CP3-09-36

The Klauder–Daubechies Construction of the Phase Space Path Integral and the Harmonic Oscillator

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The canonical operator quantisation formulation corresponding to the Klauder–Daubechies construction of the phase space path integral is considered. This formulation is explicitly applied and solved in the case of the harmonic oscillator, thereby illustrating in a manner complementary to Klauder and Daubechies' original work some of the promising features offered by their construction of a quantum dynamics. The Klauder–Daubechies functional integral involves a regularisation parameter eventually taken to vanish, which defines a new physical time scale. When extrapolated to the field theory context, besides providing a new regularisation of short distance divergences, keeping a finite value for that time scale offers some tantalising prospects when it comes to strong gravitational quantum systems.

1 Introduction

The central rôle of the phase space symplectic one-form in the quantisation programme is well known and understood. No less important and crucial to the physical properties of the quantum system however, is the rôle of an implicit phase space Riemannian metric—albeit a "shadow phase space metric" [1–6] for what the classical world is concerned. This is convincingly argued by John R. Klauder in an insightful and thought provoking paper [1] deserving to be much more widely known, which relies on prior work with Ingrid Daubechies [7–12]. Making the rôle of this phase space Riemannian metric explicit circumvents the ambiguities and difficulties inherent to the formal path integral definition of a quantised system. It even provides for a perfectly well defined functional integral over continuous paths in phase space. One requirement these two geometrical structures on phase space have to meet is that the symplectic and Riemannian metrics both define an identically normalised phase space volume form.

The Klauder–Daubechies construction is achieved through stochastic calculus methods, involving a Wiener measure of which the diffusion parameter provides a regulator which, when eventually taken

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to infinity, reproduces the correct quantum mechanical amplitudes obeying the Schrödinger equation of the system. In terms closer to a physicist's intuition perhaps, this Wiener measure is associated to the statistical Brownian motion of a particle propagating in the background Riemannian geometry of phase space, with a specific diffusion time scale taken eventually to vanish. One of the remarkable features of the Klauder–Daubechies construction of the phase space path integral is its inherent manifest covariance under general canonical transformations in phase space, much in contradistinction to all other approaches leading to path integral representations of quantum amplitudes, or even to naive canonical operator quantisation rules.

In order to make these statements somewhat more explicit, for the purpose of this introductory discussion only let us assume a system of physical units such that all relevant parameters and scales are set to unity, inclusive of $\hbar = 1$, and let us restrict to a single degree of freedom system, $q(t) \in \mathbb{R}$, with canonically conjugate momentum, $p(t) \in \mathbb{R}$, obeying at the quantum level the Heisenberg algebra, $[Q, P] = i\mathbb{I}$, $Q^\dagger = Q$, $P^\dagger = P$. If $|q, p\rangle$ denote the normalised canonical Weyl–Heisenberg quantum coherent states labelled by all classical phase space states and associated to the normalised Fock vacuum $|0\rangle$ such that $(Q + iP)\,|0\rangle = 0$ [13], the Klauder–Daubechies path integral (KD-PI) representation of the associated matrix elements of the quantum system's evolution operator of Hamiltonian $\hat{H}_0(Q, P)$ is given in the form [1],

$$\langle q_f, p_f| e^{-iT\hat{H}_0} |q_i, p_i\rangle = \lim_{\tau_0 \to 0^+} e^{C_0 \frac{T}{\tau_0}} \int_{(q(t_i),p(t_i))=(q_i,p_i)}^{(q(t_f),p(t_f))=(q_f,p_f)} \left[\frac{\mathcal{D}q(t)\mathcal{D}p(t)}{2\pi}\right] \times \\ \times e^{i\int_{t_i}^{t_f} dt\left[\frac{1}{2}(\dot{q}p - q\dot{p}) - h(q,p)\right]} \times e^{-\tau_0 \int_{t_i}^{t_f} dt \frac{1}{2}\left(\dot{q}^2 + \dot{p}^2\right)}, \quad (1)$$

