CONVEX SETS OF CONSTANT WIDTH AND 3-DIAMETER

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ABSTRACT. In this article we study 3-diameter of planar sets of constant width. We obtain analogues of the isodiametric inequality and the Blaschke-Lebesgue Theorem for 3-diameter of constant width sets. Namely, we prove that among all the sets of given constant width, disks have the smallest 3-diameter and Reuleaux triangles have the largest 3-diameter. We also discuss d_3 complete sets as well as sets of constant 3-diameter. We show that there exist infinitely many non-circular sets of constant 3-diameter and construct one such set.

1. INTRODUCTION

A convex set $E \subset \mathbb{C}$ is said to have a *constant width* $\lambda > 0$, if for each $\theta \in [0, 2\pi]$ the distance between parallel support lines orthogonal to the direction defined by θ equals λ. A convex set $E \subset \mathbb{C}$ is said to be of *constant diameter* if $\sup\{|x - z| : z \in E\} = \mathrm{diam}(E)$ for each $x \in \partial E$. It is easily shown that sets of constant width are precisely the convex sets of constant diameter.

Disks are the trivial examples of sets of constant width. Non-disk examples do exist and the Reuleaux triangle is one of the best known ones. Planar sets of constant width have been studied extensively; for instance, the perimeter of a set of constant width λ equals $\lambda\pi$. Thus, in some regards, sets of constant width are similar to disks. As for the area of a set of constant width λ, disks of diameter λ have the largest area; this follows from either the isoperimetric inequality or the isodiametric inequality. The Blaschke-Lebesgue Theorem states that Reuleaux triangles have the smallest area (see [2] and [5], for these and other properties of sets of constant width).

There are higher-order generalizations of the diameter of a set. The n-*diameter* of $E \subset \mathbb{C}$ is given by

$$d_n(E) = \sup\left(\prod_{j<k} |z_j - z_k|\right)^{\frac{2}{n(n-1)}},$$

where the supremum is taken over all n-tuples of points from E. Note that $d_2(E)$ is the ordinary diameter of E and $d_n(E)$ is weakly decreasing in n, i.e., $d_n(E) \geq d_{n+1}(E)$ [9]. The quantity $d_\infty(E) := \lim_{n\to\infty} d_n(E)$ is called the *transfinite diameter* of E. The notion of transfinite diameter plays an important role in complex analysis and relates to logarithmic capacity and the Chebycheff constant (see [10] and [14]). Some extremal problems involving the transfinite diameter and n-diameter of planar sets were studied in [3, 4, 7, 8, 13]. Recently, the n-diameter of a plane region has been used is establishing a generalization of Schwarz's Lemma [1].

In this paper we focus on the 3-diameter. It appears that the 3-diameter of planar sets of constant width has not been studied. Our results suggest that the 3-diameter reveals interesting facts about planar sets of constant width. For example, among the planar sets of constant width

λ, Reuleaux triangles have the largest 3-diameter, namely λ, and disks have the smallest 3-diameter, $\sqrt{3}\lambda/2$ (see, Theorem 3.1). Thus, the roles of the isoperimetric inequality and the Blaschke-Lebesgue Theorem are reversed when it comes to 3-diameter for sets of constant width.

As a way of extending the notion of constant width, i.e., constant 2-diameter, we introduce and study d_3-complete sets in Section 4 and sets of constant 3-diameter in Section 5. A compact set is called *complete* if the addition of any point outside the set increases its diameter. As in the case of 2-diameter, disks are both of constant 3-diameter and d_3-complete. Recall that sets of constant width are precisely the d_2-complete sets. For the 3-diameter we show that complete sets are of constant width (Theorem 5.2), and, on the other hand, many constant width sets are complete (Theorem 5.5). Unfortunately, there are some cases which we cannot handle, cf. Section 5

In many ways circular arcs are the building blocks for sets of constant width. For example, any set of constant width can be represented as a limit of Reuleaux polygons. Here the convergence is in the Hausdorff metric and the Reuleaux polygons are sets of constant width consisting of finitely many circular arcs [5, p. 128]. In contrast, the building blocks for sets of constant 3-diameter are arcs of Cassinian ovals. We show that there are infinitely many sets of constant 3-diameter which are not disks (see Theorem 5.3) and construct one such set, which we call the *Reuleaux square*, in Section 6.

There are many natural questions about the 3-diameter of sets of constant width, which we have not pursued in this paper. For example, is the analogue of the isodiametric inequality also valid for 3-diameter? In other words, do disks have the largest area among all sets of given constant 3-diameter. Similarly, one can ask for an analogue of the Blaschke-Lebesgue Theorem for sets of constant 3-diameter, i.e., which sets of given constant 3-diameter have the smallest area? It is well-known that the transfinite diameter of a disk equals half of its diameter. Since disks have the largest area among all sets of given constant width, one can easily show that disks have the largest transfinite diameter among all sets of given constant width (see, for instance, [10]). Hence a natural question is which sets of constant width have the smallest transfinite diameter. Since the disk and the Reuleaux triangle are opposite extremes in other questions, one reasonable guess is the Reuleaux triangle (whose transfinite diameter is calculated in [11, Lemma 5]).

2. PRELIMINARIES

We consider sets in the complex plane $\mathbb{C}$ and use its standard notations.

2.1. **Sets of constant width.** Let $E \subset \mathbb{C}$ be a compact convex set with non-empty interior. A line L is called a support line of E if E lies on one side of L and $L \cap E \neq \emptyset$. Hence there are exactly two parallel support lines of E in every direction. The minimal and maximal width of E are the smallest and the largest distances between the parallel support lines over all directions. The set E is said to be of constant width if the minimal width and the maximal width are equal. The common value, which is equal to the diameter of E, is referred to as the width of E. Sets of constant width are necessarily strictly convex. A compact set E is said to

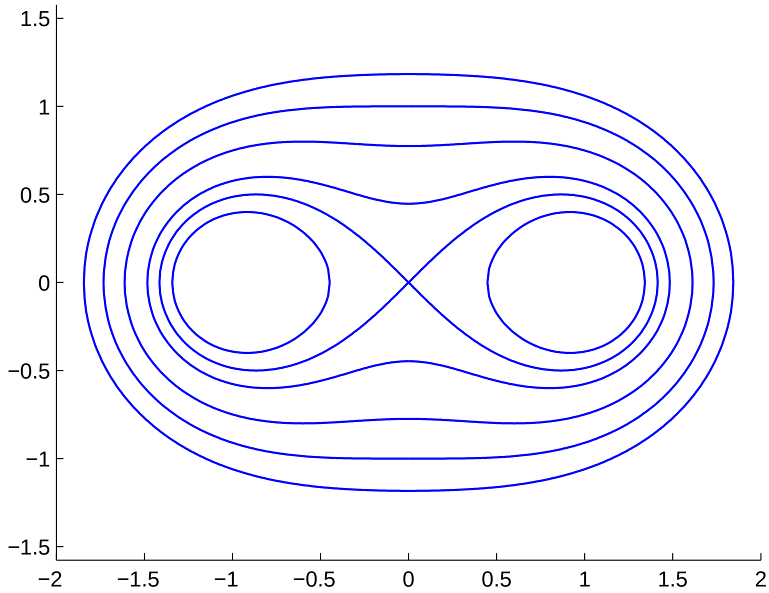

FIGURE 1. Cassinian ovals with foci 1 and -1 for $\lambda \in \{0.8, 1, 1.2, 1.6, 2, 2.4\}$

be of constant diameter if

$$\max_{y\in E} |x-y| = \operatorname{diam}(E) \qquad \text{for each} \quad x \in \partial E.$$

