5-Arc Transitive Cubic Cayley Graphs on Finite Simple Groups

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Abstract

In this paper, we determine all connected 5-arc transitive cubic Cayley graphs on the alternating group \(A_{47}\); there are only two such graphs (up to isomorphism). By earlier work of the authors, these are the only two non-normal connected cubic arc-transitive Cayley graphs for finite nonabelian simple groups, and so this paper completes the classification of such non-normal Cayley graphs.

Key words: simple group, Cayley graph, normal Cayley graph, arc-transitive graph.

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1 Introduction

Let $G$ be a group. The subset $S$ of $G$ is called a Cayley subset if $1 \notin S$ and $S^{-1} = S$. The Cayley graph $\Gamma := \text{Cay}(G, S)$ on $G$ with respect to $S$ is defined by

its vertex set $V(\Gamma) := G$, and
its edge set $E(\Gamma) := \{\{g, sg\} \mid g \in G, s \in S\}$.

Clearly, its full automorphism group $\text{Aut}(\Gamma)$ acts transitively on the vertex set $V(\Gamma)$ since $\text{Aut}(\Gamma) \geq R(G)$, the right regular representation of $G$, and hence $\Gamma$ is vertex-transitive. We always denote $R(G)$ by $G$ for short. It is well-known that $\Gamma$ is connected if and only if $\langle S \rangle = G$.

To study the symmetry properties of Cayley graphs, we need more concepts of isomorphisms between Cayley graphs and their full automorphism groups.

Denote the automorphism group of the group $G$ by $\text{Aut}(G)$. A Cayley subset $S$ of $G$ is called a CI-subset of $G$ (where CI stands for “Cayley isomorphism”), if for any isomorphism $\text{Cay}(G, S) \cong \text{Cay}(G, T)$ of Cayley graphs there exists an $\alpha \in \text{Aut}(G)$ such that $S^\alpha = T$.

Denote $\text{Aut}(G, S) = \{\alpha \in \text{Aut}(G) \mid S^\alpha = S\}$, and we easily have $\text{Aut}(\Gamma) \geq G \rtimes \text{Aut}(G, S)$. As a matter of fact, $\text{Aut}(\Gamma) = G \rtimes \text{Aut}(G, S)$ is equivalent to $G \leq \text{Aut}(\Gamma)$ (see [?]). In this case we call the Cayley graph $\Gamma = \text{Cay}(G, S)$ normal for $G$.

Let $\Gamma$ be a graph, $G \leq \text{Aut}(\Gamma)$ and $s$ a positive integer. $\Gamma$ is said to be $(G, s)$-arc transitive, if $G$ acts transitively on the set of $s$-arcs of $\Gamma$, where an $s$-arc is a sequence $(v_0, v_1, \cdots, v_s)$ in $V(\Gamma)$ satisfying $(v_{i-1}, v_i) \in E(\Gamma)$ and $v_{i-1} \neq v_{i+1}$ for all $i$. In particular, $(\text{Aut}(\Gamma), s)$-arc transitive is called $s$-arc transitive, and 1-arc transitive is simply called arc-transitive.

Sabidussi gave a construction for all vertex-transitive (not only Cayley) graphs by using group-theoretic method.

Let $G$ be a finite group and $T$ a subgroup of it. Let $D$ be a union of several double cosets of $T$ satisfying $D^{-1} = D$. He defined a graph $\Gamma$ with vertex set $V(\Gamma) = [G : T]$, the set of all right cosets of $T$, and edge set $E(\Gamma) = \{\{Tg, Tdg\} \mid g \in G, d \in D\}$. This graph is called the Sabidussi cosets graph of $G$ with respect to $T$ and $D$, denoted by $\text{Sab}(G, T, D)$.

Obviously, $\Gamma$ is connected if and only if $\langle D \rangle = G$. It is easy to check that $\text{Sab}(G, T, D)$ is $G$-arc transitive if and only if $D = TdT$ (a single double coset) for some $d \in G$. We always denote $\text{Sab}(G, T, TdT)$ by $\text{Sab}(G, T, d)$ for short.
In fact, any vertex-transitive graph $\Gamma$ is the Sabidussi coset graph of its full automorphism group $\mathcal{A} := \text{Aut} (\Gamma)$ with respect to $T = A_v$, the stabilizer of any vertex $v$, and $D := \{ \alpha \in \mathcal{A} \mid \{v, v^\alpha\} \in E(\Gamma)\}$, which is a union of several double cosets of $T$.

Let $P(G)$ be the right multiplication action of $G$ on $[G : T]$. Since $\text{Aut} (\text{Sab}(G, T, D)) \geq P(G)$, all Sabidussi coset graphs are vertex-transitive. If $T$ is core-free, that is $\cap_{g \in G} T^g = 1$, then $P(G) \cong G$. We always denote $P(G)$ by $G$.

Regarding connected cubic $s$-arc transitive graphs, the first important result due to Tutte ([?, Theorem 18.6]) claims that there is no finite $s$-arc transitive cubic graphs for $s > 5$. Also, it is easy to check that for normal cubic $s$-arc transitive Cayley graphs, we have $s \leq 2$. So, if a connected cubic Cayley graph is $s$-arc transitive $s > 2$, then it must be nonnormal.

Much excellent work has dealt with arc-transitive Cayley graphs on finite non-abelian simple groups. For example, in [?, Theorem 7.1.3], Li proved that all connected cubic arc-transitive Cayley graphs are normal except for the following exceptions listed below:

$$\mathbf{A}_5, \mathbf{PSL}_2(11), \mathbf{M}_{11}, \mathbf{A}_{11}, \mathbf{M}_{23}, \mathbf{A}_{23}, \text{ and } \mathbf{A}_{47}.$$

In [?] we proved that the only exception is $\mathbf{A}_{47}$. For all other groups listed above, we proved that their connected cubic arc-transitive Cayley graphs are normal. There we also constructed a connected 5-arc transitive cubic Cayley graph for $\mathbf{A}_{47}$.

The purpose of this paper is to classify all connected 5-arc transitive cubic Cayley graphs on the alternating group $\mathbf{A}_{47}$. By the remarks above, it is also a classification of connected 5-arc transitive cubic Cayley graphs on finite simple groups.

The rest of this paper is organized as follows. After giving some preliminary results in §2, we construct all connected 5-arc transitive cubic Cayley graphs on $\mathbf{A}_{47}$ in §3, then in the next section we determine the isomorphisms between them, and finally we complete the classification in the last section.

### 2 Preliminaries

The first lemma is about the relation between Sabidussi coset graphs and Cayley ones.
Lemma 2.1 (1) Let $\Gamma := \text{Cay}(G, S)$ be a Cayley graph, and $A := \text{Aut}(\Gamma)$. Then the vertex-stabilizer $A_1$ is a complement of $G$ in $A$, where $1$ is the identity of $G$, and we have $\Gamma \cong \Gamma := \text{Sab}(A, A_1, A_1SA_1)$. In particular, there exists an $s \in S$ such that $A_1SA_1 = A_1sA_1$ when $\Gamma$ is arc-transitive.

(2) Conversely, let $\overline{\Gamma} := \text{Sab}(A, T, D)$ be a Sabidussi coset graph and $G$ a complement of $T$ in $A$. Denote $S = G \cap D$. Then the Cayley graph $\Gamma := \text{Cay}(G, S)$ is isomorphic to $\overline{\Gamma}$, and hence $|S| = |D : T|$. In particular, $S$ contains an involution of $G$ if the valency of $\Gamma$ is odd. Also $\Gamma$ is arc-transitive if $D$ is a single double coset of $T$.

Proof (1) Obvious.