where the time interval, $T = t_f - t_i$, is such that $T > 0$, and $\tau_0 > 0$ is a time scale regularisation parameter. In this expression one integrates over all those paths in phase space possessing as end points those classical states associated to the external quantum coherent states. Furthermore, $h(q, p)$ is the coherent state symbol representing the Hamiltonian operator through [13]

$$\hat{H}_0 = \int_{(\infty)} \frac{dq\,dp}{2\pi} |q, p\rangle\, h(q, p)\, \langle q, p|, \qquad \int_{(\infty)} \frac{dq\,dp}{2\pi} |q, p\rangle\, \langle q, p| = \mathbb{I}. \quad (2)$$

To lowest order in $\hbar$, $h(q, p)$ coincides with the classical Hamiltonian, $H_0(q, p)$. One thus recognizes in the phase of the first exponential factor inside the path integral the first-order Hamiltonian action of the system, inclusive of quantum corrections in $h(q, p)$. In the absence of these corrections, that phase factor provides the usual naive formal definition of the quantised system through the phase space path integral, a definition however, which is not free of ambiguities nor difficulties, among which a lack of covariance under phase space canonical transformations. Note in particular the contribution to that Hamiltonian action in $dt(\dot{q}p - q\dot{p})/2 = (dq\,p - q\,dp)/2 = K$, which defines the symplectic one-form of the phase space symplectic geometry. The associated volume form, $\omega = dK = dp \wedge dq$, is thus normalised to unity.

However, the expression in (1) carries still two further τ_0 dependent exponential factors. Returning to the very first one later on, the very last one inside the path integral is of a purely statistical character, being purely real gaussian, in contradistinction to the previous pure phase factor of a purely quantum mechanical character. In effect the real gaussian factor plays the rôle of a phase space Wiener measure which regularises, for any finite $\tau_0 > 0$, the ordinary naive path integral based on the purely imaginary (gaussian and higher order) phase factor alone. Furthermore one recognizes in that real gaussian contribution precisely the Brownian motion of a particle in the background phase space euclidean geometry associated to the Weyl–Heisenberg algebra defined by the operators Q and P (other homogeneous geometries are also discussed in Ref. [1]). Note that the volume element associated to that Riemannian geometry with as metric a tensor given by the unit matrix, is again normalised to unity, as is the volume form associated to the symplectic one-form involved in the pure phase factor.

Note well that by having introduced the time scale τ_0, the one dimensional system with configuration space coordinate q and two dimensional phase space (q, p) has been promoted to some two dimensional system with configuration space (q, p), hence a four dimensional phase space, of which the dimensional reduction back to the space q only is achieved through the limit $\tau_0 \to 0^+$. As such the Lagrangian action for this effective two dimensional system reads,

$$\int_{t_i}^{t_f} dt \left[\frac{1}{2} i\tau_0 \left(\dot{q}^2 + \dot{p}^2\right) + \frac{1}{2}\left(\dot{q}p - q\dot{p}\right) - h(q, p)\right]. \quad (3)$$