A compact convex set E with non-empty interior is of constant width if and only if it is of constant diameter. We occasionally refer to sets of constant width as sets of constant 2-diameter. A set E is said to be complete (or d_2-complete) if

$$\operatorname{diam}(\{x\} \cup E) > \operatorname{diam}(E) \qquad \text{for each} \quad x \notin E.$$

A compact convex set E with non-empty interior is of constant width if and only if it is d_2-complete. See, for example, [5] for these and other properties of sets of constant width.

The simplest nontrivial example of a constant width set is the *Reuleaux triangle*. It is defined as follows: Let ξ_1, ξ_2 and ξ_3 be the vertices of an equilateral triangle with side lenght $\lambda > 0$. Then

$$\mathcal{R} = \bigcap_{k=1}^{3} \{z \in \mathbb{C}\colon \ |z-\xi_k| \le \lambda\}$$

is the Reuleaux triangle with vertices at ξ_1, ξ_2, ξ_3.

2.2. **Cassinian ovals.** The Cassinian oval $C(a,b;\lambda)$ with foci at $a \in \mathbb{C}$ and $b \in \mathbb{C}$ and radius $\lambda > 0$ is defined as

$$C(a,b;\lambda) = \{z \in \mathbb{C} : \ |z-a|\,|z-b| = \lambda\}.$$

If $c \in \mathbb{C}$, then we use the short-hand notation $C(a,b;c)$ for the Cassinian oval which passes through c, i.e., $C(a,b;c) = C(a,b;\lambda)$ where $\lambda = |a-c|\,|b-c|$. It is easy to see that a

Cassinian oval is connected if and only if $\sqrt{|c-a|\,|c-b|} > \frac{1}{2}|a-b|$ and convex if and only if $\sqrt{|c-a|\,|c-b|} \geq \frac{1}{\sqrt{2}}|a-b|$. Cassinian ovals for some parameter values are shown in Figure 1

2.3. 3**-diameter.** For a bounded set E in the complex plane $\mathbb{C}$, the 3-diameter of E is

$$d_3(E) = \sup\Big[|\zeta_1-\zeta_2||\zeta_2-\zeta_3||\zeta_3-\zeta_1|\Big]^{\frac{1}{3}},$$

where the supremum is taken over all triples of points from E. By compactness, we immediately obtain that

$$d_3(E) = \Big[|\zeta_1-\zeta_2||\zeta_2-\zeta_3||\zeta_3-\zeta_1|\Big]^{\frac{1}{3}} \qquad \text{for some} \quad \zeta_1,\zeta_2,\zeta_3 \in \overline{E},$$

for each bounded set $E \subset \mathbb{C}$. In particular, $d_3(\overline{E}) = d_3(E) = d_3(\partial E)$. It is also clear that $d_3(E) \leq d_2(E)$. Given $E \subset \mathbb{C}$, let $T(E)$ denote the set of all triples of points in E for which the quantity $d_3(E)$ is achieved, i.e.,

$$T(E) = \big\{(\zeta_1,\,\zeta_2,\,\zeta_3) \in E^3\colon\ d_3(E) = (|\zeta_1-\zeta_2||\zeta_2-\zeta_3||\zeta_3-\zeta_1|)^{1/3}\big\}.$$

Observe that $d_3(E) = d_2(E)$ for vertices of equilateral triangles.

As an example, and for future use, we calculate the 3-diameter of a disk (see, also [12]). The n-diameter of a disk is given in [1].

Proposition 2.1. *The* 3*-diameter of a disk of radius* r *equals* $\sqrt{3}r$.

Proof. By scale invariance of the claim, it is enough to prove that $d_3(S^1(0,1)) = \sqrt{3}$. Let $a,b,c \in S^1(0,1)$ and let a' be the mid-point of the longer of the arcs $S^1(0,1)\setminus\{b,c\}$. Then it is clear that $d_3(\{a,b,c\}) \leq d_3(\{a',b,c\})$. By rotational symmetry (a' becomes 1) this implies that

$$d_3(S^1(0,1))^3 = \max_{\xi\in S^1(0,1)} |\xi-1|\,|\overline{\xi}-1|\,|\xi-\overline{\xi}| = 2\max_{\theta\in[0,360^\circ)} \sin\theta((\cos\theta-1)^2+\sin^2\theta).$$

The maximum is attained for $\theta = 120^\circ$, and equals $3^{3/2}$, as claimed. □

The following is a combination of the isodiametric inequalities for 2-diameter and 3-diameter.

Proposition 2.2. *If* $E \subset \mathbb{C}$ *is a set with* $d_3(E) = d_2(E) = \lambda$*, then*

$$\textit{Area}(E) \leq \frac{\pi-\sqrt{3}}{2}\lambda^2.$$

Equality holds only for the Reuleaux triangle.

Proof. The assumption $d_3(E) = d_2(E)$ implies that there exist points $\xi_1,\xi_2,\xi_3 \in E$ such that $|\xi_1-\xi_2| = |\xi_2-\xi_3| = |\xi_1-\xi_3| = \operatorname{diam}(E) = \lambda$. Since $d_2(E) = \lambda$, we have $E \subset \mathcal{R}$, where

$$\mathcal{R} = \bigcap_{k=1}^{3}\{z \in \mathbb{C}\colon\ |z-\xi_k| \leq \lambda\}$$

is the Reuleaux triangle with vertices at ξ_1,ξ_2,ξ_3. Hence

$$\text{Area}(E) \leq \text{Area}(\mathcal{R}) = \frac{\pi-\sqrt{3}}{2}\lambda^2.$$

Finally, we observe that the assumption

$$\mathrm{Area}(E) = \frac{\pi - \sqrt{3}}{2}\lambda^2 = \mathrm{Area}(\mathcal{R})$$

along with $E \subset \mathcal{R}$ implies that $E = \mathcal{R}$. The proof is complete. □

Note that the 2-diameter of the convex hull of a set is equal to the diameter of the set. Hence in proving the isodiametrical inequality for 2-diameter one can assume the set to be convex. But when it comes to the 3-diameter, the convex hull of a set may have larger 3-diameter than the set itself as the following two examples show.

