you stand on the tangent plane of S at g in the positive half-space and you are looking from the elliptic domain to the hyperbolic one, then you see the right (left) branch of the flecnodal curve on your right hand side if and only if g is a positive (resp. negative) godron.

3.4 Elliptic discs and hyperbolic discs of surfaces

The following global theorem holds for any generic smooth surface:

Theorem 6. *In any hyperbolic disc (bounded by a Jordan parabolic curve), there is an odd number of special hyperbolic points* (transverse crossings of the left and right branches of the flecnodal curve).

Remark 2. Theorem 5 implies that some (global) configurations of the flecnodal curve are forbidden. So, for example, there is no surface having a hyperbolic disc in which the left and right branches of the flecnodal curve do not intersect.

The cubic surfaces in $\mathbb{R}P^3$ provide examples of surfaces having elliptic discs whose bounding parabolic curve has 0, 1, 2 or 3 negative godrons: According to Segre [15], a generic cubic surface diffeomorphic to the projective plane contains four parabolic curves (each one bounding an elliptic disc) and six godrons. According to [4], Shustin had proved that the distribution of the godrons among the four parabolic curves is $6 = 0+1+2+3$. By Corollary 2, all these godrons are negative.

Proposition. *For each natural number $n \geq 4$, there exist smooth surfaces having an elliptic disc whose bounding parabolic curve has n negative godrons:*

Example 1. Let $n \geq 3$ be a natural number. For $j = 0, 1, \ldots, n-1$, take the real numbers $c_j = \cos \frac{2\pi}{n} j$ and $s_j = \sin \frac{2\pi}{n} j$. The algebraic surface given by the equation

$$z = \prod_{j=0}^{n-1} (c_j x + s_j y - 1),$$

has an elliptic disc whose bounding parabolic curve contains n godrons, all negatives.

For a parabolic curve bounding a hyperbolic disc the situation is quite different:

Proposition 5. *The sum of the indices of the godrons on the parabolic curve bounding a hyperbolic disc (of a generic surface) equals two. In particular, such parabolic curve contains a positive even number of godrons.*

Proof. Write H for the closure of the hyperbolic disc. The asymptotic double $\tilde{H}$ is a sphere. Its Euler characteristic equals 2. By Poincaré Theorem, the sum of indices of all singular points of the direction field on $\tilde{H}$ equals 2. □

3.5 Godrons and Swallowtails

Tangential Map and Swallowtails. It is well known (c.f. [14]) that under the tangential map of S the parabolic curve of S corresponds to the cuspidal edge of $S^\vee$, the conodal curve of S corresponds to the self-intersection line of $S^\vee$ (this is evident) and a godron corresponds to a swallowtail point.

Legendrian Remark. The most natural approach to the singularities of the tangential map is via Arnold's theory of Legendrian singularities [3]. The image of a Legendrian map is called the *front* of that map. The tangential map of a surface is a Legendre map, and so it can be expected to have only Legendre singularities. Thus for a surface in general position, the only singularities of its dual surface (i.e. of its front) can be: self-intersection lines, cuspidal edges and swallowtails. So the godrons are the most complicated singularities of the tangential map of a generic surface.

Definition of Front. In this paper, a *front in general position* is a surface whose singularities, and the singularities of its dual surface, are at most: self-intersection lines, cuspidal edges and swallowtails. Moreover, we require that the parabolic curve never passes through a swallowtail point (the same requirement for the dual front).

The tangential map of S sends the elliptic (hyperbolic) domain of S to the elliptic (resp. hyperbolic) domain of $S^\vee$. Thus the hyperbolic and elliptic domains of a front in general position are separated by the cuspidal edge (and by the parabolic curve). This implies that there are two types of swallowtails:

Definition 6. A swallowtail point of a generic front is said to be *hyperbolic* (*elliptic*) if, locally, the self-intersection line of that front is contained in the hyperbolic (resp. elliptic) domain.

The proofs of the following theorems show that the configurations of the curves F, P and D at a godron have a relevant meaning for the local (projective, affine or Euclidean) differential properties of the swallowtails.

