A fast algorithm for maximum integral two-commodity flow in planar graphs

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Abstract

We consider in this note the maximum integral two-commodity flow problem in augmented planar graphs, that is with both source–sink edges added. We provide an $O(n^{3/2}\log n)$ simple algorithm for finding a maximum integral two-commodity flow in augmented planar graphs.

1. Introduction

Given an undirected graph $G = (V, E)$, a weight function $w: E \rightarrow \mathbb{Z}^+$, $q$ terminal pairs $f_i = (s_i, t_i)$ and $q$ values $r_1, \ldots, r_q$ the feasible integral multicommodity flow
The integral multicommodity flow problem was shown to be NP-complete even in the special case of the integral two-commodity flow, that is for $q = 2$, by Even, Itai and Shamir [4] and also recently in the case of integral multicommodity flow in planar augmented graphs by Middendorf and Pfeiffer [12]. If, however, the number of terminal pairs is bounded by a fixed integral, the plane integral multicommodity flow problem was shown to be solvable in polynomial time, see Sebo [13]. For other tractable cases see Frank [5].

The plane feasible half-integral multicommodity flow problem where the flows can obtain either integral or half-integral values was shown in [8] to be solvable in polynomial time and in $O(n^{3/2} \log n)$ time by Barahona [1].

The maximum integral multicommodity flow seeks, out of the feasible solutions, the one with the largest value of the total flow. We consider in this note the maximum integral two-commodity flow problem. It is well known, see for example Seymour [14], that even for the plane maximum integral two-commodity flow problem there are cases for which the maximum two-commodity flow value is greater than the maximum integral two-commodity flow value. Lomonosov [10] gave necessary and sufficient conditions for equality to hold. We provide an $O(n^{3/2})$ algorithm for finding a maximum integral two-commodity flow in augmented planar graphs.
and by two-φ if q = 2. The maximum integral multicommodity flow problem seeks, out of all feasible solutions, one with the largest value of the total flow. This value will be denoted by \( v^{(0)}(F; w) \) and by \( v^{(0)}(F; w) \), if the integrality requirement is dropped. If \( q = 2 \) the problem will be referred to as the maximum integral two-flow problem.

For a subset \( V' \subseteq V \) we denote by \( \delta(V') \) the set of edges with one end in \( V' \) and the other in \( V \setminus V' \). Any subset of edges \( C \subseteq E \) for which there exists \( V' \subseteq V \) such that \( C = \delta(V') \) will be referred to as a cut in \( G \). For a given subset of vertices \( T \subseteq V \) with \( |T| \) even, a \( T \)-join \( F \) in \( G \) is a minimal set of edges so that \( T \) is exactly the set of all vertices in \( (V, F) \) with odd valency and a \( T \)-cut is a cut \( \delta(V') \), \( V' \subseteq V \), for which \( |V' \cap T| \) is odd. A packing of \( T \)-cuts is a collection \( \mathcal{F} \) of \( T \)-cuts in \( G \) with a function \( g : \mathcal{F} \to \mathbb{R}^+ \) such that for each \( e \in E \), \( \sum \{ g(C) : e \in C, C \in \mathcal{F} \} \leq w(e) \). A packing of \( T \)-cuts will be denoted by \((\mathcal{F}, g)\). The maximum weighted packing of \( T \)-cuts problem is to find a packing of \( T \)-cuts that maximizes \( \sum \{ g(C) : C \in \mathcal{F} \} \). If, in addition, \( g \) is required to be integral, then the problem will be referred to as the maximum integral weighted packing of \( T \)-cuts. Denote by \( v^{(0)}(G, T, w) \) (respectively, \( v^{(0)}(G, T, w) \)) the value of a maximum (respectively, integral) weighted packing of \( T \)-cuts in \((G, w)\).

If \( G \) is planar we will refer to a planar embedding of \( G \) as a plane graph. The dual graph of a plane graph \( G = (V, E) \) will be denoted by \( G^* = (V^*, E^*) \) with the dual of \( e \in E \) being \( e^* \in E^* \). For \( F \subseteq E \) we denote by \( F^* \) the set of its dual edges. It is well known that a set of edges forms a cycle in \( G \) if and only if its dual edges constitute a cut in \( G^* \) and vice versa (see, e.g., [2]).

We present now a well-known result that is needed to verify the validity of the algorithm to be described in Section 3.