(2) Since $A = GT$ and $G \cap T = 1$, each coset in $[A : T]$ has only an element of $G$ as its representative. We define a bijection $\sigma$ from $\Gamma$ to $\overline{\Gamma}$ such that $g \sigma := Tg \in V(\overline{\Gamma}) = [A : T]$ for all $g \in V(\Gamma) = G$. Since

$$\{g, g'\} \in E(\Gamma) \iff g'g^{-1} \in S = G \cap D \iff \{Tg, Tg'\} \in E(\overline{\Gamma})$$

for any $g, g' \in G$, we find $\Gamma \cong \overline{\Gamma}$. \qed

By results of [?] and [?] (respectively) we easily have

Lemma 2.2 Let $G \cong A_{47}$ and $\Gamma := \text{Cay}(G, S)$ a connected 5-arc transitive cubic Cayley graph for $G$. Denote $A = \text{Aut}(\Gamma)$. Then the following hold:

(1) Let $K$ be a subgroup of $R$. Then there are $|R : K|$ $K$-orbits with length $|K|$. If $g \in R$ normalizes $K$, then $g$ induces a permutation action on the set of $K$-orbits. In particular, the action is transitive if $(K, g) = R$;

(2) If $n = 4$ and $R = \langle a \rangle \times \langle b \rangle \cong \mathbb{Z}_2^2$ such that $a^s = a$ and $b^s = ab$, then $s$ is an odd permutation. In particular, $s$ is a transposition if $s$ is an involution;

The next lemma will play a very important role in proving our theorem.

Lemma 2.3 Suppose that $R$ is a regular subgroup on $\Omega := \{1, 2, \cdots, n\}$ and $s \in S_n$. The following hold:

(1) Let $K$ be a subgroup of $R$. Then there are $|R : K|$ $K$-orbits with length $|K|$. If $g \in R$ normalizes $K$, then $g$ induces a permutation action on the set of $K$-orbits. In particular, the action is transitive if $(K, g) = R$;

(2) If $n = 4$ and $R = \langle a \rangle \times \langle b \rangle \cong \mathbb{Z}_2^2$ such that $a^s = a$ and $b^s = ab$, then $s$ is an odd permutation. In particular, $s$ is a transposition if $s$ is an involution;
(3) If \( n = 8 \) and \( R = \langle a \rangle \times \langle b \rangle \cong D_8 \) such that \( \langle a \rangle^s = \langle a \rangle \) and \( b^s = ab \), then \( s \) is an odd permutation. In particular, \( s \) is a product of three disjoint transpositions if \( s \) is an involution.

**Proof** (1) Clearly, \( K \) is semiregular on \( \Omega \) and each \( K \)-orbit has the same length \( |K| \). Since \( |\Omega| = |K| \), then there are \( |R : K| \) \( K \)-orbits on \( \Omega \). Let \( \Delta \) be a \( K \)-orbit. If \( g \) normalizes \( K \), then \( \Delta^g \) is an orbit of \( K^g = K \), and hence \( g \) may act on the set of \( k \)-orbits. Furthermore, if \( (K, g) = R \), which is transitive on \( \Omega \), then \( \langle g \rangle \) is also transitive on the set of \( K \)-orbits.

(2) As \( R = \langle a, b \rangle \) is regular on \( \Omega \), we may let \( a = (12)(34), b = (13)(24) \in A_4 \). Since \( s \) commutes with \( a \) but not with \( b \), then \( s \) is not a 3-cycle on \( \Omega \) and \( s \notin \langle a, b \rangle \), either. But \( \langle a, b \rangle \) contains all involutions of \( A_4 \), then \( s \) is either \( (i_1i_2) \) or \( (i_1i_2i_3i_4) \), and hence \( s \) is an odd permutation. In particular, \( s = (i_1i_2) \) if its order is 2.

(3) Being semiregular on \( \Omega \), \( \langle a \rangle \) has two orbits with length 4, denoted by \( \Delta_1, \Delta_2 \). Without loss of generality, we may let \( \Delta_1 = \{1, 2, 3, 4\}, \Delta_2 = \{5, 6, 7, 8\} \), and \( a = (1234)(5678) \).

Since each of \( b \) and \( ab \) normalizes \( \langle a \rangle \), we find \( \Delta_1^b = \Delta_1^{ab} = \Delta_2 \) by (1). This means that as two permutations with order 2 on \( \Omega \), \( b = (1i_1)(2i_2)(3i_3)(4i_4) \) and \( ab = (1j_1)(2j_2)(3j_3)(4j_4) \) where \( i_r, j_r \in \Delta_2 \). Since \( a^b = a^{ab} = a^{-1} \), both arrangements \( i_1i_2i_3i_4 \) and \( j_1j_2j_3j_4 \) on \( \Delta_2 \) are in the set \( \{8765, 7658, 5687, 5876\} \).

Clearly \( s \) also normalizes \( \langle a \rangle \), and hence \( \Delta_1^s = \Delta_1 \) or \( \Delta_2 \). We deal with these two cases separately:

**Case 1:** \( \Delta_1^s = \Delta_1 \). We may let \( r^s = k_r \) where \( r, k_r \in \Delta_1 \). Then \( k_1k_2k_3k_4 \) is an arrangement on \( \Delta_1 \). Note that \( a^s = ab \) and \( b = (1i_1)(2i_2)(3i_3)(4i_4) \), only \( (r i_r)^s = (k_r i_r)^s \) is a transposition of \( ab \), and hence \( i_r^s = j_k \). Thus \( s = (1^{234} i_1 i_2 i_3 i_4 k_1 k_2 k_3 k_4 i_1 i_2 i_3 i_4 k_1 k_2 k_3 k_4)^s \).

Denote \( u := \frac{1^{234} i_1 i_2 i_3 i_4 k_1 k_2 k_3 k_4}{i_1 i_2 i_3 i_4 k_1 k_2 k_3 k_4}, w := u^{ab} \) and \( x := s(uw)^{-1} \). We will finish the proof of case 1 by the following steps:

(i) \( w = (1^{234} i_1 i_2 i_3 i_4 k_1 k_2 k_3 k_4) \).

In fact, since \( w = u^{ab} = abuab \), then for any \( r \in \Delta_1, r^w = (r^{ab})uab = j_r^{ab} = j_r^s = r \), and for any \( j_r \in \Delta_2, j_r^w = j_r^{ab} = r^{ab} = k_r^{ab} = j_k \).

(ii) \( uw = (1^{234} i_1 i_2 i_3 i_4 k_1 k_2 k_3 k_4) \), and \( uw \) commutes with \( ab \).

First, \( uw = (1^{234} i_1 i_2 i_3 i_4 k_1 k_2 k_3 k_4) = (1^{234} i_1 i_2 i_3 i_4 k_1 k_2 k_3 k_4) \). Then \((ab)^{uw} = ab \) since \((r, j_r)^{uw} = (k_r, j_k) \) is still a transposition of \( ab \).
(iii) \[ j_r = i_{r+1}, \] where \( 4 + 1 \equiv 1 \pmod{4} \).

In fact, for any \( r \in \Delta_1 \), \( j_r = r^{ab} = (r^a)^b = (r + 1)^b = i_{r+1} \).

(iv) \( x = (i_1i_2i_3i_4) \).

In fact, by (ii) and (iii), \( x = s(uw)^{-1} = (1 2 3 4 i_1 i_2 i_3 i_4)(k_1 k_2 k_3 k_4 j_1 j_2 j_3 j_4) = (1 2 3 4 i_1 i_2 i_3 i_4) = (i_1i_2i_3i_4) \).

(v) \( s \) is an odd permutation.

In fact, since \( uw = u(u^{ab}) \) is obviously an even permutation, then by (iv), \( s = x(uw) = (i_1i_2i_3i_4)uw \) is an odd permutation.

Case 2: \( \Delta_1^s = \Delta_2 \). In this case, \( \Delta_1^{bs} = \Delta_1 \). Then \( bs \), which obviously satisfies the assumption as \( s \) does in case 1, is an odd permutation, and so is \( s \).

In particular, as an odd permutation with order 2 of \( S_8 \), \( s \) is a transposition or a product of three disjoint transpositions. If \( s \) is a transposition, then it has 6 fixed points on \( \Omega \), and hence \( b = (1i_1)(2i_2)(3i_3)(4i_4) \) and \( ab = (1i_2)(2i_3)(3i_4)(4i_1) \) are not conjugate under \( s \). It follows that \( s \) is a product of three disjoint transpositions.

\[ \square \]

3 How to construct the graphs

In this section, we construct all connected 5-arc transitive cubic Cayley graphs on \( A_{47} \).