Example 2.3. Let $E = \{-1,\, a,\, 1\}$, $a > 0$. Then $\hat{E} = [-1,\, 1]$ is its convex hull. Clearly,

$$d_3(E) = (|-1-1||1-a||1+a|)^{1/3} < (|-1-1||1-0||-1-0|)^{1/3} \le d_3(\hat{E}).$$

One can also find examples of more regular domains in $\mathbb{C}$ whose convex hulls have bigger d_3-diameter. For example, let E be the triangle given by the convex hull of the points $\xi_1 = 0,\, \xi_2 = i/3,\, \xi_3 = 2$. Put $\xi_0 = 1 + i/6$, $\xi_0 \in E$. Then

$$\big(|\xi_1-\xi_3||\xi_1-\xi_2||\xi_2-\xi_3|\big)^{1/3} = \big(\frac{2\sqrt{37}}{9}\big)^{1/3} < \big(\frac{37}{18}\big)^{1/3} = \big(|\xi_1-\xi_3||\xi_1-\xi||\xi-\xi_3|\big)^{1/3} \le d_3(E).$$

It follows that the triple $(\xi_1, \xi_2, \xi_3) \notin T(E)$. In particular, in each triple in $T(E)$ at least one point is not a vertex of E. Around each such point $\xi \in \partial E$ we define a disk D_ξ, whose radius is chosen so that the set

$$E_0 = E \setminus \bigcup_{\xi} D_\xi$$

is connected. Clearly, $E = \hat{E}_0$ and by definition $d_3(E_0) < d_3(E)$.

3. 3-DIAMETER AND SETS OF CONSTANT 2-DIAMETER

In this section we discuss analogues of the isodiametric inequality and the Blaschke-Lebesgue Theorem for 3-diameter rather than for area. Recall that the isodiametric inequality says that disks have the largest area among all sets of given 2-diameter, while the Blaschke-Lebesgue Theorem says that Reuleaux triangles have the smallest area among all sets of constant 2-diameter. Surprisingly, the situation is reversed when it comes to the 3-diameter of sets of constant 2-diameter. Namely,

Theorem 3.1. *Let D be a set of constant 2-diameter $\lambda > 0$. Then*

$$\frac{\sqrt{3}}{2}\lambda \le d_3(D) \le \lambda.$$

The upper and lower bounds are attained only for Reuleaux triangles and disks, respectively.

The upper bound is easily derived. Indeed, it is obvious that $d_3(E) \le d_2(E) = \lambda$ and the assumption $d_3(E) = \lambda$ implies, as in the proof of Lemma 2.2, that there exist points $\xi_1, \xi_2, \xi_3 \in E$ such that $|\xi_1 - \xi_2| = |\xi_2 - \xi_3| = |\xi_1 - \xi_3| = \mathrm{diam}(E) = \lambda$. Since $d_2(E) = \lambda$, we have $E \subset \mathcal{R}$, where

$$\mathcal{R} = \bigcap_{k=1}^{3}\{z \in \mathbb{C}\colon\ |z - \xi_k| \le \lambda\}$$

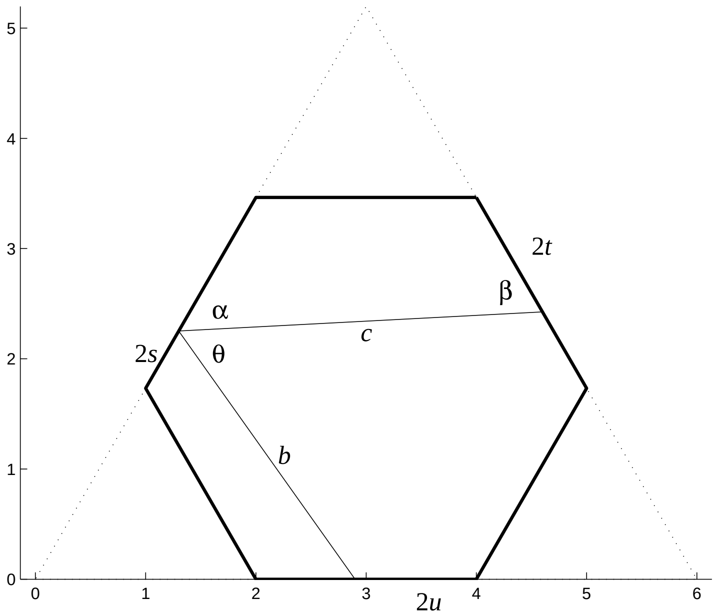

FIGURE 2. The circumscribed hexagon.

is the Reuleaux triangle with vertices at ξ_1, ξ_2, ξ_3. Since E is of constant width λ, we conclude that $E = \mathcal{R}$ as required.

In the remainder of this section we provide a proof of the first inequality of the theorem. We will need the following two lemmas. Recall that a hexagon is said to be *regular* if all its sides are the same length and opposite sides are parallel.

Lemma 3.2 (p. 127, [5])**.** *A set E of constant width $\lambda > 0$ can be inscribed in a regular hexagon H. Moreover, every side of the hexagon is a support line of E.*

Lemma 3.3. *Let H be a regular hexagon and T be a triangle with vertices on non-adjacent edges of H. Denote the side length of the hexagon by l and the distances of the vertices of T from three non-adjacent corners of H by $2s_0, 2t_0, 2u_0$, as shown in Figure 2. We do not consider the vertices of the hexagon, thus we restrict the variables to the open range: $s_0, t_0, u_0 \in (0, l/2)$. The triangle T can be rotated in both directions inside the hexagon if and only if $s_0 + t_0 + u_0 = 3l$.*

Proof. Due to scale invariance, we may normalize the situation and consider only a hexagon with side-length 2. We set up a coordinate system so that the hexagon is inscribed in an equilateral triangle with vertices $(0,0)$, $(6,0)$ and $(3, 3\sqrt{3})$. Let us denote the vertices of the triangle T by T_s, T_t and T_u.

Considering T as a rigid object we move T_s and T_t along the sides of the hexagon. Since the side lengths are fixed, we may consider the positions of the vertices after the move as a function of only s. By the sine rule we derive the expression

$$\frac{4-2s}{\sin\beta(s)} = \frac{c}{\sin 60^\circ}$$

from which we have

$$\sin(120^\circ - \alpha(s)) = \frac{4-2s}{c}\sin 60^\circ. \tag{3.4}$$

We denote by $\big(x(s), y(s)\big)$ the coordinates of T_u after the move. Then we easily calculate that

$$\big(x(s), y(s)\big) = (1+s)\big(1, \sqrt{3}\big) + b\big(\sin(\hat\theta(s)), \cos(\hat\theta(s))\big),$$

where $\hat\theta(s) = \alpha(s) + \theta + 30^\circ$.