Theorem 7. *The dual of a surface at a positive godron is an elliptic swallowtail. The dual of a surface at a negative godron is a hyperbolic swallowtail.*

Proof. By Proposition 3, a godron g is positive (negative) if and only if its cr-invariant satisfies $\rho(g) > 1$ (resp. $\rho(g) < 1$). By Theorem 3, $\rho(g) > 1$ (resp. $\rho(g) < 1$) if and only if the conodal curve at g lies locally in the elliptic (hyperbolic) domain. Finally, since the tangential map sends the elliptic (hyperbolic) domain to the elliptic (resp. hyperbolic) domain of the dual surface, it is evident that the conodal curve at g lies locally in the elliptic (hyperbolic) domain if and only if the dual surface is an elliptic (resp. hyperbolic) swallowtail. □

Theorem 8. *In the neighbourhood of a swallowtail point s of a front in general position, the flecnodal curve F has a cusp whose tangent direction coincides with that of the cuspidal edge. The point s separates F locally into its left and right branches. There are four possible generic configurations of F in the neighbourhood of s* (see Fig. 6):

(e) *For an elliptic swallowtail the flecnodal curve is a cusp lying in the small domain bounded by the cuspidal edge.*

There are 3 *different generic types of hyperbolic swallowtails.*

(h_1) *Each branch of the cuspidal edge is separated from the self-intersection line by one branch of the flecnodal curve.*
(h_2) *The self-intersection line lies between the two branches of the flecnodal curve and separates them from branches of the cuspidal edge. The cusp of the flecnodal curve points in the same direction as the cusp of the cuspidal edge.*
(h_3) *The cusp of the flecnodal curve and the cusp of the cuspidal edge are pointing to opposite directions.*

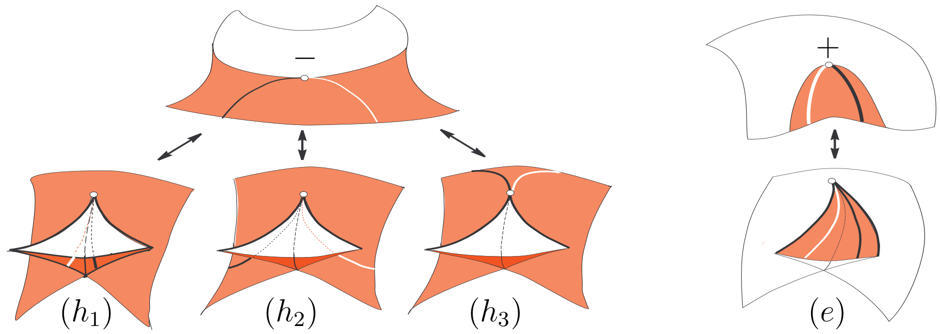

Figure 6: A godron of a smooth surface and its dual surface: a swallowtail.

4 The proofs of the theorems

Preparatory conventions and results. In the sequel, we will consider the surface S as the graph of a smooth function $z = f(x, y)$, where x, y, z form an affine coordinate system. The asymptotic directions satisfy the equation:

$$f_{xx}(dx)^2 + 2f_{xy}dxdy + f_{yy}(dy)^2 = 0.$$

For $dy = pdx$, this equation takes the form

$$G^f(x, y, p) = f_{xx} + 2f_{xy}p + f_{yy}p^2 = 0. \tag{1}$$

Equation (1) is called the *asymptote-equation* of f.

In what follows, we will assume without loss of generality that the point under consideration in the (x, y, p)-space is the origin: by a translation and a rotation in the (x, y)-plane, we can take $(x, y) = (0, 0)$ and $p = 0$, respectively.

Moreover, we will take an affine coordinate system x, y, z such that the (x, y)-plane is tangent to S at the point under consideration. Thus we will have the conditions

$$f(0, 0) = f_x(0, 0) = f_y(0, 0) = 0. \tag{2}$$

The parabolic curve of the surface $z = f(x, y)$ is the restriction of the graph of f to the discriminant curve (in the (x, y)-plane) of equation (1). That is, the parabolic curve is determined by the equations

$$G^f(x, y, p) = 0 \text{ and } G^f_p(x, y, p) = 0. \tag{$*$}$$

The fact that a godron is a folded singularity of eq. (1) implies that

$$G^f_x(0, 0, 0) = 0. \tag{$**$}$$

The conditions $(*)$ and $(**)$, at the origin in the (x, y, p)-space, imply that

$$f_{xx} = f_{xy} = f_{xxx} = 0 \tag{3}$$

at the origin in the (x, y)-plane.

The choice of a coordinate system such that the x-axis is an asymptotic direction of S at the origin is equivalent to our assumption that the point under consideration in the (x, y, p)-space is the origin.