Let \((G, w)\) be a plane graph and \((G^*, w^*)\) its dual graph with \( w^*(e^*) = w(e) \) for all \( e^* \in E^* \). For \( F \subseteq E \), the set of demand edges, let \( F^* \) be the set of its dual edges with \( w(f) = w^*(f^*) \forall f^* \in F^* \) and \( |F^*| = q \). In addition, let \( T^* \) be the set of all vertices \((V^*, F^*) \) with odd valency and \((\mathcal{F}^*, g^*) \) a maximum integral weighted packing of \( T^* \)-cuts in \((G^*, w^*)\).

**Lemma 1** [14]. If \( F^* \) is a minimum \( T^* \)-join and \((\mathcal{F}^*, g^*) \) a maximum integral weighted packing of \( T^* \)-cuts in \((G^*, w^*)\) with value \( w^*(F^*) \), then for \( i = 1, \ldots, q \), \( \phi_i(e) = \sum \{ g^*(C^*) : \{ e^*, f^* \} \in C^*, C^* \in \mathcal{F}^* \} \forall e \in E \) and multi-φ = \((\phi_1, \ldots, \phi_q)\) is a feasible integral multicommodity flow in \((G, w)\).

3. Maximum two-flow algorithm

Based on Lemma 1 above and Lemma 2 as described in the sequel, we present herein an \( O(n \sqrt{\log n}) \) algorithm for finding a maximum integral two flow in an augmented planar graph \( G \). Our method is to transform the given maximum two-flow problem into a certain feasibility problem in the augmented graph.

Let \( \bar{G} = (\bar{V}, \bar{E}) \) be an undirected, connected, loopless graph, \( w : \bar{E} \to \mathbb{Z}^+ \) a weight function and \( \{ \bar{s}_1, \bar{s}_2, t_1, t_2 \} \subseteq \bar{V} \). Let \( f_1 = (s_1, \bar{t}_1), f_2 = (s_2, \bar{t}_2) \) be two new edges called
the demand edges and $F = \{f_1, f_2\}$. Let $G = (V, E) = (\bar{V}, \bar{E} \cup F)$ be the augmented graph of $\bar{G}$ and assume it is planar.

Let $G^* = (V^*, E^*)$ be the dual graph of $G = (V, E)$, $\bar{G}^* = (V^*, E^* \setminus F^*)$ and $(G^*, w^*_m)$ be the dual graph of $G$ with $w^*_m$, the modified dual weight function with respect to $F^*$, given by

$$w^*_m(e^*) = \begin{cases} w(e), & \text{if } e^* \in E^* \setminus F^*, \\ w^*_m(p^*_{12}), & \text{if } e^* = f^*_{1}, \\ v^{\alpha}(F; w) - w^*_m(f^*_{1}), & \text{if } e^* = f^*_{2}, \end{cases}$$

where $f^*_1 = (t^*_1, t^*_2)$ and $f^*_2 = (t^*_3, t^*_4)$ are the dual edges of $f_1$ and $f_2$ respectively, $p^*_{12}$ is a shortest path from $t^*_1$ to $t^*_4$ in $G^*$, $w^*_m(p^*_{12})$ is its length in $(G^*, w^*_m)$ and $v^{\alpha}(F; w)$ is given by the following claim:

**Claim.**

$$v^{\alpha}(F; w) = \min \{w^*_m(p^*_{12}) + w^*_m(p^*_{24}), w^*_m(p^*_{13}) + w^*_m(p^*_{23})\}.$$

**Proof.** Observe that Hu’s [7] well-known result that the value of a maximum two flow is equal to the value of a minimum cut separating the two source–sink pairs, and the fact that a set of edges forms a cycle in $G^*$ if and only if its dual edges constitute a cut in $G$ imply $v^{\alpha}(F; w) = \min \{w^*_m(C^* \setminus F^*) : F^* \subset C^*, C^* \text{ a cycle in } G^*\}$. Moreover, one can easily verify that $\min \{w^*_m(C^* \setminus F^*) : F^* \subset C^*, C^* \text{ a cycle in } G^*\} = \min \{w^*_m(p^*_{12}) + w^*_m(p^*_{24}), w^*_m(p^*_{13}) + w^*_m(p^*_{23}), w^*_m(p^*_{14}) + w^*_m(p^*_{23})\}$ and the proof is completed. □

In a similar manner one should note that the value of a minimum cut separating the first source–sink pair, i.e., $s_1$ from $t_1$, but not separating the second pair is $w^*_m(p^*_{12})$ and will be referred to as $M_1$.