Since T_u touches the boundary in the initial position, we see that $y(s)$ must have a minimum there in order that it be possible to rotate the triangle in both directions. Thus $y'(s_0) = 0$, where $s = s_0$ is the initial value. A calculation gives

$$y'(s) = \sqrt{3} - b\sin(\hat\theta(s))\hat\theta'(s) = \sqrt{3} - b\sin(\hat\theta(s))\alpha'(s).$$

From (3.4) we calculate α', which produces

$$y'(s) = \sqrt{3} - b\sin(\hat\theta(s))\frac{2\sin 60^\circ}{c\cos(120^\circ - \alpha(s))}.$$

Next we use the expression for $\sin(120^\circ - \alpha(s))$ to derive

$$c\cos(120^\circ - \alpha(s)) = \sqrt{c^2 - \tfrac{3}{4}(4-2s)^2}.$$

For the angle $\hat\theta$ we get

$$-\cos(\hat\theta(s_0)) = \sin(\hat\theta(s_0) - 90^\circ) = \frac{2+2s}{b}\sin 60^\circ,$$

from the sine rule in the lower left hand triangle. Thus we conclude that

$$b\sin(\hat\theta(s_s)) = \sqrt{b^2 - b^2\cos(\hat\theta(s_0))^2} = \sqrt{b^2 - \tfrac{3}{4}(2+2s_0)^2}.$$

With our previous expression for the derivative of y this gives

$$y'(s_0) = \sqrt{3} - \sqrt{3}\sqrt{b^2 - \tfrac{3}{4}(2+2s_0)^2}\frac{1}{\sqrt{c^2 - \tfrac{3}{4}(4-2s_s)^2}}.$$

Since the derivative is supposed to equal zero, we thus have

$$\sqrt{b^2 - \tfrac{3}{4}(2+2s_0)^2} = \sqrt{c^2 - \tfrac{3}{4}(4-2s_0)^2}.$$

Squaring and rearranging gives

$$b^2 - c^2 = \tfrac{3}{4}\big((2+2s_0)^2 - (4-2s_0)^2\big) = 9(2s_0 - 1).$$

By symmetry, we conclude that the equations

$$c^2 - a^2 = 9(2t_0 - 1) \quad \text{and} \quad a^2 - b^2 = 9(2u_0 - 1)$$

also hold. Adding these three equations gives $9(2(s_0 + t_0 + u_0) - 3) = 0$, which is the claim of the lemma, since $l = 2$ in our case. □

Proof of Theorem 3.1. It is clear that $d_3(D) \le \lambda$, and the claim regarding equality in the upper bound follows as in the proof of Lemma 2.2. Thus we move on to the lower bound.

By Lemma 3.2 we circumscribe a regular hexagon of width λ about D. Let the points of intersection between D and H be marked by $a_1, \dots, a_6$, in order along the perimeter. Since the 3-diameter is defined as a supremum over triples of boundary points, it is clear that

$$d_3(D) \ge \max\big\{d_3\big(\{a_1, a_3, a_5\}\big), d_3\big(\{a_2, a_4, a_6\}\big)\big\}.$$

We can therefore prove the claim by showing that

$$\max\big\{d_3\big(\{a_1, a_3, a_5\}\big), d_3\big(\{a_2, a_4, a_6\}\big)\big\} \ge= \frac{\sqrt{3}}{2}\lambda.$$

Due to scale invariance, we may normalize the situation and consider only $\lambda = 2\sqrt{3}$, in which case the hexagon has side-lengths 2. We set up a coordinate system so that the hexagon is inscribed in an equilateral triangle with vertices $(0,0)$, $(6,0)$ and $(3, 3\sqrt{3})$. As in Figure 2 we may describe a triple $\{a_1, a_3, a_5\}$ of points on the hexagons edges by a triple $\{s, t, u\}$ of numbers, each in the interval $[0,1]$. We define a function $f\colon [0,1]^3 \to [0,\infty)$ by

$$f(s,t,u) = d_3(\{a_1, a_3, a_5\}).$$

Since a_4 is diametrically opposite of a_1, etc., the triple $\{a_2, a_4, a_6\}$ is then described by the triple $\{1-s, 1-t, 1-u\}$. Thus

$$\max\big\{d_3\big(\{a_1, a_3, a_5\}\big), d_3\big(\{a_2, a_4, a_6\}\big)\big\} = \max\big\{f(s,t,u), f(1-s, 1-t, 1-u)\big\} =: g(s,t,u).$$

We now proceed to show that $\min_{s,t,u} g(s,t,u) = 3$, which will conclude the proof.

Suppose first that least one vertex, say a_1, of the triangle coincides with a vertex of the hexagon. Then the closed ball $\overline{B}(a_1, 2\sqrt{3})$ intersects the hexagon edges opposite to a_1 only in the corner points, and D is contained in the ball, since $\operatorname{diam} D = 2\sqrt{3}$. Since every edge of the triangle contains a point from the constant width set by construction, we find that these vertices have to be the alternating corners of the hexagon, see Figure 3. But then D is a Reuleaux triangle, so $D_3(D) = \lambda$, which is certainly not a minimum.

Suppose that $s, t, u \in (0,1)$ are such that $g(s,t,u) = \min_{x,y,z} g(x,y,z)$. Then by Lemma 3.3 we must have $s + t + u = \frac{3}{2}$. For if this were not the case then we could rotate the rigid triangle slightly so that two corner points remain on the hexagon and one corner point is outside the hexagon. But in that situation it is obvious that the outside point can be moved to the boundary of the hexagon decreasing the d_3-diameter of the triangle. Thus the original triangle was not minimal.

We can express the vertices of the triangle in terms of the parameters s, t, u as follows

$$T_s = (1+s)\big(1, \sqrt{3}\big), T_t = \big(4 + t, (2-t)\sqrt{(3)}\big), T_u = \big(4 - 2u, 0\big).$$

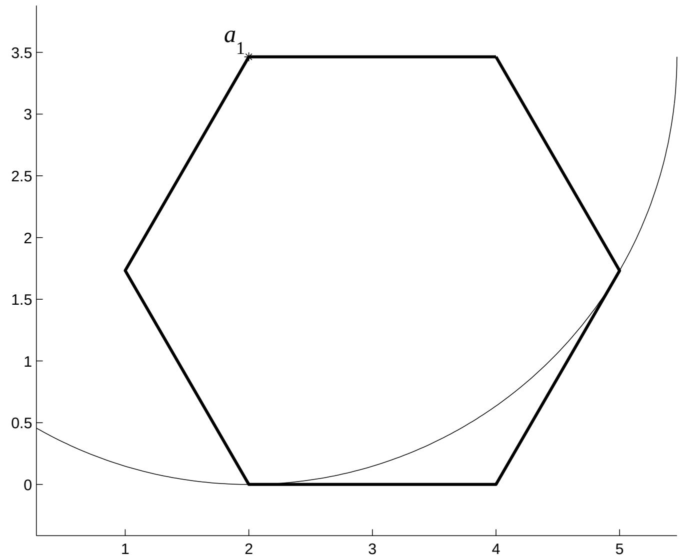

FIGURE 3. The minimum is not attained at the vertices.