So the x-axis is tangent to the parabolic curve at the godron.

4.1 Proof of Theorem 2

Let $\gamma(t) = (x(t), y(t), z(t))$ be a curve on S, where $z(t) = f(x(t), y(t))$, which is tangent to the parabolic curve at the origin, that is,

$$\dot{y}(0) = 0. \tag{4}$$

Since all our calculations and considerations take place at the origin $(x, y) = (0, 0)$ and at $t = 0$, we will omit to write this explicitly.

Evidently conditions (2) imply $\dot{z} = f_x\dot{x} + f_y\dot{y} = 0$. The equality

$$\ddot{z} = f_x\ddot{x} + f_y\ddot{y} + (f_{xx}\dot{x}^2 + 2f_{xy}\dot{x}\dot{y} + f_{yy}\dot{y}^2)$$

together with conditions (2), (3) and (4) imply that $\ddot{z} = 0$. This proves that the plane $z = 0$ is osculating.

Finally, the equality

$$\begin{aligned} \dddot{z} &= f_x\dddot{x} + f_y\dddot{y} + 3(f_{xx}\dot{x}\ddot{x} + f_{xy}(\dot{x}\ddot{y} + \ddot{x}\dot{y}) + f_{yy}\dot{y}\ddot{y}) \\ &\quad + f_{xxx}\dot{x}^3 + 3f_{xxy}\dot{x}^2\dot{y} + 3f_{xyy}\dot{x}\dot{y}^2 + f_{yyy}\dot{y}^3 \end{aligned}$$

together with conditions (2), (3) and (4), imply that $\dddot{z} = 0$, proving that the first three derivatives of γ at $t = 0$ are linearly dependent (all of them lie in the (x, y)-plane). So γ has a flattening or an inflection at the origin, according to the linear independence or dependence, respectively, of its first two derivatives at $t = 0$. □

4.2 Preliminary remarks and computations

We recall that Platonova's Theorem [13] implies that at a godron of a generic smooth surface S, there is an affine coordinate system such that S is locally given by

$$z = \frac{y^2}{2} - x^2y + lx^4 + \varphi(x, y) \quad \text{(for some } l \neq \tfrac{1}{2}, 0), \tag{G1}$$

where φ is the sum of homogeneous polynomials in x and y of degree greater than 4 and (possibly) of flat functions.

The information we need about S (for the proofs of our theorems) is contained in its 4-jet. The term φ in (G1) only breaks slightly the symmetry, but it does not contain additional information. Thus, in the proofs of our theorems, we will systematically use Platnova's normal form of the the 4-jet of S. The reader can easily verify that the term φ has no influence in our arguments.

First we need to calculate the curves F, P and D. For we need the second partial derivatives of the functions $f(x,y;l)=\frac{y^2}{2}-x^2y+lx^4$:

$$f_{xx}=-2y+12lx^2, \quad f_{xy}=-2x, \quad f_{yy}=1. \tag{H}$$

The asymptote-equations of the surfaces $z=\frac{y^2}{2}-x^2y+lx^4$ are therefore given by

$$G^f(x,y,p;l)=(12lx^2-2y)-4xp+p^2=0. \tag{5}$$

We are interested in the configurations of the curves F, P and D, at the godron g. According to Theorem 2, these curves have at least 4-point contact with the (x,y)-plane. We will thus consider the curves $\bar{F}$, $\bar{P}$ and $\bar{D}$, on the (x,y)-plane, whose images by f are F, P and D, respectively. These plane curves have the same 2-jet that F, P and D, respectively.

The parabolic curve. The conditions $(*)$ of **4.1** imply that $\bar{P}$ is given by the Hessian of f, $f_{xy}^2-f_{xx}f_{yy}=0$. From (H), one obtains that $\bar{P}$ is a parabola:

$$y=2(3l-1)x^2.$$

The flecnodal curve. According to [16, 18], the curve $\bar{F}$ associated to the surface $z=f(x,y)$ is obtained from the intersection of the surfaces

$$G^f(x,y,p)=0 \quad \text{and} \quad I^{G^f}(x,y,p):=(G_x^f+pG_y^f)(x,y,p)=0,$$

in the (x,y,p)-space, by the projection of this intersection to the (x,y)-plane, along the p-direction. From eq. (5) one obtains

$$I^{G^f}(x,y,p)=6(4lx-p).$$

Combining the equation $p=4lx$ with eq. (5) one obtains that $\bar{F}$ is a parabola:

$$y=2l(4l-1)x^2.$$

The conodal curve. Since Platonova's normal form is symmetric with respect to the x-direction, the bitangent planes in the neighbourhood of g are invariant under the reflection $(x,y,z)\mapsto(-x,y,z)$. Thus the points of the conodal curve satisfy $f_x(x,y;l)=0$. That is, $-2x(y-2lx^2)=0$. Thus the curve $\bar{D}$ is a parabola:

$$y=2lx^2.$$

4.3 Proof of Theorem 3

We consider the parabolas $\bar{F}$, $\bar{P}$ and $\bar{D}$ as graphs of functions $y=y(x)$. The Legendrian curves L_F, L_P and L_D in the (x,y,p)-space $J^1(\mathbb{R},\mathbb{R})$ (which is the space of 1-jets of the real functions $y(x)$ of one real variable) are tangent to the contact plane Π at the origin (parallel to the plane $y = 0$). The slope of the tangent line at the origin, of each of these Legendrian curves, equals twice the second derivative at zero of the function $y = y(x)$ associated to the corresponding parabola, that is, equals twice the coefficient of that parabola (note that the term φ in $(G1)$ will contribute with higher order terms which will have no influence on these coefficients).

The Legendrian curve consisting of the contact elements tangent to the origin is vertical. Write l_g for its tangent line. The cross-ratio of the tangent lines l_F, l_P, l_D and l_g is given in terms of the coefficients c of the parabolas $\bar{F}$, $\bar{P}$ and $\bar{D}$ by

$$\rho(g) = (l_F, l_P, l_D, l_g) = \frac{c(F) - c(D)}{c(P) - c(D)} = \frac{2l(4l-1) - 2l}{2(3l-1) - 2l} = 2l.$$

This proves Theorem 3. □

Rewriting the equations in terms of ρ. After Theorem 3, we rewrite Platonova's normal forms of the 4-jet of S at a godron and the equations of the curves $\bar{F}$, $\bar{P}$ and $\bar{D}$ in terms of the cr-invariant ρ:

$$z = \frac{y^2}{2} - x^2 y + \rho\frac{x^4}{2} \qquad (\rho \neq 1, 0). \tag{R}$$

$$y = (3\rho - 2)x^2; \tag{P}$$

$$y = \rho(2\rho - 1)x^2; \tag{F}$$

$$y = \rho x^2. \tag{D}$$

4.4 Proof of Separating 2-jet Lemma

An easy way to compute (and to see) the dual surface of $S \subset \mathbb{R}P^3$, viewed as a surface lying in the same space $\mathbb{R}P^3$, is by the 'polar duality map' with respect to a quadric. With this map, the calculations are simpler if the considered quadric is a paraboloid of revolution (see [17]). Moreover, if the surface S is the graph of a function $z = f(x, y)$, then the polar duality map with respect to the paraboloid $z = \frac{1}{2}(x^2 + y^2)$ coincides with the classical Legendre transform of f. So the dual surface has the following parametrisation:

$$\tau_f : (x, y) \mapsto (f_x(x, y),\ f_y(x, y),\ xf_x(x, y) + yf_y(x, y) - f(x, y)).$$

In the case of the surfaces S_ρ given in eq. (R), one obtains

$$\tau_\rho : (x, y) \mapsto \left(-2xy + 2\rho x^3,\ y - x^2,\ \frac{y^2}{2} - 2x^2 y + 3\rho\frac{x^4}{2}\right).$$

The images of our plane curves $\bar{F}$, $\bar{P}$ and $\bar{D}$, under τ_ρ, are exactly the flecnodal curve, the cuspidal edge and the self-intersection line of the dual surface $S_\rho^\vee$, respectively. Since $\bar{F}$, $\bar{P}$ and $\bar{D}$ are parabolas, we state the proposition:

Lemma 1. *The image of the parametrised parabola* $t \mapsto (t, ct^2)$*, under* τ_ρ*, is the parametrised space curve (lying on* $S^\vee$*):*

$$\alpha_\rho^c : t \mapsto \left(2(\rho - c)t^3,\ (c-1)t^2,\ \left(\frac{c^2}{2} - 2c + \frac{3}{2}\rho \right) t^4 \right).$$