In this expression one recognizes the action of a particle of pure positive imaginary mass, $m_0 = i\tau_0$, moving in a two dimensional euclidean plane, subjected to a potential energy $h(q,p)$, as well as a velocity dependent, hence magnetic, coupling defined by the symplectic one-form of the Hamiltonian formulation of the original system, as if the particle were coupled to a static homogeneous magnetic field perpendicular to the two dimensional space (q,p). Except for the mass factor which is not real, this is precisely a generalised Landau problem in phase space with interaction energy $h(q,p)$ [2, 3]. As is well known, in the absence of this interaction energy, the energy levels of the quantised Landau problem are organised in infinitely degenerate discrete Landau levels, with a gap set by the ratio of the magnetic coupling to the mass. In the presence of the interaction energy $h(q,p)$, the Landau level degeneracies are lifted but states are still organised in discrete Landau sectors with a gap set by the same ratio. In the limit of a vanishing mass, namely in the present context the limit $\tau_0 \to 0^+$, this gap grows infinite. Hence in order that not all Landau sectors decouple one has to adjust the quantum vacuum energy of the lowest Landau sector such that the energy of all states in that sector retain a finite energy in that limit. This is precisely the reason for the very first exponential factor in (1) multiplying the path integral, C_0 being some normalisation factor to be adjusted accordingly (which may be done up to an arbitrary finite contribution even when $h(q,p) = 0$ [1]). And as a consequence, the surviving quantum states of the lowest Landau sector span the quantum Hilbert space of the original quantum system with the single degree of freedom q. Dimensional reduction in phase space is achieved for the quantum system by projecting onto its lowest Landau sector the extended quantised system, as defined by (1). Note that by the same token noncommutativity in the (q,p) space is induced once again through that projection, out of commuting operators (q,p) as configuration space coordinates for the extended dynamics. In essence, this is the genesis of the noncommutative Moyal plane of noncommutative quantum mechanics as well.

Incidentally, besides this intriguing possibility of having "extra dimensions" introduced in a dynamics which are neither of a space- nor a time-like character (as in Kaluza-Klein or string theory contexts) but are rather of a phase space character, the mixture of both purely quantum and statistical behaviours present in the formulation of a quantum dynamics as provided by the KD-PI construction in (1), reminds one of progress made by G. 't Hooft [14] with precisely such motivations in mind towards a deterministic formulation of quantum dynamics displaying at the same time a stochastic behaviour.

Until recently [15, 16] to the present authors' best knowledge, and in spite of all the potential interest offered by this approach to quantum dynamics, if only to illustrate explicitly the workings of (1) no actual evaluation of the KD-PI was available—certainly not for a finite value for τ_0—even for as simple a test-bed system as the harmonic oscillator, the basis for all of perturbative relativistic quantum field theory. Certainly to the authors of the KD-PI is it clear—having proved it—that the quantum dynamics of the original system is recovered in the limit $\tau_0 = 0$. But if the formulation is to find practical applications, some explicit evaluations with finite τ_0 are most presumably useful. Furthermore, besides the path integral point of view on which the construction of (1) is based, a complementary understanding of the quantum properties of the extended system associated to the action in (3) from the canonical operator quantisation point of view should prove to be of relevance as well, and could lead to further insight into the workings of the limit $\tau_0 = 0$. Finally, keeping the value for τ_0 finite may also be of interest in the context of deformations of algebraic structures associated to quantum dynamics in a more general setting, for instance that of quantum gravity and noncommutative geometries of spacetime [15–17].

The purpose of the present paper is not to justify the result in (1), but rather, by starting from it, to show explicitly that it indeed reproduces the correct quantum dynamics of the harmonic oscillator, and thereby acquire greater familiarity with the meaning of the Klauder–Daubechies approach and the prospects it may offer. And since this has already been done in Refs. [15,16] through a direct saddle point evaluation of the path integral (1) for a finite τ_0 and in the limit $\tau_0 = 0$, the same issue is addressed here directly from the canonical operator quantisation point of view, based on the τ_0 deformed effective action (3) of the system defined over the original phase space promoted to a two dimensional configuration space. In the case of the one dimensional harmonic oscillator, one is thus dealing with a Landau problem with pure positive imaginary mass subjected to a harmonic potential well. Even though the quantum solution for that system should be straightforward enough, its lack of unitarity and its properties under the limit $\tau_0 = 0$ are sufficiently instructive to deserve a detailed analysis. At the same time, a broader and perhaps clearer understanding of the relevance and potential interest of the Klauder–Daubechies construction of the phase space path integral is achieved.