From this we calculate

$$\begin{aligned} a^2 &= |T_t - T_u|^2 = (t+2u)^2 + 3(2-t)^2, \\ b^2 &= |T_s - T_u|^2 = (3-s-2u)^2 + 3(1+s)^2, \\ c^2 &= |T_s - T_t|^2 = (3+t-s)^2 + 3(s+t-1)^2. \end{aligned}$$

We denote $2s = 1 - z - w$ and $2t = 1 + z - w$ (i.e. $z = t - s$ and $w = 1 - s - t$) which together with the condition $s + t + u = \frac{3}{2}$ gives $2u = 1 + 2w$. With this notation we get

$$\begin{aligned} 4a^2 &= (2t+4u)^2 + 3(4-2t)^2 = (3+z+3w)^2 + 3(3-z+w)^2, \\ 4b^2 &= (6-2s-4u)^2 + 3(2+2s)^2 = (3+z-3w)^2 + 3(3-z-w)^2, \\ c^2 &= (3+z)^2 + 3w^2. \end{aligned}$$

Expanding a^2 we find that

$$4a^2 = (3+z)^2+6(3+z)w+9w^2+3(3-z)^2+6(3-z)w+3w^2 = 36-12z+4z^2+36w+12w^2;$$

hence, $a^2 = 9 - 3z + z^2 + 9w + 3w^2$. Similarly $b^2 = 9 - 3z + z^2 - 9w + 3w^2$. Thus we conclude that

$$f(s,t,u)^2 = a^2b^2c^2 = \big((z^2 - 3z + 9 + 3w^2)^2 - (9w)^2\big)\big((3+z)^2 + 3w^2\big).$$

Next we denote $p = z/3$ and $q = w^2/3$ and expand the previous expression:

$$f(s,t,u)^2 = 3^6\big(q^2 + (2p^2 - 2p - 1)q + (p^2 - p + 1)^2\big)\big((p+1)^2 + q\big).$$

Thus we have

$$f(s,t,u)^2 = 3^6\big(q^3 + 3p^2q^2 + 3(p^4 - 2p)q + (p^3+1)^2\big).$$

Next we use the expressions for z and w to derive $p = (t - s)/3$ and $q = (1 - s - t)^2/3$. Swapping $1 - t$ for t and $1 - s$ for s leaves q invariant, whereas p is changed to $-p$. Therefore we see that

$$f(1-s, 1-t, 1-u)^2 = 3^6\big(q^3 + 3p^2q^2 + 3(p^4 + 2p)q + (-p^3 + 1)^2\big).$$

We further conclude that

$$\begin{aligned} g(s,t,u)^2 &= \max\{f(s,t,u)^2, f(1-s,1-t,1-u)^2\} \\ &= 3^6\big(q^3 + 3p^2q^2 + 3(p^4 + 2|p|)q + (|p|^3 + 1)^2\big). \end{aligned}$$

Now every coefficient of q is positive. Therefore the least value of the expression occurs for $q = 0$. In this case $s + t = 1$ so that $u = \frac{1}{2}$ and we have

$$g(s,t,u) \geq g(s,t,\tfrac{1}{2}) = 3^3(|p|^3 + 1).$$

Again the left hand side takes the least value at $p = 0$, i.e. $t = s$, so finally

$$g(s,t,u) \geq g(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}) = 3^3,$$

from which we get the 3-diameter by taking the cube root. □

4. d_3-COMPLETE SETS

Recall that a set $E \subset \mathbb{C}$ is called d_2*-complete* if

$$d_2(E \cup \{z\}) > d_2(E) \qquad \text{for any} \quad z \in \mathbb{C} \setminus E.$$

Definition 4.1. A set $E \subset \mathbb{C}$ containing at least three points is called d_3*-complete* if

$$d_3(E \cup \{z\}) > d_3(E) \qquad \text{for any} \quad z \in \mathbb{C} \setminus E.$$

Clearly, d_3-complete sets are closed.

Proposition 4.2. *Closed disks are* d_3*-complete.*

Proof. Clearly, it is enough to show that $\overline{\mathbb{D}}$ is d_3-complete. Given $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$, let ξ_1 and ξ_2 be the unique points on $\partial\mathbb{D}$ with $|\xi_1 - \xi_2| = \sqrt{3}$ and equidistant from z. Obviously, $|\xi_1 - z| = |\xi_2 - z| > \sqrt{3}$. Then

$$d_3(\overline{\mathbb{D}} \cup \{z\}) \geq \Big[|\xi_1 - \xi_2||\xi_1 - z||\xi_2 - z|\Big]^{\frac{1}{3}} > \sqrt{3} = d_3(\overline{\mathbb{D}}).$$

Thus $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$ cannot be added to $\overline{\mathbb{D}}$ without increasing its d_3-diameter, which completes the proof. □

Recall that every set of diameter λ is contained in a d_2-complete set of diameter λ (see, for instance, [5, Theorem 54]). By modifying the proof of Theorem 54 [5], we show that similar result holds for d_3-complete sets.

Theorem 4.3. *Every set* E *with* $d_3(E) = \lambda$ *is contained in a* d_3*-complete set* $\tilde{E}$ *with* $d_3(\tilde{E}) = \lambda$.

Proof. Let

$$\mathcal{U}(E) = \{z \in \mathbb{C}\colon\ d_3(E \cup \{z\}) = d_3(E)\} \quad \text{and} \quad \mathcal{B}(E) = \{z \in \mathcal{U}(E)\colon\ \operatorname{dist}(z, E) = \delta(E)\}$$

where

$$\delta(E) = \sup\{\operatorname{dist}(z, E)\colon\ z \in \mathcal{U}(E)\}.$$

Next, we inductively construct sets E_n, $n = 1, 2, \ldots$, with $d_3(E_n) = \lambda$. We select $z_1 \in \mathcal{B}(E)$ and define $E_1 = E \cup \{z_1\}$. By definition we have $d_3(E_1) = d_3(E) = \lambda$. If E_n has been defined, we select $z_{n+1} \in \mathcal{B}(E_n)$ and define $E_{n+1} = E_n \cup \{z_{n+1}\}$. Clearly, $d_3(E_{n+1}) = d_3(E_n)$ and hence $d_3(E_n) = \lambda$ for each n. We define

$$\tilde{E} = \overline{\bigcup_{n=1}^{\infty} E_n}$$

and note that this is a bounded set, say $\tilde{E} \subset B(0, R)$. We have $d_3(\tilde{E}) = \lambda$ by the continuity of d_3-diameter.

Suppose that we can choose $z \in \mathbb{C} \setminus \tilde{E}$ with $d_3(\tilde{E} \cup \{z\}) = d_3(\tilde{E})$. Then it follows that

$$\lambda = d_3(E_i) \le d_3(E_i \cup \{z\}) \le d_3(\tilde{E} \cup \{z\}) = \lambda$$

so we conclude that $z \in \mathcal{U}(E_i)$. We define $\delta = \operatorname{dist}(z, \tilde{E}) > 0$ and note that

$$\delta(E_i) \ge \operatorname{dist}(z, E_i) \ge \operatorname{dist}(z, \tilde{E}) = \delta$$

for every $i \ge 1$. Let $j > i$. Since $z_i \in E_i \subset E_{j-1}$ and since $z_j \in \mathcal{B}(E_{j-1})$, we have

$$|z_j - z_i| \ge \operatorname{dist}(z_j, E_{j-1}) = \delta(E_{j-1}) \ge \delta,$$

for all i, j, which is a contradiction since the points $\{z_i\}$ all lie in $B(0, R)$. Therefore no point $z \in \mathbb{C} \setminus \tilde{E}$ with $d_3(\tilde{E} \cup \{z\}) = d_3(\tilde{E})$ can be found, as was to be shown. □

The previous theorem allows us to show the existence of nontrivial d_3-complete sets, as the following propositions illustrate.