Let \( G \) be a finite nonabelian simple group and \( \Gamma := \text{Cay}(G, S) \) a connected arc-transitive cubic Cayley graph. We know from [5] that \( \Gamma \) is nonnormal for \( G \) if and only if \( G \cong A_{47} \), and \( A := \text{Aut}(\Gamma) \) is isomorphic to \( A_{48} \). Recall that this means, (1) \( \Gamma \) is 5-arc transitive; (2) the vertex-stabilizer \( T \) of \( A \) is isomorphic to \( S_4 \times \mathbb{Z}_2 \); (3) there exists an involution \( s \) in \( G \) such that \( |T : T \cap T^s| = 3 \), \( \langle T, s \rangle = A \), and furthermore the coset graph \( \overline{\Gamma} := \text{Sab}(A, T, s) \) is isomorphic to \( \Gamma \).

To construct all these Cayley graphs, we first let \( A \cong A_{48} \), and \( A = GT \), where \( G \cong A_{47} \) and \( T \cong S_4 \times \mathbb{Z}_2 \). Secondly, we choose involutions \( s \) of \( G \) which satisfy that \( |T : T \cap T^s| = 3 \) and \( \langle T, s \rangle = A \). Then we examine the structure of \( T \) and its subgroups with index 3.

(A) The structure of \( T \)

We will find out generators of \( T \). Without loss of generality, we may let \( T = S_4 \times \mathbb{Z}_2 \). Note that \( A_4 \leq S_4 \), we take \( K \in \text{Syl}_2(A_4) \) which is a Klein four-group
and $L \in \text{Syl}_3(A_4)$ which is a cyclic group of order 3 such that $A_4 = K \times L$. Thus there exist $b \in K$, $t \in L$ such that $K = \{1, b, b^2, b^3\}$ where $b^2 = bb^4$, and hence $A_4 = \langle b, t \rangle$. Note that $|S_4 : A_4| = 2$, then we may take an element $a$ with order 4 of $S_4$ such that $S_4 = \langle A_4, a \rangle = \langle a, b, t \rangle$.

Consider the relations between $a$, $b$ and $t$. Note that $a^2 = b^{i-1} = bb^4$, and accordingly $b^t = a^2b$, $(a^2)^t = b$. Further, $D := K \langle a \rangle$ is an order 8 Sylow 2-subgroup of $S_4$, and then $a^b = a^{-1}$. That is $D = \langle a \rangle \times \langle b \rangle \cong D_8$, and accordingly each Sylow 2-subgroup of $T$ is isomorphic to $D_8 \times Z_2$.

By the way, $T$ has 4 Sylow 3-subgroups all of which are in $A_4$ since $A_4 \triangleleft S_4 \triangleleft T$ and $|T : A_4|$ is divisible by 3, and hence all 8 elements of order 3 of $T$ exactly make up the right coset union $Kt \cup Kt^{-1}$.

Represent $a^i$ by $a$, $b$ and $t$. First, $a^i$ has order 4 and is not in $A_4$, and hence in the left coset $aA_4 = S_4 \setminus A_4$. Secondly, $a^i \inDt \cup Dt^{-1}$ since $t$ does not normalize $D$. Thus we may let $a^i = a^tb^{i^k}$ where $i = \pm 1$, $j = 0$ or $1$, $k = \pm 1$. We claim that $i = k = -1$ and $j = 0$, namely $a^i = a^{-t}t^{-1}$. In fact, (i) If $k = 1$, then $t^{-1}at = a^tb^t$ and $t^{-1} = a^tb^{-1}a^{-1} \in D$, a contradiction. Hence $k = -1$. (ii) If $i = 1$, then $b = (a^2)^t = (a^i)^2 = (ab^t)(ab^t)^{-1}(ab^t(1-b^t))^{-1}b^t = a(a^t)(b^t)^{-1} = a^{-1}a$. In this case, if $j = 0$, then $b = a^2$, a contradiction, and if $j = 1$, $b = aa^{-1}b^2 = a^2$, still a contradiction. Hence $i = -1$. (iii) If $j = 1$, we may let $t' := a^2t \in Kt$, then $t'$ is still an element of order 3, and $a^t = a = a^{-1}b^{-1} = a^{-1}b^tt^{-1} = a^{-1}b(a^2)^t = a^{-1}b(a^2)^tt^{-1} = a^{-1}b(a^2)^tt^{-1} = a^{-1}b(a^2)^tt^{-1} = a^{-1}b(a^2)^tt^{-1} = a^{-1}b(a^2)^tt^{-1}$. We may replace $t$ by $t'$ such that $a^t = a^{-1}t^{-1}$.

Finally, let $Z_2 = \langle c \rangle$. Then $T = \langle a, b, c, t \mid a^4 = b^2 = c^2 = t^3 = 1, a^b = a^{-1}, a^c = a, b^c = b, a^t = a^{-1}t^{-1}, b^t = a^2b, c^t = c \rangle$.

(B) The subgroup of index 3 in $T$

Let $R$ be a subgroup of $T$ with index 3. Clearly, $R$ is a Sylow 2-subgroup of $T$, and hence by (A) $R$ is isomorphic to $D_8 \times Z_2$. Without loss of generality, we may let $R = \langle a, b, c \rangle$, where $a$, $b$, $c$ are the same as in (A). We are easy to check that $R$ has 7 subgroups of order 8 as follows:

- **Type 1**, $Z_2^3 : \langle a^2, b, c \rangle$ and $\langle a^2, ab, c \rangle$;
- **Type 2**, $Z_4 \times Z_2 : \langle a, c \rangle$;
- **Type 3**, $Z_4 \times Z_2 : \langle a, b \rangle, \langle a, bc \rangle, \langle ac, b \rangle, \langle ac, bc \rangle$.

We shall choose an involution $s$ of $G$ such that $|T : T \cap T^s| = 3$ and $\langle T, s \rangle = A$. Clearly, those involutions $s$ which normalize one of Sylow 2-subgroups of $T$ except
for \( T \) itself must satisfy that \( |T : T \cap T^*| = 3 \). Note that all three Sylow 2-subgroups of \( T \), including \( R \), are conjugate to each other. Without loss of generality, we may let \( R \) be normalized by \( s \), and accordingly \( R = T \cap T^* \). However, we are difficult to check if \( \langle T, s \rangle = A \). But \( \langle T, s \rangle \) should be a simple group, then \( s \) does not normalize any nontrivial normal subgroup of \( T \). Then \( s \) belongs to the set defined by

\[
\Pi_G(R, T) := \{ s \in N_G(R) \mid o(s) = 2 \text{ and } \forall 1 \neq K \trianglelefteq T, K^s \neq K \},
\]

where \( T = \langle a, b, c, t \rangle \) and \( R = \langle a, b, c \rangle \) defined in (A) and (B).

We still denote \( S_4 = \langle a, b, t \rangle \), its subgroup \( D = \langle a, b \rangle \) as in (A) and denote \( \langle a, c \rangle \) by \( K \).

The next lemma shows us some properties of involutions in \( \Pi_G(R, T) \):

**Lemma 3.1** For \( s \in \Pi_G(R, T) \), the following hold:

1. \((a^2)^s = a^2\) and \(c^s = a^2c\);
2. \(b^s = ab\);
3. \(a^s = a^{-1}\) and \((ac)^s = ac\);
4. \((c^j D)^s = c^j D\) and \((b^j K)^s = b^j K\) for every \( j \in \{0, 1\} \).

**Proof** (1) First, by (B), \( s \) normalizes \( K = \langle a, c \rangle = \{1, a, a^2, a^3, c, ac, a^2c, a^3c\} \) and hence \( \langle a \rangle^s \) equals one of \( \langle a \rangle \) and \( \langle ac \rangle \). In any case, \((a^2)^s = a^2\). But \( s \) normalizes neither \( \langle a^2, b, c \rangle \) nor \( \langle c \rangle \) since \( \langle a^2, b, c \rangle, \langle c \rangle \triangleleft T \) and \( c^s = a^2c \).