Proof. This is a direct application of the above Legendre duality map τ_ρ. ☐

The above parametrisation implies that the curves α_ρ^c have at least 4-point contact with the (x, y)-plane at $t = 0$. In order to study the behaviour of the curves α_ρ^c for different values of c (for a fixed value of the cr-invariant ρ), we will consider their projection to the (x, y)-plane along the z-direction:

$$\gamma_\rho^c : t \mapsto \left(2(\rho - c)t^3,\ (c-1)t^2 \right). \tag{6}$$

Lemma 2. *Fix a value of the godron invariant* ρ*. The images of all parabolas* $y = cx^2$*,* $c \neq 1$*, under the composition of* τ_ρ *with the projection* $(x, y, z) \mapsto (x, y)$*, are cusps pointing down if* $c > 1$ *and pointing up if* $c < 1$*. These cusps are semi-cubic if* $c \neq \rho$ *and (very) degenerate if* $c = \rho$*.*

The image of the parabola $y = x^2$*, under the above composition, is the* x*-axis if* $c \neq \rho$ *and is the origin if* $c = \rho$*.*

Proof. Lemma 2 is evident from parametrisation (6). ☐

Remark 3. It is clear from Lemma 2 that the behaviour of the curve $\tau_\rho(\bar{F})$, $\tau_\rho(\bar{P})$ or $\tau_\rho(\bar{D})$ in $S_\rho^\vee$, changes drastically when the coefficient $c_F(\rho)$, $c_P(\rho)$ or $c_D(\rho)$, respectively, passes through the value 1.

4.5 Proof of Theorem 4

The projection of S_ρ to the (x, y)-plane, along the z-axis, is a local diffeomorphism. So the configuration of the curves F, P and D with respect to the asymptotic line and the separating 2-jet at g, on the surface S, is equivalent to the configuration of the parabolas $\bar{F}$, $\bar{P}$ and $\bar{D}$ with respect to the parabolas $y = 0 \cdot x^2 = 0$ and $y = 1 \cdot x^2$ (see Remark 1), on the (x, y)-plane.

Given a value of ρ, this configuration is determined by the order, in the real line, of the coefficients of these five parabolas:

$$c_F = \rho(2\rho - 1), \quad c_P = (3\rho - 2), \quad c_D = \rho, \quad c_{al} = 0, \quad c_\sigma = 1.$$

The graphs of these coefficients, as functions of ρ, are depicted in Fig. 7.

Using the formulas of the coefficients c_F, c_P and c_D (or from Fig. 7) one obtains by straightforward and elementary calculations that :

$$\begin{aligned}
\rho \in (1, \infty) \quad &\Longleftrightarrow \quad 1 < c_D < c_P < c_F;\\
\rho \in (\tfrac{2}{3}, 1) \quad &\Longleftrightarrow \quad 0 < c_P < c_F < c_D < 1;\\
\rho \in (\tfrac{1}{2}, \tfrac{2}{3}) \quad &\Longleftrightarrow \quad c_P < 0 < c_F < c_D < 1;\\
\rho \in (0, \tfrac{1}{2}) \quad &\Longleftrightarrow \quad c_P < c_F < 0 < c_D < 1;\\
\rho \in (-\tfrac{1}{2}, 0) \quad &\Longleftrightarrow \quad c_P < c_D < 0 < c_F < 1;\\
\rho \in (-\infty, -\tfrac{1}{2}) \quad &\Longleftrightarrow \quad c_P < c_D < 0 < 1 < c_F.
\end{aligned}$$

This proves Theorem 4. ☐

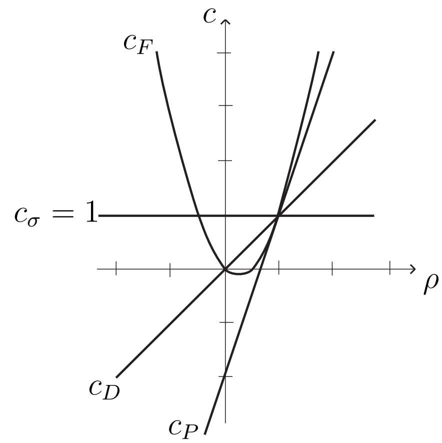

Figure 7: The coefficients c_F, c_P and c_D as functions of the invariant ρ.