Proposition 4.4. *Let E be a triangle with vertices at the points $1, -1, i$ and $\tilde{E}$ be a d_3-complete set containing E with $d_3(\tilde{E}) = d_3(E)$. Then $\tilde{E}$ is not a disk.*

Proof. First, we observe that $(1, -1, i) \in T(E)$ and hence $d_3(E) = 4^{1/3}$. Since $d_3(\tilde{E}) = d_3(E)$, we have $(1, -1, i) \in T(\tilde{E})$. In particular, $(1, -1, i) \in \partial\tilde{E}$. The only disk containing the points $1, -1, i$ in its boundary is the unit disk $\mathbb{D}$. By Proposition 2.1, we have $d_3(\mathbb{D}) = \sqrt{3}$. Hence $\tilde{E} \ne \mathbb{D}$ completing the proof. □

Proposition 4.5. *There exist d_3-complete sets with two components.*

Proof. Let $E = \{-1,\ a,\ 1\}$, $a > 0$ and let $\tilde{E}$ be its completion. Then $\tilde{E}$ does not contain any point of the imaginary axis:

$$d_3(E)^3 = |-1-1|\,|1-a|\,|1+a| < |-1-1|\,|1-0|\,|-1-0|) \le |-1-1|\,|1-iy|\,|1+iy|,$$

for every real y. In fact, $\tilde{E}$ lies in the Cassinian oval $C(-1, 1; a)$, which has two components containing $\{-1\}$ and $\{1, a\}$, respectively. □

5. Sets of constant 3-diameter

Recall that a set is called *regular* if it is the closure of its interior.

Definition 5.1. A regular compact set $E \subset \mathbb{C}$ is said to be of constant d_3-diameter if

$$\sup_{a,b\in E}\Big[|a-b||a-\xi_0||b-\xi_0|\Big]^{\frac{1}{3}} = d_3(E)$$

for every $\xi_0 \in \partial E$.

It is easy to see that every disk is a set of constant d_3-diameter. The purpose of this section is to show that there are other such sets, as well.

Theorem 5.2. *If $E \subset \mathbb{C}$ is d_3-complete, then it is of constant d_3-diameter.*

Proof. We show first that completeness of E implies that it is of constant d_3-diameter. Indeed, given $z_0 \in \partial E$ assume to the contrary that

$$\sup_{a,b\in E}\Big[|a-b||a-z_0||b-z_0|\Big]^{\frac{1}{3}} < d_3(E).$$

Then by continuity there exists $\varepsilon > 0$ such that

$$\sup_{a,b\in E}\Big[|a-b||a-z||b-z|\Big]^{\frac{1}{3}} < d_3(E)$$

for all $z \in B(z_0,\varepsilon)$. Since z_0 is a boundary point of E we find $z_1 \in B(z_0,\varepsilon)\setminus E$. Then $E\cup\{z_1\}$ is an expansion of E which does not increase the d_3-diameter, a contradiction. Hence E is of constant d_3-diameter. □

As a consequence of Proposition 4.4 and Theorem 5.2 we obtain the following result.

Theorem 5.3. *There exist infinitely many non-circular sets of constant 3-diameter.*

Although we have proven that there are infinitely many sets of constant 3-diameter, we only know the disk explicitly, so far. In the next section we will construct a non-trivial example of a set with constant 3-diameter.

For the converse of Theorem 5.2 we do not have a complete theorem. We content ourselves by giving two results with simple proofs, and some conjectures. First, however, a technical lemma.

Lemma 5.4. *If $d_3(E) \ge 2^{-1/3}d_2(E)$, then for every triple $(a,b,c) \in T(E)$ the oval $C(a,b;c)$ is convex.*

Proof. Let $(a,b,c) \in T(E)$ and suppose that c lies inside $C(a,b;|a-b|^2/2)$. Then

$$d_3(E)^3 = |a-b|\,|a-c|\,|b-c| < \tfrac{1}{2}|a-b|^3 \le \tfrac{1}{2}d_2(E)^3,$$

contrary to assumption. Therefore $c \in C(a,b;\lambda)$ for some $\lambda \ge |a-b|^2/2$. But, as was said above, such an oval $C(a,b;\lambda)$ is convex. □

Proposition 5.5. *Let $E \subset \mathbb{C}$ be a constant d_3-diameter set. Then it is d_3-complete if either*

(1) $d_2(E) \le 2^{1/3}d_3(E)$; *or*

(2) *E is connected and has C^1 boundary.*

Proof. Let $E \subset \mathbb{C}$ be of constant d_3-diameter $\lambda > 0$ with $d_2(E) \leq 2^{1/3} d_3(E)$. We define

$$\hat{E} = \bigcap_{a,b \in E} C\Big(a, b; \frac{\lambda^3}{|a-b|}\Big).$$

The oval $C(a, b; \frac{\lambda^3}{|a-b|})$ is convex, since $|a-b| \leq d_2(E) \leq 2^{1/3}\lambda$. Therefore $\hat{E}$ is convex and in particular connected. If $z \in \partial E$, then z lies on the boundary of one of the ovals in the definition of $\hat{E}$. Therefore $z \in \partial \hat{E}$. Since $\hat{E}$ is connected and $\partial E \subset \partial \hat{E}$ we see that $E = \hat{E}$. It is clear that $\hat{E}$ is complete, so this concludes the proof in the first case.

Suppose next that E is connected and has C^1 boundary. Let $x \in \partial E$, and choose $a, b \in E$ for which $d_3(a, b, x) = d_3(E)$. Since E is connected, the Cassinian oval $C\big(a, b; \frac{\lambda^3}{|a-b|}\big)$ is necessarily connected. Let R_x be the exterior normal ray of this oval at x. An easy calculation shows that R_x intersects the oval only at x. Therefore we conclude that no point can be added to R_x without increasing the 3-diameter. On the other hand, R_x sweeps out all of $\mathbb{C} \setminus E$ as x varies over ∂E, because of the C^1 assumption. Thus E is complete, in this case also. □

Although we are not able to prove it, we expect constant diameter and complete sets to be equivalent notions for 3-diameter.

Conjecture 5.6. *A set of constant* 3*-diameter is complete.*

We actually have a proof which shows that a connected set of constant 3-diameter is complete. This proof is very long and technical, so we do not include it here. However, if one could show the following conjecture, then it would imply the previous conjecture, in view of the $\hat{E}$-construction presented in the previous proof.