(2) Since \((a^2, b, c)^s = \{1, a^2, b, c, a^2b, a^2c, bc, a^2bc\}^s = (a^2, ab, c) = \{1, a^2, ab, c, a^2b, a^2c, abc, a^3bc\} \) then \( b^s = a^\pm 1 bc^j \) (\( j \in \{0, 1\} \)). Let \( a' = a^\pm 1 c^j \). Then \( R = (\langle a' \rangle \times \langle b \rangle) \times \langle c \rangle \) and \( b^s = a'b \). We may replace \( a \) by \( a' \), and hence (2) holds.

(3) We know from (1) that \( \langle a \rangle^s = \langle a \rangle \) or \( \langle ac \rangle \). If \( \langle a \rangle^s = \langle ac \rangle \), then \( a^s = a^\pm 1 c \) and further \( a = a^s = (a^\pm 1 c)^s = (a^s)^\pm 1 c^s = (a^\pm 1 c)^\pm 1 (a^2c) = (ac)^\pm 1 (ac)(a^2c) = (c^2)(a^2c) = a^3 \neq a \). This contradiction shows that \( \langle a \rangle^s = \langle a \rangle \) and hence \( a^s = a^\pm 1 \). If \( a^s = a \), then \( b = b^s = (ab)^s = a^s b^s = a(ab) = a^2b \neq b \). This contradiction shows that \( a^s = a^{-1} \) and consequently \( (ac)^s = a^s c^s = a^{-1} a^2 c = ac \).

(4) Obvious. \( \square \)

Of course, we can not confirm if \( \langle T, s \rangle = A \) for \( s \in \Pi_G(R, T) \) and so we need some additional assumptions to help us choose \( s \).

With the right multiplication permutation representation of \( A \) on \( \Omega := [A : G] \) being faithful, we may assume \( A = \text{Alt}(\Omega) \). As a complement of \( G \) in \( A, T \) is a
regular subgroup on $\Omega$. Its subgroups $R$, $S_4$ and $D$ are semiregular. So there are $|T : R| = 3$ $R$-orbits denoted by $\Omega_0, \Omega_1, \Omega_2$, and $|T : S_4| = 2$ $S_4$-orbits denoted by $\Delta_0, \Delta_1$ in $\Omega$. By Lemma ?? (1) and $T = S_4 \rtimes \langle c \rangle$, $c$ interchanges $\Delta_0$ and $\Delta_1$. Furthermore, for all $i \in \{0, 1, 2\}$ and $j \in \{0, 1\}$, $\Omega_{ij} := \Omega_i \cap \Delta_j$ are 6 $D$-orbits. $c$ interchanges $\Omega_{i0}$ and $\Omega_{i1}$ for each $i$.

The $R$-orbits $\Omega_i (i = 0, 1, 2)$, $S_4$-orbits $\Delta_j (j = 0, 1)$ and their intersection orbits $\Omega_{ij}$ are depicted in following figure:

According to the action of $s \in \Pi_G(R, T)$ on $S_4$-orbits, we say $s$ is of the first type if it fixes each $\Delta_j$ setwise and $s$ is of the second type if it is not of the first type.

Since $s$ normalizes $R$, $s$ may act on the $R$-orbit’s set $\{\Omega_0, \Omega_1, \Omega_2\}$. As an involution, $s$ must fix one of them. Without loss of generality, we always assume $\Omega_0^s = \Omega_0$. Thus $\Omega_1^s = \Omega_1$ or $\Omega_2$. Analogously, from Lemma ?? (4) $s$ normalizes $D$ and hence fixes the sets $\{\Omega_{00}, \Omega_{01}\}$ and $\{\Omega_{10}, \Omega_{11}, \Omega_{20}, \Omega_{21}\}$ of $D$-orbits setwise. But $s$ does not normalize $S_4$, so the equation $\Delta_0^s = \Delta_1$ may be not true even if $s$ is of the second type.

The next lemma is related to the type of the involutions in $\Pi_G(R, T)$.

**Lemma 3.2** Let $s \in \Pi_G(R, T)$. Then the following hold:

1. The element $cs$ has order 4, and has no fixed point in $\Omega$, and contains no transposition on $\Omega$. In fact, $cs$ is a product of 12 disjoint 4-cycles;
2. If $s$ fixes some $\Omega_i$ setwise, then $s$ fixes each of $\Omega_{i0}$ and $\Omega_{i1}$ setwise, and
further $s^{\Omega_i}$ is a product of 6 disjoint transpositions on $\Omega_i$. In particular, by our assumption, so is $s^{\Omega_0}$;

(3) The involution $s$ does not interchange $\Omega_{i0}$ and $\Omega_{i1}$ for any $i \in \{0, 1, 2\}$;

(4) If $s$ fixes $\Omega_1$ or $\Omega_2$ setwise, then $s$ fixes every $\Omega_{ij}$ setwise and hence $s$ is of the first type;

(5) If $s$ interchanges $\Omega_{10}$ and $\Omega_{20}$, then $s$ is of the first type.

Proof

(1) Since $(cs)^2 = c(cs) = c(a^2c) = a^2 \neq 1$ and $(cs)^4 = (a^2)^2 = 1$, we find $o(cs) = 4$. In particular, $(cs)^2 = a^2$ has no fixed point in $\Omega$ so that $cs$ has no fixed point and contains no transposition. Hence $cs$ with order 4 is a product of 12 disjoint 4-cycles.

(2) Since $s$ fixes $\Omega_i$, then $\Omega_{i0}^s = \Omega_{i0}$ or $\Omega_{i1}^s = \Omega_{i1}$. Assume $\Omega_{i0}^s = \Omega_{i1}$. Then $\Omega_{i0}^{cs} = \Omega_{i0}$. As components of $\Omega_{i0}$, $a' := a^{\Omega_{i0}}, b' := b^{\Omega_{i0}}$ and $s' := (cs)^{\Omega_{i0}}$ satisfy, by easily checking, the assumption of Lemma ?? (3), that is, $\langle a' \rangle \times \langle b' \rangle \cong D_8$, $\langle a' \rangle s' = \langle a' \rangle$ and $(b')^{s'} = a'b'$. So $s' = (cs)^{\Omega_{i0}}$ is an odd permutation on $\Omega_{i0}$. By (1), $cs$ with order 4 has no fixed point and contains no transposition, then $s' = (cs)^{\Omega_{i0}}$ is a product of 2 disjoint 4-cycles on $\Omega_{i0}$, contradicting the oddness of it. Hence $s$ fixes each of $\Omega_{i0}$ and $\Omega_{i1}$ setwise. In this case, using the way of dealing with $(cs)^{\Omega_{i0}}$ on $\Omega_{i0}$ above to deal with $s^{\Omega_{i0}}$ on $\Omega_{i0}$ and $s^{\Omega_{i1}}$ on $\Omega_{i1}$, we finally have that $s^{\Omega_i} = s^{\Omega_{i0}} s^{\Omega_{i1}}$ is a product of 6 disjoint transpositions on $\Omega_i$.

(3) and (4) hold by (2).

(5) Since $s$ always fixes $\Omega_0$ setwise, and by (2), we find $s$ also fixes $\Omega_{00}$ setwise. It follows that $s$ fixes $\Delta_0$ setwise, and consequently $s$ fixes $\Delta_1$ setwise. So $s$ is of the first type.

Theorem 3.3

Let $s \in \Pi_G(R, T)$. Then the following statements are equivalent:

(1) $\langle T, s \rangle = A$;

(2) The involution $s$ is of the second type;

(3) The involution $s$ interchanges $\Omega_{10}$ and $\Omega_{21}$.

Proof

(1)⇒(2): Since $\langle T, s \rangle = A$ is primitive on $\Omega$ but each $\Delta_j$ is obviously nonprimitive block of $T$, then $s$ does not fix $\Delta_j$ anymore, namely $s$ is of the second type.

(2)⇒(3): By Lemma ?? (3), (4) and (5), $s$ must interchange $\Omega_{10}$ and $\Omega_{21}$.