4.6 Proof of Proposition 3

Consider the family of surfaces S_ρ given by eq. (R). By eq. (P), the slope m of the tangent lines of the curve $\bar{P}$ is given by:

$$m(x) = 2(3\rho - 2)x.$$

The slope p of the (double) asymptotic lines on the parabolic curve, projected to the (x, y)-plane, is given by the equation $G_p^f(x, y, p; l) = 0$, that is,

$$p(x) = 2x.$$

The points of the positive y-axis, near the origin, are hyperbolic points of the surface S_ρ of eq. (R). So the hyperbolic domain of S_ρ lies locally in the upper side of the parabolic curve. Therefore g is a positive (negative) godron if and only if the difference of slopes $(p-m)$ is a decreasing (resp. increasing) function of x, at $x = 0$.

Consequently, the equation $(p-m)'(0) = -6(\rho-1)$ implies that the godron g is positive for $\rho > 1$ and negative for $\rho < 1$, proving Proposition 3. □

4.7 Proof of Proposition 2 and of Theorem 5

As we mentioned above, the asymptotic double is a surface which doubly covers the closure of the hyperbolic domain, and whose projection to S, $\pi : \tilde{H} \to S$, has a fold singularity over the parabolic curve P. Write $\tilde{P}$ for the curve of $\tilde{H}$ which projects over P.

The surface $\tilde{H} \setminus \tilde{P}$, has two (not necessarily connected) components which are foliated by the integral curves of the lifted asymptotic direction field on $\tilde{H}$. One component corresponds to the left asymptotic curves and the other corresponds to the right ones. We call these components the *left component* and the *right component*, respectively, of $\tilde{H} \setminus \tilde{P}$.

Now the asymptote-equation $G^f(x,y,p) = f_{xx} + 2f_{xy}p + f_{yy}p^2 = 0$, associated to the surface defined by the equation $z = f(x,y)$, defines a surface in the (x,y,p)-space (that is, in the space of 1-jets $J^1(\mathbb{R},\mathbb{R})$) which realise the asymptotic double of the closure of the hyperbolic domain of the initial surface $z = f(x,y)$.

The curve $\tilde{P}$ is the criminant curve (see c.f. [2]) of the implicit differential equation $G^f(x,y,p) = 0$ and it is determined by the pair of equations $G^f(x,y,p) = 0$ and $G^f_p(x,y,p) = 0$.

Proof of Proposition 2. Write $\tilde{F}$ for the curve in $\tilde{H}$ which is the intersection of the surfaces given by the equations $G^f(x,y,p) = 0$ and $I^{G^f}(x,y,p) = 0$. As we mentioned in **4.2** the flecnodal curve correspond, in $\tilde{H}$, to the curve $\tilde{F}$. At a godron g of a generic surface, the curve $\tilde{F}$ intersects $\tilde{P}$ tansversally in $\tilde{H}$ at a point which projects over g. So the curve $\tilde{F}$ is locally separated by $\tilde{P}$. That is, $\tilde{F}$ has one branch on the left component of $\tilde{H}$ and other branch on the right component. This implies that a godron separates locally the the left and right branches of the flecnodal curve, proving Proposition 2. □

Proof of Theorem 5. Consider a godron g, with cr-invariant ρ, of a generic smooth surface. To prove Theorem 5 we need to know the values of ρ for which the curves $\tilde{P}$ and $\tilde{F}$ are tangent. For this we only need to know their tangent directions at g. So it suffices to take the 4-jet of S at g.

Take the normal form considered above

$$z = \frac{y^2}{2} - x^2y + \rho\frac{x^4}{2} \qquad (\rho \neq 1, 0). \tag{R}$$

The coordinates (x,y,z) of this normal form satisfy the conditions considered in Theorem 5.

The surface $G^f_p(x,y,p) = 0$ is the plane given by the equation $p = 2x$, which is independent of ρ. The surface $I^{G^f}(x,y,p) = 0$ is the plane given by the equation $p = 2\rho x$. So the curves $\tilde{P}$ and $\tilde{H}$ are tangent only for $\rho = 1$ (in this case we have the collapse of two godrons and, in order to have a picture of the curves F, P and D, the 5-jet of the surface have to be considered).

By Proposition 3, this implies that the side on which the right branch of the flecnodal curve will lie depends only on the index of the godron.

To see explicitly on which side of the x-axis the right branch of the flecnodal curve lies for a negative godron, it is enough to look at an example. We will take a godron of a cubic surface (whose index is -1, after Corollary 2).

The osculating plane of an asymptotic curve at a point of a surface is the tangent plane to the surface at that point. Using this fact, one defines the "osculating plane" of a straight line lying in a surface.