Conjecture 5.7. *A set of constant* 3*-diameter has at most* 2 *components.*

6. The Reuleaux square

The simplest non-trivial constant width set is the Reuleaux triangle. Similarly one can construct regular Reuleaux polygons with $2k+1$ vertices. In this section we show that we can construct a constant 3-diameter set on the vertices of a square.

We define the planar set E by

$$E = C(1+i, 1-i; 2^{7/6}) \cap C(1-i, -1-i; 2^{7/6}) \cap C(-1-i, -1+i; 2^{7/6}) \cap C(-1+i, 1+i; 2^{7/6}).$$

This set is shown in Figure 4. From the definition is is immediately clear that if we add a point to E, then we get a set with 3-diameter greater than $2^{7/6}$. However, it is not immediately clear that $d_3(E) = 2^{7/6}$. Certainly, the lower bound $d_3(E) \geq 2^{7/6}$ is clear; for instance, it is realized by any triple of distinct corners of the square.

Consequently, the rest of the section will be dedicated to showing that $d_3(E) \leq 2^{7/6}$. Thus we will prove

Theorem 6.1. *The "Reulaux" square,*

$$E = C(1+i, 1-i; 2^{7/6}) \cap C(1-i, -1-i; 2^{7/6}) \cap C(-1-i, -1+i; 2^{7/6}) \cap C(-1+i, 1+i; 2^{7/6})$$

has constant d_3-diameter equal to $2^{7/6}$.

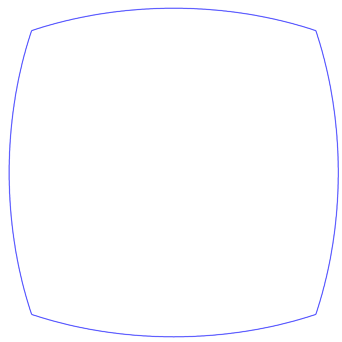

FIGURE 4. A set of constant 3-diameter.

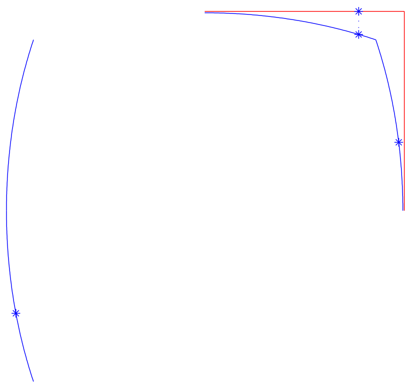

FIGURE 5. A set of constant 3-diameter.

Proof. As has been noted before, we need only check triples of boundary points of E. It is easy to see that a configuration with two or three points on one side of the square is not d_3-maximal. Thus, by symmetry, we assume that the three points lie one on each of the left, top and right sides of the square. We denote these points by L, T and R, respectively. Further, we may assume that the T lies in the (closed) right half-plane.

Assume that L, T and R forms a critical configuration, i.e. that $d_3(E) = d_3(\{L, T, R\})$. We start by showing that R is not in the first quadrant. We first note that $d_3(\{L, T, R\})^3 \leq d(E)^2|T - R|$. Thus the configuration is not critical if $|T - R| < \sqrt{2}$. Define $\theta = \sqrt{4\sqrt{2} - 1} - 1 < 7/6$, and note that $(0, \theta)$ is the topmost point in E, and similarly for other directions. We can move the points T and R to the lines $[7i/6, 7/6 + 7i/6)]$ and $[7/6, 7/6 + 7i/6]$, respectively, and this will only make the d_3 distance larger, see Figure 5.

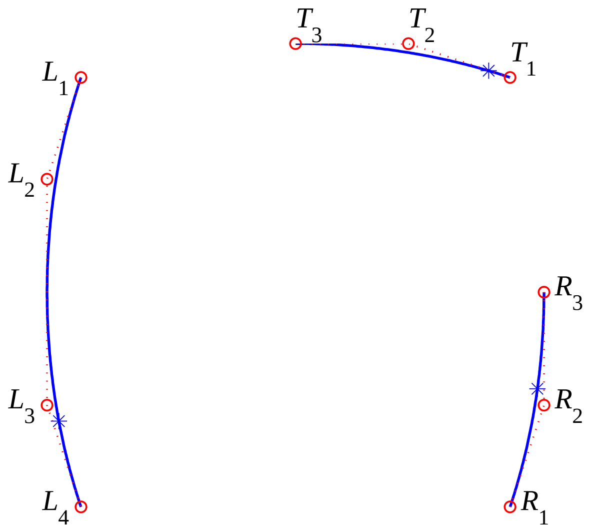

FIGURE 6. A set of constant 3-diameter.

Fix $R = (7/6, s)$ and $T = (t, 7/6)$. It is clear that L should be chosen in the lower left corner in order to minimize $d_3(\{L,T,R\})$. Then we have

$$d_3(\{L,T,R\})^6 = \big((t+1)^2 + \tfrac{169}{36}\big)\big((s+1)^2 + \tfrac{169}{36}\big)\big((\tfrac{7}{6}-s)^2 + (\tfrac{7}{6}-t)^2\big).$$

Since we want to show that our points were not critical, we will show that this expression is smaller than $(2^{7/6})^6 = 128$, under the previously derived constraint $|T-R| \geq \sqrt{2}$. The last condition implies that $(\frac{7}{6})^2 + (\frac{7}{6}-t)^2 \geq 2$, from which we get $t \leq 0.4$; similarly $s \leq 0.4$. Next, we calculate

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}&\Big[\big((t+1)^2+\tfrac{169}{36}\big)\big((\tfrac{7}{6}-s)^2+(\tfrac{7}{6}-t)^2\big)\Big]\\ &= (t+1)\big((\tfrac{7}{6}-s)^2+(\tfrac{7}{6}-t)^2\big) - (\tfrac{7}{6}-t)\big((t+1)^2+\tfrac{169}{36}\big)\\ &\leq 2\cdot\tfrac{7}{5}\cdot\tfrac{49}{36} - (\tfrac{7}{6}-\tfrac{2}{5})\big(1+\tfrac{169}{36}\big) = -\tfrac{599}{1080}.\end{aligned}$$

The same estimate holds for the partial derivative with respect to s. Thus we should chose $s = t = 0$ to get the largest d_3-diameter:

$$d_3(\{L,T,R\})^6 \leq 2\big(1+\tfrac{169}{36}\big)^2(\tfrac{7}{6})^2.$$

Then a simple computation shows that $d_3(\{L,T,R\}) < 2^{7/6}$, which means that these points are not critical.

We have shown that R is in the fourth quadrant in a critical configuration. Let us now define $\theta_2 = 4-3\theta$ and the points $R_1 = 1-i$, $R_2 = \theta - \theta_2 i$ and $R_3 = \theta$, see Figure 6. The points T_i and L_i are defined similarly, as indicated in the figure. We note that the segments $S_1 = [R_1, R_2]$ and $S_2 = [R_2, R_3]$ lie outside E. If R is on the Cassinian oval, and R' has the same y coordinate as R, but lies on one of these line segments, then $d_3(\{L,T,R\}) < d_3(\{L,T,R'\})$. Therefore it suffices to show that $d_3(\{L,T,R'\}) \leq 2^{7/6}$.