(3)⇒(1): See Corollary ?? later. 

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Theorem ?? shows that the Sabidussi coset graph \( \Gamma := \text{Sab}(A, T, s) \) is a connected 5-arc transitive cubic graph if and only if the involution \( s \) is of the second type.

4 Finding the graphs

In this section, we will find out all connected 5-arc transitive cubic Cayley graphs for \( A_{47} \).

We first denote each coset \( G_a \in \Omega = [A : G] \) by \( a \). Then \( \Omega = T := \{ \overline{h} \mid h \in T \} \) and \( G \) is the point stabilizer of \( 1 \) in \( A \). For any subgroup \( L \) of \( T \) and its left coset \( hL \), the set \( hL \) is obviously an \( L \)-orbit in \( \Omega \). But \( \overline{R} = GR = GsR = GR = \overline{R} \), then \( s \) fixes \( \overline{R} \) setwise. So we may let \( \overline{R} = \Omega_0 \).

Without loss of generality, we may assume that \( \Omega_s = t_iR \) and \( \Omega_{ij} = t_i\sigma D \) for \( i \in \{0, 1, 2\}, j \in \{0, 1\} \).

By Theorem ??, we need only to investigate those \( s \in \Pi_G(R, T) \) for which \( \Omega^s_{10} = \Omega^s_{21} \). In this case, \( \Omega^s_1 = \Omega_2 \) and hence \( s^{\Omega_1 \cup \Omega_2} \) is a product of 16 disjoint transpositions on \( \Omega_1 \cup \Omega_2 \). By Lemma ?? (2), \( s^{\Omega_0} \) is a product of 6 disjoint transpositions on \( \Omega_0 \), and then \( s = s^{\Omega_0} s^{\Omega_1 \cup \Omega_2} \) is a product of 22 disjoint transpositions on \( \Omega \), and \( s \) has only 4 fixed-points all belong to \( \Omega_0 = \overline{R} \).

To find out all these involutions, we will examine the permutations induced by them on \( \Omega_0 \) and \( \Omega_1 \cup \Omega_2 \) respectively.

First, the action by \( s \) on \( \Omega_0 \) is conjugation since for every \( \overline{r} \in \overline{R} = \Omega_0 \), \( \overline{r}^s = Grs = Gs^{-1}rs = Grs = \overline{r} \). By Lemma ?? there is only one choice for \( s^{\Omega_0} \).

Secondly, in \( \Omega_1 \cup \Omega_2 \), since \( s \) forces \( \overline{r} \in \overline{R} = \Omega_0 \) to be in \( \Omega_2 = \overline{t \sigma D} \), there exists \( d \in D \) such that \( \overline{r}^s = \overline{r^2cd} \), or \( Gts = Gt^2cd \). Immediately, for each \( \overline{r} \in \Omega_1 = \overline{tR} \), \( \overline{r}^s = Gtr = Gtss^{-1}rs = (Gt^2cd)r^s = \overline{t^2cdr} \in \overline{t^2R} = \Omega_2 \). Thus there are 8 choices for \( s^{\Omega_1 \cup \Omega_2} \) since \( |D| = 8 \).

Let \( d_0 = 1, d_1 = ab, d_2 = a, d_3 = a^2b, d_4 = a^2, d_5 = a^3b, d_6 = a^3 \) and \( d_7 = b \) which make up \( D \), then we have 8 involutions \( s_0, s_1, \cdots, s_7 \) to make \( \overline{t}^s_k = \overline{t^2cd_k} \) and \( \Omega^s_{10} = \Omega_2 \) \((k = 0, 1, \cdots, 7) \).

Accordingly, we have 8 Sabidussi coset graphs:

\[
\Gamma_k := \text{Sab}(A, T, s_k),
\]

where \( \text{Val}(\Gamma_k) = |T_{sk}T : T| = |T : T \cap T^{s_k}| = 3 \).
**Remark:** Each $Γ_k$ here may not be connected because we don’t know if $⟨T, s_k⟩ = A$ yet.

From Theorem 4.1, we immediately have

**Corollary 4.1** Let $s ∈ Π_G(R, T)$. If the Sabidussi coset graph $Γ := Sab(A, T, s)$ is connected 5-arc transitive cubic, then $s ∈ \{s_0, s_1, \cdots, s_7\}$, and further, $Γ ∈ \{Γ_0, Γ_1, \cdots, Γ_7\}$.

Due to Lemma 3.2 (2) we also have 8 Cayley graphs of $G$

$$Γ_k := Cay(G, S_k) ∼= Γ_k,$$

where the Cayley subset $S_k = G ∩ (T s_k T)$, and $|S_k| = Val(Γ_k) = Val(Γ) = 3$.

We will prove soon that $S_0$ is conjugate to $S_1, S_2, S_3$ and $S_4$ is conjugate to $S_5, S_6, S_7$. Moreover, we will prove that each $S_k$ generates $G ∼= A_{47}$.

Clearly, the three-element set $S_k$ contains $s_k$. To find other two elements of $S_k$, we denote $u_k := t^2 d_k s_k t^2 ∈ T s_k T$. Note that $G t s_k = G t^2 c d_k$, then $u_k = t^2 d_k s_k t^2 ∈ G$ and hence $u_k ∈ S_k = G ∩ (T s_k T)$. We claim that $u_k$ is not an involution on $Ω$.

Otherwise, if $u_k$ is an involution, then $1 = u_k^2 = t^2 d_k s_k t^2 c d_k s_k t^2 = t^2 d_k c s_k c d_k s_k t^2 = d_k s_k (a^2 t) d_k s_k t = d_k s_k (tb) d_k s_k t$, that is, $G t d_1 = G t d_1^1 (d_k s_k t b d_k s_k t) = G t b d_k s_k t$.

For $k = 0$, $G t b d_0 s_0 t = G t b s_0 t = G t s_0 a b t = (G t^2 c d_0) a b t = G t^2 a b c t ≠ G = G d_0^1$, a contradiction;

For $k = 4$, $G t b d_4 s_4 t = G t b a^3 t = G t s_4 a^3 b t = (G t^2 c d_4) a^3 b t = G t^2 a b c t ≠ G = G d_4^2$, a contradiction;

Analogously, for $k = 1, 2, 3, 5, 6$ or 7, it also deduces a contradiction (the details are omitted). Thus, $u_k$ is not an involution so that $u_k^{-1} ∈ S_k \setminus \{s_k, u_k\}$, and hence

$$S_k = \{s_k, u_k, u_k^{-1}\}.$$

**Lemma 4.2** Assume $σ ∈ G$ such that $a^σ = a, b^σ = a^2 b, c^σ = c$, and $t^σ = t^2 a^2$.

Then $S_0^σ = S_k$ and $S_4^σ = S_{k+4}$ for $k ∈ \{1, 2, 3\}$.

**Proof** We easily have

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(t^2)^a = tb, (t^2)^{a_2} = ta_2b, (t^2)^{a_3} = t^2a_2b, (t^2)^{a_4} = ta^2, and t^{a_4} = t.

Consequently, a_4 \in G centralizes T, and hence a_4 = 1.

Since \sigma \in G, then (Gt)\sigma = Gt a^2. Analogously, (Gt^2)\sigma = Gt^2 a^2. Consequently, (Gt)\sigma^2 = Gt^2 b, (Gt^2)\sigma^2 = Gt^2 b, (Gt)\sigma^3 = Gt^2 a^2b and (Gt^2)\sigma^3 = Gt^2 a^3.

First, we will prove that for each k \in \{1,2,3\}, s_k = s_0^k or s_k s_0^k = 1. Equivalently, we manage to prove s_k s_0^k fixing \Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2 pointwise. We deal with these three cases separately.

Case 1: r \in \overline{R} = \Omega_0.

We claim that s_k s_0^k commutes with each r \in R.