Note that if $w^*_m(f^*_2) < 0$, then $v^{\alpha}(F; w) < M_1$. Hence, there is an integral flow of value $v^{\alpha}(F; w)$ from $s_1$ to $t_1$ and this flow is a maximum two flow. Thus we may assume $w^*_m(f^*_2) \geq 0$.

In [9] we have proved the following key observation:

**Lemma 2.** Let $(G^*, w^*_m)$ be the dual graph of $G$ with $w^*_m$, the modified dual weight function such that $w^*_m(f^*_2) \geq 0$, then $F^*$ is an optimal $T^*$-join in $(G^*, w^*_m)$ for $T^* = \{t^*_i \in V^* : t^*_i \text{ adjacent to exactly one edge of } F^*\}$.

**Proof.** Let $C^*_2 = p^*_{12} \cup \{f^*_2\}$; then $w^*_m(C^*_2) = 2w^*_m(f^*_2)$. Clearly $w^*_m(C^* \cap F^*) \leq w^*_m(C^* \setminus F^*)$ for each cycle $C^*$ such that $f^*_2 \notin C^*$. Let $C^*_2$ be a cycle such that $f^*_2 \in C^*_2$ and $f^*_2 \notin C^*_2$ then

(i) $w^*_m(C^*_2) \geq 2w^*_m(f^*_2)$. If not, then

$$w^*_m(C^*_1) < 2w^*_m(f^*_2) = w^*_m(C^*_2 \Delta C^*_1) \leq w^*_m(C^*_2) + w^*_m(C^*_1) < 2w^*_m(f^*_2) + 2w^*_m(f^*_1) = 2v^{\alpha}(F; w).$$
Since $F^* \subset (C^* \Delta C^*_{1})$ and $w_m^*(F^*) = w_m^*((\tilde{p}^*_1) + v_{\partial}(F; w) - w_m^*(\tilde{p}^*_2) = v_{\partial}(F; w)$ we have that $w_m^*((C^* \Delta C^*_{1}) \setminus F^*) \leq v_{\partial}(F; w) = \min\{w_m^*(C^* \setminus F^*); F^* \subset C^*, C^* a \text{ cycle in } G^*\}$, a contradiction to the definition of $v_{\partial}(F; w)$.

Obviously for each cycle $C^*$ such that $F^* \subset C^*$ and for every cycle $C^*$ such that $C^* \cap F^* = f^*$ the inequality $w_m^*(C^* \cap F^* \setminus F^*) \leq w_m^*(C^* \setminus F^*)$ holds. Moreover, (i) implies that the above inequality holds also for every cycle $C^*$ such that $C^* \cap F^* = f^*$. Therefore, for every cycle $C^*$, $w_m^*(C^* \cap F^*) \leq w_m^*(C^* \setminus F^*)$. Using the simple observation, due to Mei Gu Guan [11], that $F^*$ is an optimal $T^*$-join if and only if for every cycle $C^*$, $w_m^*(C^* \cap F^*) \leq w_m^*(C^* \setminus F^*)$ we obtain that $F^*$ is an optimal $T^*$-join.

Before describing the fast algorithm for finding a maximum integral two-flow we state the main ideas underlying it. The algorithm is based on the following observations:

1. $F^* = \{f^*_1, f^*_2\}$ is an optimal $T^*$-join, as was proven in Lemma 2.
2. Seymour [14] proved that there is an integral packing of $T^*$-cuts of value $w_m^*(f^*_1) + w_m^*(f^*_2)$ if and only if
   
   $$w_m^*(\tilde{p}^*_1) + w_m^*(\tilde{p}^*_2) \geq w_m^*(f^*_1) + w_m^*(f^*_2),$$
   $$w_m^*(\tilde{p}^*_3) + w_m^*(\tilde{p}^*_4) \geq w_m^*(f^*_1) + w_m^*(f^*_2)$$

   and if equality holds in the two inequalities, then $w_m^*(\tilde{p}^*_1) + w_m^*(\tilde{p}^*_2) + w_m^*(\tilde{p}^*_3) + w_m^*(\tilde{p}^*_4)$ is even. This last result together with observation (1) imply that there is an integral packing of $T^*$-cuts of value $w_m^*(f^*_1) + w_m^*(f^*_2)$ if and only if there is a nonnegative solution to the linear system of Step 4 of the algorithm below, and if there is no such a solution then there is a solution to the system when we replace $w_m^*(f^*_1)$ by $w_m^*(f^*_1) - 1$. Let $T^*_0 \subset T^*$ be the set of all $T^*$-cuts in $T^*$ which separates $t^*_i$ from $T^* \setminus t^*_i$. Observe that if $|T^*| = 4$, then every $T^*$-cut belongs to some $T^*_i, i = 1, \ldots, 4$. Observe, further, that if $x$ is a nonnegative solution to the linear system of Step 4 of the algorithm below, then $x$ satisfies:
   $$x_i = \sum\{g^*(C^*); C^* \in T^*_i\}, i = 1, \ldots, 4$$

   for some $(T^*_*, g^*)$, an optimal packing of $T^*$-cuts.