In this way, a segment of a straight line lying in a surface is said to be a left (right) curve, if the tangent plane to the surface along that segment twists like a left (resp. right) screw.

The x-axis is an asymptotic (and flecnodal) curve of the cubic surface $z = y^2/2 - x^2y$. One verify easily that the positive half axis is a left asymptotic curve. This proves Theorem 5. □

4.8 Proof of Theorem 6

First, we will prove Theorem 6 for the case in which the parabolic curve bounding the hyperbolic disc has only two godrons.

Lemma 3. *If the parabolic curve bounding a hyperbolic disc H (of a generic smooth surface) has exactly two godrons, then the disc H contains an odd number of special hyperbolic points.*

Write g_1 and g_2 for the godrons lying on ∂H. By Proposition 5, both g_1 and g_2 are positive godrons.

Claim 1. *If two vectors v_1 and v_2 are tangent to F at g_1 and g_2, respectively, and both are pointing from F_l to F_r, then v_1 and v_2 orient the parabolic curve ∂H in the same way.*

Proof. Since all neighbouring elliptic points of the parabolic curve ∂H belong to the same connected component of the elliptic domain, they have the same "natural" co-orientation (given by the tangent plane). Since both godrons are positive, Claim 1 follows from Theorem 5. □

Proof of Lemma 3. Write f_r for the connected component of F_r starting in g_1. Since there is only two godrons on ∂H, f_r is a segment ending in g_2. This segment separates H into two parts that we denote by A and B. The connected component of F_l starting in g_1, f_l, is also a segment ending in g_2. Claim 1 implies that if in the neighbourhood of g_1 the segment f_l lies in A, then, in the neighbourhood of g_2, it lies in B. Thus f_l crosses f_r an odd number of times.

If H contains other connected components of F_l and F_r, then there are (possibly) additional special hyperbolic points in H. Apart from f_l and f_r, the only connected components of F_l and F_r in H are closed curves (possibly empty). But the number of intersection points of a closed curve of F_r (lying H) with f_l, or with a closed curve of F_l, is even. Thus the number of intersection points of F_l with F_r is odd. □

Proof of Theorem 6. To prove the general case of Theorem 6, we will consider the closure of the hyperbolic disc, the parabolic curve ∂H and the connected components of F_l and F_r lying in H as a diagram Δ. We will prove that the number of intersection points of F_l with F_r is odd in a purely combinatorial manner. For this, we will transform the diagram Δ using two "moves", which are elementary changes (of two types) of local diagrams:

These moves are depicted in Fig. 8 where the diagram lying in the dotted box represents just an intermediate singular diagram. These moves preserve the number of intersection points of F_l with F_r.

Write G^+ and G^- for the number of positive and negative godrons on ∂H, respectively. Since the asymptotic covering of $\bar{H}$ is a sphere, $G^+ - G^- = 2$.

If $G^- = 0$, the theorem is proved in Lemma 3. So suppose $G^- > 0$.

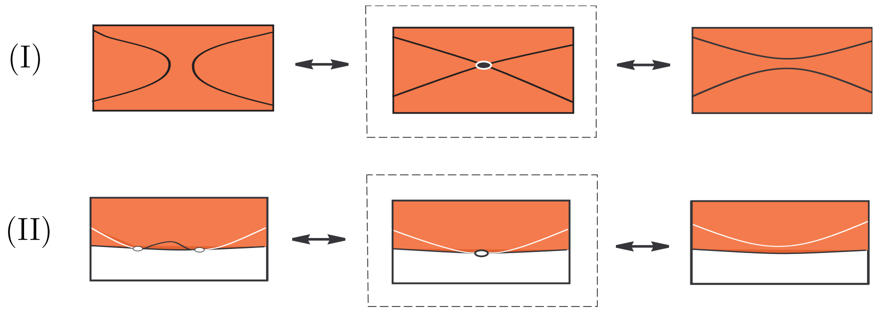

Figure 8: The two elementary moves of diagrams. The moves with opposite choice of colours of the flecnodal curve are also possible.

Consider a pair of godrons g_+ and g_- of opposite index, which are consecutive on ∂H. Two vectors tangent to ∂H and pointing from F_l to F_r, one at g_+ and the other at g_-, provide different orientations of ∂H (see Claim 1).

Consider the segment of parabolic curve joining g_+ to g_-, and which does not contain other godrons. The local diagram in the tubular neighbourhood of this segment of the parabolic curve is depicted in the left side of Fig. 9.