In fact, let r \in b^j K (j = 0 or 1). Then r^a = a^2r, r^{a_2} = r and r^{a_3} = a^2r. That is r^{a_3} = a^2j r. Note that r^{s_k} = r^{s_0} and r^{s_k} \in b^j K by Lemma ?? (4). We have r s_k s_0^k = r s_k \sigma^{-k} s_0 \sigma^k = s_k r s_k \sigma^{-k} s_0 \sigma^k = s_k \sigma^{-k} a^2j (2-k) r s_k s_0 \sigma^k = s_k \sigma^{-k} a^2j (2-k) r s_0 \sigma^k

= s_k \sigma^{-k} a^2j (2-k) s_0 r \sigma^k = s_k \sigma^{-k} a^2j (2-k) r \sigma^k = s_k \sigma^{-k} s_0 \sigma^k r = s_k s_0^k r. Thus s_k s_0^k commutes with each r \in R, and hence s_k s_0^k fixes \overline{R} = \Omega_0 pointwise.

Case 2: \overline{r} \in \overline{t R} = \Omega_1.

Let \overline{t R} := Gtr(s_k \sigma^{-k} s_0 \sigma^k) = Gt(s_k \sigma^{-k} s_0 \sigma^k)r = (Gt s_k) \sigma^{-k} s_0 \sigma^k r = Gt^2 c d_k \sigma^{-k} s_0 \sigma^k r, then

1. \overline{t_1} = Gt^2 c(ab) \sigma^{-1} s_0 \sigma r = Gt^2 (abc) \sigma^3 s_0 \sigma r = Gt^2 \sigma^3 (a^3bc s_0 \sigma r = (Gt a^2) a^3 c s_0 \sigma r = Gt a s_0 \sigma r = Gt a^2 c s_0 \sigma r = Gt^2 (bc) \sigma^2 c s_0 \sigma r = Gt^2 (c a) \sigma^2 c s_0 \sigma r = Gt^2 (a^3 b) s_0 \sigma^3 r

2. \overline{t_2} = Gt^2 (c a) \sigma^2 c s_0 \sigma^2 r = Gt^2 c a \sigma^2 c s_0 \sigma^2 r = Gt^2 c (b) \sigma^2 c s_0 \sigma^2 r = Gt^2 c (a^3 b) s_0 \sigma^3 r = (Gt^2) s_0 (a^2 b) \sigma^2 r = Gt^2 (a^2 b) \sigma^2 r = (Gt) (a^2 b) \sigma^2 r = Gt \sigma^2 (a^2 b) r = Gt \sigma^2 (a^2 b) r = Gt r;

3. \overline{t_3} = Gt^2 (a^2 b) \sigma s_0 \sigma^3 r = Gt^2 \sigma c b s_0 \sigma^3 r = (Gt b) c b s_0 \sigma^3 r = Gt c b s_0 \sigma^3 r = Gt a^2 \sigma^3 r = a^2 \sigma^3 r = Gt r.

Thus s_k s_0^k fixes \overline{t R} = \Omega_1 pointwise.

Case 3: \overline{t_2} \in \overline{t R} = \Omega_2.

Let \overline{t R} := Gt^2 (s_k \sigma^{-k} s_0 \sigma^k) = (Gt^2 c s_k) (cd_k^{-1}) = (Gt s_k) (s_k \sigma^{-k} s_0 \sigma^k) (cd_k^{-1} r) = (Gt^2 c s_k) \sigma^{-k} s_0 \sigma^k (cd_k^{-1} r) = Gt^2 \sigma^{-k} s_0 \sigma^k (cd_k^{-1} r), then
With the above notation, we have

(1) \( T_1 = G\sigma^3 s_0 \sigma(cab)r = (Gt^2 a^2 b) s_0 \sigma(cab)r = Gt^2 c(a^2 b)s_0 \sigma(cab)r = Gt^2 c s_0 (ab)c \sigma(cab)r = (Gt) \sigma(a^3 bc)(cab)r = (Gt) \sigma(a^2) a^2 r = Gt^2 r; \)

(2) \( T_2 = G\sigma^2 s_0 \sigma^2 (ca^3)r = (Gt^2 b) s_0 \sigma^2 (ca^3)r = Gt s_0 a^3 b \sigma^2 (ca^3)r = (Gt^2 c) \sigma^2 a^3 b(c^3) \sigma = Gt^2 c \sigma^2 c(bcr) = (Gt^2 b)(br) = Gt^2 r; \)

(3) \( T_3 = G\sigma s_0 \sigma^3 (ca^2 b)r = (Gt^2 a^2) s_0 \sigma^3 (ca^2 b)r = (Gt^2 c) (ca^2 s_0)^3 (ca^2 b)r = (Gt s_0)(s_0 c) \sigma^3 (ca^2 b)r = Gt (c a^3)(ca^2 b)r = Gt^3 c ca^2 b r = (Gt^2 a^2 b) a^2 b r = Gt^2 r. \)

Thus \( s_k s_0^k \) fixes \( t^2 R = \Omega^2 \) pointwise.

Therefore, \( s_k s_0^k = 1 \) or \( s_0^k = s_k \).

Secondly, depending on \( u_k = t^2 c d_k s_k t^2 \), we have:

(1) \( u_0^2 = (t^2 c d_0 s_0 t^2)^2 = (t^2 c s_0 t^2)^2 = (tb) c s_1 (tb) = t(b c s_1) t b = t s_1 (a b)(a^2) c t = t s_1 (a b) c t = t s_1 (a b) c t = u_1^{-1}; \)

(2) \( u_0^2 = (t^2 c s_0 t^2)^2 = (t^2 b) c s_2 (t^2 b) = t^2 b c s_2 (a^2 b t^2) = t^2 b c (a^3 b s_2) t^2 = t^2 c a s_2 t^2 = u_2; \)

(3) \( u_0^3 = (t^2 c s_0 t^2)^3 = (t^2 a) c s_3 (t a^2) = t(a^2 c s_3) t a^2 = t(s_3) (a^2 b t) = t s_3 (a^2 b) c t = (t^2 c d s_3 t^2)^{-1} = u_3^{-1}. \)

To sum up, \( S_0^k = S_k (k = 1, 2, 3). \)

Similarly, \( S_1^k = S_{k+4}(the \ detail \ proof \ is \ omitted). \)

\[ \square \]

**Theorem 4.3** With the above notation, we have

(1) The graph \( \Gamma_0 \) is isomorphic to \( \Gamma_1, \Gamma_2, \Gamma_3, \) and \( \Gamma_4 \) is isomorphic to \( \Gamma_5, \Gamma_6, \Gamma_7; \)

(2) Each graph \( \Gamma_k (k = 0, 1, \cdots, 7) \) is connected 5-arc transitive cubic graph and its full automorphism group is isomorphic to \( A; \)

(3) Each set \( S_k (k = 0, 1, \cdots, 7) \) is a CI-subset of \( G; \)

(4) The graphs \( \Gamma_0 \) and \( \Gamma_4 \) are not isomorphic to each other.

Before our proof we need to represent the permutations of \( A \) in another way.

For each element of \( \Omega = [A : G] \) which is a right coset of \( G \) with the unique representative from \( T \), we will rearrange these representatives. First, we denote all elements of \( R \) in turn by

\[
\begin{align*}
    r_1 := 1, & \quad r_2 := a, \quad r_3 := a^2, \quad r_4 := a^3, \quad r_5 := a^3 b, \quad r_6 := a^2 b, \quad r_7 := a b, \quad r_8 := b, \\
    r_9 := c, & \quad r_{10} := a c, \quad r_{11} := a^2 c, \quad r_{12} := a^3 c, \quad r_{13} := a^3 b c, \quad r_{14} := a^2 b c, \quad r_{15} := a b c, \quad \text{and} \quad r_{16} := b c.
\end{align*}
\]

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Then, from \( T = R \cup tR \cup t^2R \), we denote all other elements of \( T \) by \( r_{16i+j} := t^ir_j \) where \( i \in \{1, 2\} \) and \( j \in \{1, 2, \cdots, 16\} \). Thus we may depict \( \Omega = \{\Omega_1, \Omega_2, \cdots, \Omega_{48}\} \) as follows:

\[
\begin{array}{ccccccccccccccccccc}
1 & a & d^2 & d^3 & d & d^2d & ab & b & 9 & c & ac & d^2c & d & d^2bc & ab & abc & 16 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\
33 & 34 & 35 & 36 & 37 & 38 & 39 & 40 & 41 & 42 & 43 & 44 & 45 & 46 & 47 & 48 \\
\end{array}
\]