3. Based on the correspondence between the feasible integral multicmodity flow problem and the maximum integral weighted packing of $T$-cuts problem as given by Lemma 1, on the above observation and on Lemma 2, we show that the maximum integral two-flow problem in $(G, w)$ can be solved by solving the maximum integral weighted packing of $T^*$-cuts problem in $(G^*, w_m^*)$. This is so, because there is a maximum two flow that is integral in $(G, w)$ if and only if there is a maximum weighted packing of $T^*$-cuts which is integral in $(G^*, w_m^*)$.

**Maximum integral two-flow algorithm.**

**Input:** Planar graph $(G, w)$, $F \subseteq E$, $|F| = 2$, $w: E \rightarrow Z^+$, $w(F) = 0$.

**Output:** Maximum integral two flow.

**Step 1.** Using Booth and Lueker's algorithm [3] find $G^* = (V^*, E^*)$ the dual graph of $G = (V, E)$ and $G^* = (V^*, E^* \setminus F^*) = (V^*, E^*)$. Obtain $w_m^*$, the modified dual weight function, for all $\tilde{e}^* \in \tilde{E}^*$ by: $w_m^*(\tilde{e}^*) = w(e) \forall \tilde{e}^* \in \tilde{E}^*$. 

**Maximum integral two-commodity flow.**
Step 2. Let \( f^*_1 = (t^*_1, t^*_2) \) and \( f^*_2 = (t^*_3, t^*_4) \). For every \( v^* \in V^* \) and \( i \in \{1, 2, 3, 4\} \) find using Frederickson's shortest path algorithm [6], \( w^*_m(\tilde{p}^*_i) \), the length of a shortest path from \( v^* \) to \( t^*_i \) in \( (G^*, w^*_m) \) and \( P^* = \{ \tilde{p}^*_i \mid 1 \leq i < j \leq 4 \} \), a collection of six shortest paths from \( t^*_i \) to \( t^*_j \), with \( w^*_m(\tilde{p}^*_i) \) their lengths for \( 1 \leq i < j \leq 4 \).

Step 3 (Extension of \( w^*_m \)). Let \( u^*(F; w) = \min \{ w^*_m(\tilde{p}^*_1), w^*_m(\tilde{p}^*_2), w^*_m(\tilde{p}^*_3), w^*_m(\tilde{p}^*_4), w^*_m(\tilde{p}^*_5), w^*_m(\tilde{p}^*_6) \} \) and let \( w^*_m(f^*_1) = w^*_m(\tilde{p}^*_1), w^*_m(f^*_2) = w^*_m(\tilde{p}^*_2) \).

Step 4. Let \( V(F^*) = \{ v^* \in V^* \mid v^* \text{ adjacent to an edge in } F^* \} \). If \( |V(F^*)| \leq 3 \), let \( T^* = \{ t^*_i \in V^* \mid t^*_i \text{ adjacent to exactly one edge of } F^* \} \), and set \( x_1 = w^*_m(f^*_1) \) and \( x_2 = w^*_m(f^*_2) \). Go to Step 5.

If, on the other hand, \( |V(F^*)| = 4 \), then let \( T^* = \{ t^*_1, t^*_2, t^*_3, t^*_4 \} \) and find two-\( \phi \) by applying the following procedure. Assuming the following set of linear equations and inequalities with nonnegative \( x_i, i = 1, \ldots, 4 \).

\[
\begin{align*}
x_1 + x_2 &= w^*_m(f^*_1), \\
x_3 + x_4 &= w^*_m(f^*_2), \\
x_1 + x_3 &\leq w^*_m(\tilde{p}^*_1), \\
x_1 + x_4 &\leq w^*_m(\tilde{p}^*_3), \\
x_2 + x_3 &\leq w^*_m(\tilde{p}^*_2), \\
x_2 + x_4 &\leq w^*_m(\tilde{p}^*_4).
\end{align*}
\]

Solve the linear system by integrals \( x_1, x_2, x_3, x_4 \). If there is no integral solution then the modified system where \( w^*_m(f^*_1) \) is replaced by \( w^*_m(f^*_1) - 1 \) has an integral one.