Suppose that for some L and T the Cassinian oval $C(L,T;2^{7/2}/|L-T|)$ does not contain $S_1\cup S_2$. Since this oval is convex, we see that it does not contain all of the points $1-i$, $\theta-\theta_2 i$ and θ. Therefore the problem is reduced to showing $d_3(\{L,T,R\})\le 2^{7/6}$ when R is one of these three points. By the same argument as before, we see that

$$d_3(\{L,T,R_l\})\le \max_{j\in[1,4],k\in[1,3]} d_3(\{L_j,T_k,R_l\}).$$

A simple but tedious calculation shows that $\max_{j\in[1,4],k\in[1,3]} d_3(\{L_j,T_k,R_l\}) < 2^{7/6}$ when $l=2$ and $l=3$.

Therefore, it only remains to consider the case $R=R_1$, i.e. when R is the lower right corner. In this case it is easy to see that it is not possible to use the previous polygonal approximation of the domain. Define the function $f\colon[-1,1]\to[1,\theta]$ by $f(x)=\sqrt{2\sqrt{8+x^2}-x^2-1}-1$. Note that the arcs of the Cassinian ovals are of the form $(\pm t,\pm f(t))$ and $(\pm f(t),\pm t)$. Let y denote the imaginary part of L and let x denote the real part of T. Then $L=f(y)+yi$ and $T=x+f(x)i$. Since $R=1-i$ we have

$$d_3(\{L,T,R\})^6=\big((f(y)-1)^2+(y-1)^2\big)\big((x-1)^2+(f(x)+1)^2\big)\big((x+f(y))^2+(y+f(x))^2\big).$$

Denote the expression on the right by $g(x,y)$. Note that $g(x,1)\equiv 2^7$, since in this case T and R are the foci of the Cassinian oval over which L is varying. Next we calculate

$$\frac{\partial \ln g}{\partial x}=\frac{x-1+(f(x)+1)f'(x)}{(x-1)^2+(f(x)+1)^2}+\frac{x+f(y)+(f(x)+y)f'(x)}{(x+f(y))^2+(y+f(x))^2}.$$

We claim that the second term takes on a minimum at $y=1$ (see below). Thus $\frac{\partial \ln g}{\partial x}(x,y)\ge \frac{\partial \ln g}{\partial x}(x,1)=0$. This in turn implies that $g(x,y)\le g(x,1)=2^7$.

Therefore, we need only show that $\frac{\partial \ln g}{\partial x}(x,y)\ge \frac{\partial \ln g}{\partial x}(x,1)$ in order to complete the proof. Writing out this inequality, and clearing the denominators, gives the equivalent inequality

$$\begin{aligned}&\big(x+f(y)+(f(x)+y)f'(x)\big)\big((x+1)^2+(1+f(x))^2\big)\\ &\quad\ge \big(1+x+(1+f(x))f'(x)\big)\big((x+f(y))^2+(y+f(x))^2\big).\end{aligned}$$

We will prove a stronger inequality in which we replace $f(y)$ on the left hand side by 1, which can be done, since $f(y)\ge 1$. The claim then is that

$$\begin{aligned}h(x,y)=&\big(1+x+(f(x)+y)f'(x)\big)\big((x+1)^2+(1+f(x))^2\big)\\ &-\big(1+x+(1+f(x))f'(x)\big)\big((x+f(y))^2+(y+f(x))^2\big)\end{aligned}$$

is non-negative. Note that $h(x,1)=0$. Next we calculate

$$\begin{aligned}\frac{\partial h}{\partial y}&=f'(x)\big((x+1)^2+(1+f(x))^2\big)\\ &\quad-2\big(1+x+(1+f(x))f'(x)\big)\big((x+f(y))f'(y)+y+f(x)\big).\end{aligned}$$

Since $f'(x)\le 0$, the first term is non-positive. Moreover, from the definition of f we calculate $1+x+(1+f(x))f'(x)=\frac{x}{\sqrt{8+x^2}}+1>0$. Thus $\frac{\partial h}{\partial y}\le 0$ provided

$$H(x,y)=(x+f(y))f'(y)+y+f(x)\ge 0.$$

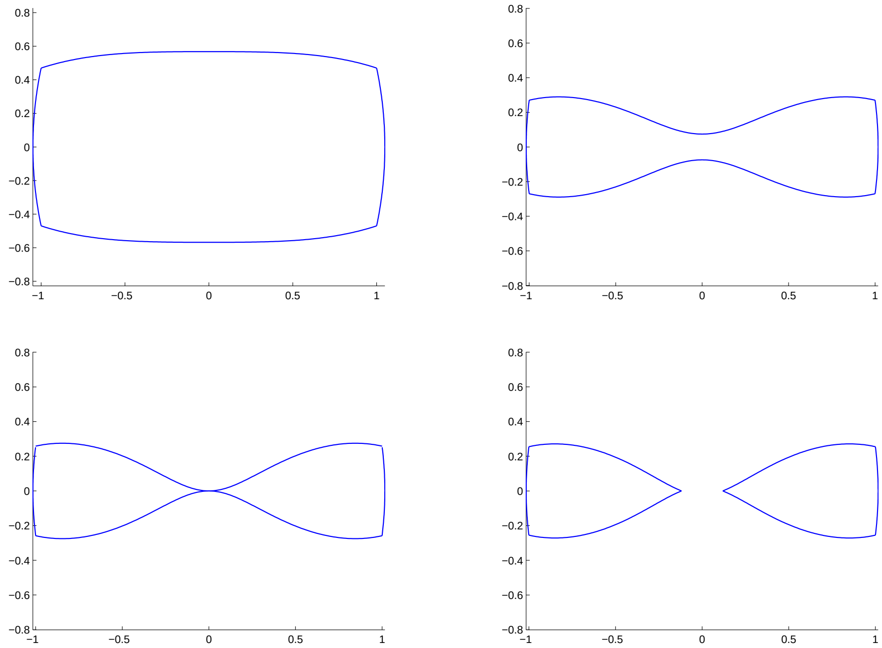

FIGURE 7. Reulaux rectangles?

As before, $H(1,y) = \frac{y}{\sqrt{8+y^2}} + 1 > 0$. On the other hand, $\frac{\partial H}{\partial x} = f'(y) + f'(x) \leq 0$. Therefore $H(x,y) \geq H(1,y) > 0$. This implies that $\frac{\partial h}{\partial y} \leq 0$, which, combined with the fact that $h(x,1) = 0$, implies that $h(x,y) \geq 0$. As was already shown, this gives the estimate for g that we needed, so the proof is done. □

Question 6.2. Let $a \in (0,1)$. We construct a Reuleaux rectangle on the set of vertices $(\pm 1, \pm a)$ by drawing arcs of Cassinian ovals with foci $(\pm 1, a)$ through $(\pm 1, -a)$, etc. Four such rectangles are shown in Figure 7. Is it true that these sets are always of constant 3-diameter?

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