Step 1. In this tubular neighbourhood we deform the black curves starting in g_+ and g_-, in order to approach one to the other (the central diagram of Fig. 9). Now we apply a move of type I to this diagram in order to obtain a new diagram in which the connected component of F_r starting in g_+ will be a segment ending in g_- and lying in the tubular neighbourhood of the considered segment of the parabolic curve.

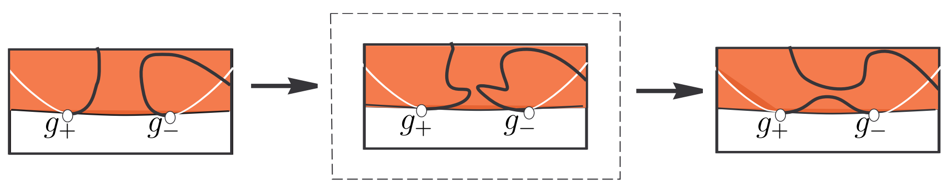

Figure 9: A deformation of F_r and a move if type I.

Step 2. Applying a move of type II to the local diagram obtained in Step 1, one obtains a new diagram without the pair of godrons g_+ and g_-.

Applying G^- times the above process, one obtains a final diagram having only two positive godrons. Theorem 6 is proved applying Lemma 3 to this final diagram (note that also the proof of Lemma 3 depends only on the combinatorial properties of the initial diagram). □

4.9 Proof of Theorem 8

To prove Theorem 8 we will use the fact that *the tangent planes of S along the flecnodal curve form the flecnodal curve of the dual surface $S^\vee$*, that is, *the* *tangential map of S sends the flecnodal curve of S onto the flecnodal curve of $S^\vee$.*

The dual of a front $\tilde{S}$ in general position at a swallowtail point s is a godron of a generic (locally) smooth surface. So Theorem 1 and Separating Lemma imply that the flecnodal curve of $\tilde{S}$ has a cusp at s having the same tangent line that the cuspidal edge of $\tilde{S}$. Now, by Proposition 2, the swallowtail point separates the flecnodal curve into its left and right branches.

The configuration formed by the flecnodal curve, the cuspidal edge and the self-intersection line of $\tilde{S}$ at the swallowtail point s, is determined by the configuration formed by the curves F, P, D and the separating 2-jet on the (locally smooth) dual surface $\tilde{S}^\vee$, at its godron $g = s^\vee$.

Since the asymptotic line is not considered in the concerned configurations, one can eliminate the number 0 (corresponding to the asymptotic line) from the list of six inequalities of the proof of Theorem 4. One obtains that the number of distinct inequalities reduces to four, corresponding to the following four open intervals for the values of ρ :

$$\begin{array}{ccc} \rho \in (1, \infty) & \Longleftrightarrow & 1 < c_D < c_P < c_F; \\ \rho \in (0, 1) & \Longleftrightarrow & c_P < c_F < c_D < 1; \\ \rho \in (-\frac{1}{2}, 0) & \Longleftrightarrow & c_P < c_D < c_F < 1; \\ \rho \in (-\infty, -\frac{1}{2}) & \Longleftrightarrow & c_P < c_D < 1 < c_F. \end{array}$$

Using Separating Lemma and the configurations of Theorem 4 (not considering the asymptotic line) one obtains that these four configurations correspond to the four configurations (of Theorem 8) for the flecnodal curve, the cuspidal edge and the self-intersection line in the neighbourhood of a swallowtail point of a front in general position. □

Remark 4. When this paper was almost finished, I visited l'École Normale Supérieure de Lyon to give a talk about the results of [18] and about this paper. Few days before my talk, E. Ghys and D. Serre have discovered the book [12] about the history of thermodynamics in Netherlands. This book describes some part of the work of Korteweg ([10, 11]) about the godrons (called plaits in [12]), the parabolic curve and the conodal curve. According to [12], Korteweg had also decribed the bifurcations of the parabolic and conodal curves when two godrons are born or disappear, for an evolving surface. So Korteweg could be considered as one of the founders of catastroph theory. His mathematical work on the theory of surfaces was motivated by the solution of thermodynamical problems (theoretical and practical). It would be interesting to know his contributions, but the references [10, 11] seems to be difficult to find. It seems, according to [12], that the mathematical comunity is not awere about these works and that the physical comunity had forget them.

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