Step 5. For every \( e^* = (u^*, v^*) \in E^* \) and every \( i \in \{1, 2, \ldots, |T^*|\} \) let

\( g^*_i(e^*) = \min \{ |w^*_m(\tilde{p}^*_m) - w^*_m(\tilde{p}^*_n)|, \max \{ 0, x_i - w^*_m(\tilde{p}^*_m), x_i - w^*_m(\tilde{p}^*_n) \} \} \).

Step 6. Set \( \phi_1 = (\varphi_1(e_1), \ldots, \varphi_1(e_m)) \) and \( \phi_2 = (\varphi_2(e_1), \ldots, \varphi_2(e_m)) \) where for every \( e_i \in E \) if \( |T| = 4 \) then

\[
\begin{align*}
\varphi_1(e_i) &= g^*_1(e^*_i) + g^*_2(e^*_i), \\
\varphi_2(e_i) &= g^*_3(e^*_i) + g^*_4(e^*_i),
\end{align*}
\]

and if \( |T| = 2 \)

\[
\begin{align*}
\varphi_1(e_i) &= g^*_1(e^*_i) \quad \text{for } j = 1, 2.
\end{align*}
\]

Step 7. Terminate with two-\( \phi = (\phi_1, \phi_2) \) a maximum integral two flow.

**Theorem 3.** The maximum integral two-flow algorithm runs in \( O(n\sqrt{\log n}) \) time and terminates with maximum integral two flow.

**Proof.** Recall from the Claim that \( u^*(F; w) = \min \{ w^*_m(\tilde{p}^*_1), w^*_m(\tilde{p}^*_2), w^*_m(\tilde{p}^*_3), w^*_m(\tilde{p}^*_4) \} \) and \( \{ \tilde{p}^*_1, \tilde{p}^*_2, \tilde{p}^*_3, \tilde{p}^*_4 \} \) and this is as claimed in Step 3. Observe that there is an optimal integral packing of \( T^* \)-cuts \( (F^*, g^*) \) that can be calculated in
Maximum integral two-commodity flow

O(n \log n) time using the xi as computed in Step 4. This solution is used to obtain the required flow. However, we do not need to have explicitly \( \mathcal{F}^* \) and \( g^* \) since we only need to know \( \varphi_1(e) \) and \( \varphi_2(e) \) for every \( e \in E \). This can be calculated as follows: By planar duality, \( \varphi_i(e) \) is equal to the sum of the weights of the \( T^* \)-cuts in \((\mathcal{F}^*, g^*)\) that contain \( e^* \) and \( f^*_i \) for \( i = 1, 2 \). One can see that if \( |T| = 4 \) then \( \varphi_1(e) = g_1^*(e^*) + g_2^*(e^*) \) and \( \varphi_2(e) = g_1^*(e^*) + g_2^*(e^*) \) while if \( |T| = 2 \) then \( \varphi_1(e) = g_1^*(e^*) \). This is done in Steps 5 and 6. Observe as well that the optimality of \((\mathcal{F}^*, g^*)\) implies the optimality of \((\varphi_1, \varphi_2)\).

It remains to verify that the time complexity of the algorithm is indeed \( O(n \sqrt{\log n}) \). This will be done by calculating the complexity of each step separately. Booth and Lueker's algorithm in Step 1 runs in \( O(n) \) time; see [3]. The complexity of Step 2 is determined by Frederickson's [6] \( O(n \sqrt{\log n}) \) algorithm for computing a shortest path in a planar graph and Step 3 runs in \( O(1) \) time. Step 4 runs in \( O(1) \) time; this is since observation (2) implies that one can check easily if the linear system has an integer solution for \( w_x(f^*_1) \) or for \( w_x(f^*_1) - 1 \) and this in turn leads to an easy way for solving the system. The time complexity of Steps 5 and 6 is \( O(|E|) = O(n) \) each. Thus, the overall time complexity of the algorithm is determined by Step 2 and equals \( O(n \sqrt{\log n}) \). ☐

Note that there is always a fractional (in fact \( \frac{1}{2} \mathbb{Z}^+ \)) solution to the linear system as presented in Step 4 of the algorithm. Hence, there is an \( O(n \sqrt{\log n}) \) time algorithm for computing a maximum (\( \frac{1}{2} \) integral) two flow in an augmented planar graph.

References