\( \Delta_0 \)

Now, acting by its right multiplication, each element of \( A \) may simply be denoted as the permutation on \( \{1, 2, \cdots, 48\} \), such as

\[
a = (1, 2, 3, 4)(5, 6, 7, 8)(9, 10, 11, 12)(13, 14, 15, 16)(17, 18, 19, 20)(21, 22, 23, 24) \\
(25, 26, 27, 28)(29, 30, 31, 32)(33, 34, 35, 36)(37, 38, 39, 40)(41, 42, 43, 44)(45, 46, 47, 48), \\
b = (1, 8)(2, 7)(3, 6)(4, 5)(9, 16)(10, 15)(11, 14)(12, 13)(17, 24)(18, 23)(19, 22)(20, 21) \\
\]

For the \( t \), we easily have that \( bt = ta^2b \) and \( ct = tc \). In addition, since \( a^t = a^{-1}t^{-1} \), then \( at^2 = ta^3 \), and hence \( at = (at^2)t^2 = (ta^3)t^2 = ta^2(at^2) = t^2t^{-1}a^2ta^3 = t^2ab \). So we also obtain the following permutation for \( t \) :

\[
t = (1, 17, 33)(2, 39, 20)(3, 24, 38)(4, 34, 23)(5, 37, 21)(6, 19, 40)(7, 36, 18)(8, 22, 35) \\
\]

Proof of Theorem ??:

1. From Lemma ?? we need only to find out a \( \sigma \in G \) such that \( \sigma \) satisfies the assumptions there. Take
\( \sigma := (5,7)(6,8)(13,15)(14,16)(17,35,22,38)(18,36,23,39)(19,33,24,40) \\
(20,34,21,37)(25,43,30,46)(26,44,31,47)(27,41,32,48)(28,42,29,45) \in A_1 = G. \\
It is easy to check that \( a^\sigma = a, b^\sigma = a^2b, c^\sigma = c, t^\sigma = t^2a^2 \), and hence (1) holds.

(2) We will prove that \( \langle T, s_l \rangle = A \) for \( l \in \{0,4\} \). This will imply that \( \Gamma_l \) is a connected 5-arc transitive cubic graph for any \( l \), and so are all other \( \Gamma_k \) from (1) and also are \( \Gamma_k \cong \Gamma_0 \).

To prove the above assertion, we first determine the permutations of \( s_0, s_4, u_0 \) and \( u_4 \) on \( \{1,2,\ldots,48\} \):

For every \( r \in \Omega_0 \), since \( r^s = (Gr)s = Gr^s = r' \) and \( a^s = a^{-1}, b^s = ab, c^s = a^2c \), we easily have that \( s_k^{(0)} = (2,4)(5,6)(7,8)(9,11)(13,16)(14,15) \), being independent of \( k \).

For every \( r \in \Omega_1 \), since \( r^s_k = (Gr)s_k = (Gt_0s_k)r^s_k = (Gt^2cd_0)r^s_k = r'c_0d_0s_k \in \Omega_2 \), we have the following

since \( d_0 = 1 \),
\[
\begin{align*}
s_0 &= (2,4)(5,6)(7,8)(9,11)(13,16)(14,15)(17,41)(18,44)(19,43)(20,42)(21,46) \\
\text{and since } d_4 &= a^2c, \\
s_4 &= (2,4)(5,6)(7,8)(9,11)(13,16)(14,15)(17,43)(18,42)(19,41)(20,44)(21,48) \\
\end{align*}
\]

Accordingly,
\[
\begin{align*}
u_0 &:= t^2c_0s_0t^2 = (2,7,4,24,41,25,33,22,29,11,42,20,3,37,43,13,32,38,5,19,26, \\
&\quad 47,23,39,28,16,9,14,45,40,17,46,10,12,27,21,8,6,34,31,44,35)(18,36,48,30), \\
u_4 &:= t^2c_4a^2s_4t^2 = (2,7,4,24,46,10,12,27,18,36,43,13,32,33,22,26,47,20,3,37,48, \\
&\quad 30,21,8,6,34,28,16,9,14,45,35)(5,19,29,11,42,23,39,31,44,40,17,41,25,38).
\end{align*}
\]

Denote \( H_l := \langle T, s_l \rangle \) for \( l \in \{0,4\} \). We claim that \( H_l \) is 2-transitive on \( \Omega \).

In fact, \( H_l \) is obviously transitive on \( \Omega \). Examining all orbits of \( \langle u_l \rangle \) in \( \Omega \setminus \{r_l\} \) as follows, we easily have that \( \langle s_l, u_l \rangle \) is transitive on \( \Omega \setminus \{r_l\} \) and so is \( H_l \cap G \) since \( s_l, u_l \in H_l \cap G \).

3 orbits of \( u_0 \):
\[
\{r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_9, r_{10}, r_{11}, r_{12}, r_{13}, r_{14}, r_{16}, r_{17}, r_{19}, r_{20}, r_{21}, r_{22}, r_{23}, r_{24}, r_{25}, \\
r_{26}, r_{27}, r_{28}, r_{29}, r_{31}, r_{32}, r_{33}, r_{34}, r_{35}, r_{37}, r_{38}, r_{39}, r_{40}, r_{41}, r_{42}, r_{43}, r_{44}, r_{45}, r_{46}, r_{47}\},
\]

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\{ r_{15} \}, \{ r_{18}, r_{36}, r_{38}, r_{48} \};

3 orbits of \( u_4 \):
\begin{align*}
&\{ r_2, r_3, r_4, r_6, r_7, r_8, r_9, r_{10}, r_{12}, r_{13}, r_{14}, r_{16}, r_{18}, r_{20}, r_{21}, r_{22}, r_{24}, r_{26}, r_{27}, r_{28}, r_{30}, r_{32}, \\
&\quad r_{33}, r_{34}, r_{35}, r_{36}, r_{37}, r_{43}, r_{45}, r_{46}, r_{47}, r_{48} \}, \\
&\{ r_5, r_{11}, r_{17}, r_{19}, r_{23}, r_{25}, r_{29}, r_{31}, r_{38}, r_{39}, r_{40}, r_{41}, r_{42}, r_{43} \}, \{ r_{15} \}.
\end{align*}

But \( H_t \cap G \) is exactly the point stabilizer of \( r_7 \) in \( H_t \), then each \( H_t \) is 2-transitive on \( \Omega \), and hence is primitive. Besides, by direct checking we have:

\( u_0^{10} = (2, 42, 26, 17, 44, 29, 5, 45, 34, 33, 32, 9, 8, 41, 43, 28, 27, 4, 3, 23, 10)(6, 25, 13, 16, 21, \\
24, 37, 39, 12, 7, 20, 47, 46, 35, 11, 19, 40, 31, 22, 38, 14)(18, 48)(30, 36) \) with order 42,

\( s_1 u_0^{10} = (2, 3, 23, 18, 29, 31, 14, 15, 6, 45, 38, 22, 34, 17, 43, 40, 5, 25, 11, 8, 20, 26, 33, 4, 42, \\
47, 37, 9, 19, 28, 30, 12, 7, 41, 44, 48, 10)(13, 21, 35)(24, 46)(27, 32, 39, 36) \) with order 444,

\( (s_0 u_0^{10})^{148} = (13, 21, 35) \in H_0 \), and

\( s_4 u_4 = (2, 24, 35, 18, 23, 10, 12, 27)(3, 37, 21, 30, 48, 8, 4, 7, 6, 19, 25, 22, 20, 40, 44)(5, 34, \\
16, 32, 31, 17, 13, 9, 42, 36, 47, 26, 43, 29)(11, 14, 15, 45, 46, 39, 33, 38) \) with order 120,

\( (s_4 u_4)^3 = (2, 18, 12, 24, 23, 27, 35, 10)(3, 30, 4, 19, 20)(5, 32, 13, 36, 43)(6, 22, 44, 21, 8, \\
7, 25, 40, 37, 48)(9, 47, 41, 34, 31)(11, 45, 33, 14, 46, 38, 15, 39)(16, 17, 42, 26, 29) \) with order 40,

\( s_4 u_4^{-1} = (2, 7, 21, 37, 48, 30, 3, 20, 31, 44, 47, 33, 41, 5, 8)(4, 35, 12, 10, 46, 42, 27, 45) \\
(6, 38, 19, 17, 36, 22, 26, 18, 11, 16, 43, 40, 39, 13, 28)(9, 29, 25, 32, 23, 24, 14, 15) \) with order 120,

\( (s_4 u_4^{-1})^2 = (2, 21, 48, 3, 31, 47, 41, 8, 7, 37, 30, 20, 44, 33, 5)(4, 12, 46, 27)(6, 19, 36, 26, 11, \\
43, 39, 28, 38, 17, 22, 18, 16, 40, 13)(9, 25, 23, 14)(10, 42, 45, 35)(15, 29, 32, 24) \) with order 60,

\( (s_4 u_4^{-1})^2 s_4 u_4^{-1} = (2, 16, 22, 33, 9, 41, 34, 47, 8, 19, 44, 48, 37, 3, 20, 31, 25, 13, 26, 32, 6, 18, \\
46, 17, 45, 5, 24, 14, 27, 10, 21, 7, 23, 4, 36, 39, 43)(11, 35, 42)(12, 15, 28, 38, 29, 40, 30) \) with order 777,

\( ((s_4 u_4)^3(s_4 u_4^{-1})^2)^{259} = (11, 35, 42) \in H_4 \).

It follows from Jordan’s theorem that \( H_t = \langle T, s_t \rangle = A \) for \( t = 0, 4 \).
(3) Since $A = \text{Aut}(\Gamma_k) \cong A_{48}$, then we need only to prove that for any $\sigma \in \text{Sym}(G)$ satisfying $G^\sigma \leq A$, $G^\sigma$ has to be conjugate with $G$ in $A$. In this case, $S_k$ is a CI-subset by Babai’s criterion (see [?]).

In fact, we see that $G^\sigma$ has at most one fixed point on $\Omega$, since $G^\sigma \cong A_{47}$ contains a 47-cycle. We claim that $G^\sigma$ has exactly one fixed point. If not, then $G^\sigma$ is transitive on $\Omega$, and the point stabilizer is a maximal subgroup since its index in $G^\sigma$ is 48. Consequently $G^\sigma$ is primitive on $\Omega$. But $G^\sigma$ has 43-cycles in $\Omega$, and $|\Omega| - 43 \geq 3$, it follows from Jordan’s theorem that $G^\sigma \geq \text{Alt}(\Omega) \cong A$, a contradiction.

Let $r \in \Omega (r \in T)$ is this fixed point of $G^\sigma$. It is easy to check that $G^\sigma$ is the point stabilizer of $r$ in $A$. Note that $G$ is also the point stabilizer of $I$ and hence $G^\sigma$ is conjugate with $G$ in $A$.

(4) By direct calculation we have $o(u_0) = o(u_0^{-1}) = 84$ for $u_0 \in S_0$, and $o(u_4) = o(u_4^{-1}) = 224$ for $u_4 \in S_4$. Therefore, for any $\alpha \in \text{Aut}(G)$, $S_0^\alpha \neq S_4$. By (3), $\Gamma_0$ is not isomorphic to $\Gamma_4$. \hfill \Box

By the proof of Theorem ??, we have directly a corollary below and hence complete the proof of Theorem ??:

**Corollary 4.4** Let $s \in \Pi_G(R, T)$. Assume that $s$ interchanges $\Omega_{10}$ and $\Omega_{21}$. Then the Sabidussi coset graph $\Gamma := \text{Sab}(A, T, s)$ is isomorphic to one of $\Gamma_0$ and $\Gamma_4$. In particular, $\langle T, s \rangle = A$.

\hfill \Box

By the proof of Theorem ?? (3), We have the following corollary.

**Corollary 4.5** In the alternating group $A_{48}$, all subgroups which are isomorphic to $A_{47}$ are conjugate.

\hfill \Box

The following lemma shows that no matter how to choose the involution $s$, the Sabidussi coset graph $\text{Sab}(A, T, s_k)$ is, up to isomorphism, independent of $T$.

**Lemma 4.6** Any two mutually isomorphic regular subgroups of $S_n$ are conjugate in $S_n$.

**Proof** If $X$ and $Y$ are regular permutation groups on $\Omega = \{1, 2, \ldots, n\}$, and $\alpha : \Omega \to X$ and $\beta : \Omega \to Y$ are bijections with the property that $1^{\alpha(i)} = 1^{\beta(i)}$ for
1 \leq i \leq n, and \( \theta : X \rightarrow Y \) is isomorphism, then \( \alpha \theta \beta^{-1} : \Omega \rightarrow \Omega \) is a permutation in \( S_n \) that conjugates \( X \) to \( Y \).

5 The main result

Now, we are able to give a complete classification of the nonnormal connected \( s \)-arc transitive cubic Cayley graphs of finite nonabelian simple groups:

**Theorem 5.1** Let \( G \) be a finite nonabelian simple group and \( \Gamma := \text{Cay}(G, S) \) a nonnormal connected \( s \)-arc transitive cubic Cayley graph for \( G \). Then \( \Gamma \) is isomorphic to one of \( \Gamma_0 \) and \( \Gamma_4 \).

**Proof** We know from the paper [1] that \( G \) here must be isomorphic to \( A_{47} \), and the full automorphism group \( \Gamma := \text{Aut}(\Gamma) \) of the Cayley graph \( \Gamma \) is isomorphic to \( A_{48} \), and that its point stabilizer \( A_1 \) is isomorphic to \( S_4 \times \mathbb{Z}_2 \) which is the complement of \( G \) in \( A \), that is \( A = GA_1 \).

Thus, by Lemma ?? there exists an involution \( s \) in \( G \) such that \( \Gamma \cong \Gamma := \text{Sab}(A, A_1, s) \) with \( |A_1sA_1 : A_1| = |A_1 : A_1 \cap A_1^s| = 3 \) and \( \langle A_1, s \rangle = A \).

According to Corollary ??, we may assume the pair of simple groups \( (A, G) \) to be the same as in Theorem ???. In this case, \( A \) has two subgroups \( A_1 \) and \( T \) (\( T \) is defined as in Theorem ??), both are complements of \( G \) and they are regular on \( \Omega := [A : G] \). In light of Lemma ??, there exists \( \sigma \in \text{Sym}(\Omega) \) such that \( A_1^\sigma = T \), and then \( A = A^\sigma = (GA_1)^\sigma = G^\sigma T \). It follows by Corollary ?? that \( G^\sigma \in A \) is conjugate to \( G \) in \( A \), and we may take \( h \in T \) such that \( G^{\sigma h} = G \).

Denote \( \alpha = \sigma h \in \text{Sym}(\Omega) \). Then \( A = A^\alpha = G^\alpha A_1^\alpha = GT \), \( |A_1^\alpha s^\alpha A_1^\alpha : A_1^\alpha| = |Ts^\alpha T : T| = 3 \) and \( \langle A_1^\alpha, s^\alpha \rangle = \langle T, s^\alpha \rangle = A \). It implies that \( s^\alpha \in \Pi_G(R, T) \). From Corollary ??, we have \( s^\alpha \in \{ s_0, s_1, \ldots, s_7 \} \) and \( \Gamma' := \text{Sab}(A, T, s^\sigma) \in \{ \Gamma_0, \Gamma_1, \ldots, \Gamma_7 \} \).

So we know from Theorem ?? that \( \Gamma' \) is isomorphic to \( \Gamma_0 \) or \( \Gamma_4 \), and hence we complete the proof since \( \Gamma = \text{Sab}(A, A_1, s) \) is obviously isomorphic to \( \Gamma' = \text{Sab}(A, T, s^\alpha) \).

\[ \square \]

References