The paper is organised as follows. In Section 2, the canonical formulation associated to the extended action (3) is constructed. Section 3 then applies this formalism to the one dimensional harmonic oscillator to construct the canonical quantisation of its extended formulation, and its quantum solution, enabling thereby an explicit analysis of the limit $\tau_0 = 0$ corresponding to the effective projection onto the lowest Landau sector of the system. Section 4 then addresses the evaluation of the projected quantum evolution operator for a finite value of τ_0, to compare with the saddle point evaluation of Refs. [15,16]. Finally, some conclusions are presented in Section 5.

2 Canonical Formulation of the Extended System

First let us still consider an arbitrary one degree of freedom system, with canonically conjugate phase space variables (q, p) and classical Hamiltonian $H_0(q, p)$, and reinstate dimensionful quantities, inclusive of all explicit factors of $\hbar$. In order to account for the different physical dimensions of the configuration space variable, q, and its conjugate momentum, p, let us also introduce a constant factor λ_0 having the dimension of mass times angular frequency. It proves then useful to work in terms of the following rescaled phase space coordinates, ϕ^a $(a = 1, 2)$, with

$$\phi^1 = \frac{1}{\sqrt{\lambda_0}} p, \qquad \phi^2 = q\sqrt{\lambda_0}, \tag{4}$$

having the canonical Poisson brackets, $\left\{\phi^a, \phi^b\right\} = -\epsilon^{ab}$, ϵ^{ab} being the two dimensional antisymmetric symbol with $\epsilon^{12} = +1$. Hence the classical Hamiltonian first-order action of the system reads,

$$S_0[\phi^a] = \int dt \left[\frac{1}{2}\left(\dot{q}p - q\dot{p}\right) - H_0(q,p)\right] = \int dt \left[\frac{1}{2}\epsilon_{ab}\phi^a\dot{\phi}^b - H_0(\phi^a)\right]. \tag{5}$$

For what the extended system is concerned, the function $H_0(\phi^a)$ gets replaced by the symbol $h(\phi^a) = h(q, p)$, while the associated Lagrangian action reads,

$$S[\phi^a] = \int dt \left[\frac{1}{2} i\tau_0 \delta_{ab}\dot{\phi}^a\dot{\phi}^b + \frac{1}{2}\epsilon_{ab}\phi^a\dot{\phi}^b - h(\phi^a) + E_0\right], \tag{6}$$

δ_{ab} being the phase space euclidean metric, and E_0 some ($\hbar$ dependent) constant to be adjusted later on in order to retain quantum states of finite energy in the limit $\tau_0 \to 0^+$ in the manner explained previously.

Developing a classical canonical formulation corresponding to this Lagrangian action as such is problematic. Indeed, even when initial or boundary conditions for ϕ^a are specified to be real valued, because of the pure imaginary mass term trajectories solving the associated classical Euler–Lagrange equations of motions are bound to become complex valued, hence also the momentum variables, p_a, conjugate to the configuration space ones, ϕ^a. At the quantum level it would therefore appear to be unjustified to associate to both these quantities operators that are self-adjoint.

However, one should keep in mind that the above Lagrangian action for the extended system only contributes inside a quantum path integral of the form,

$$\int [\mathcal{D}\phi^a(t)]\; e^{\frac{i}{\hbar}S[\phi^a]}, \tag{7}$$

where integration is taken over real paths in the real valued configuration space, $\phi^a(t)$, not involving therefore the complex valued classical trajectories (unless one considers an evaluation of the integral through contour deformations into the complex plane, as is done effectively in a saddle point evaluation [15,16]). In terms of this path integral it becomes possible to introduce real valued variables, p_a, canonically conjugate to the real configuration space ones, ϕ^a, as auxiliary variables for some well defined real gaussian integrals, thereby bringing the path integral into canonical first-order form, namely,

$$\int [\mathcal{D}\phi^a(t)]\; e^{\frac{i}{\hbar}S[\phi^a]} = \int [\mathcal{D}\phi^a(t)\mathcal{D}p_a(t)]\; e^{\frac{i}{\hbar}\int dt\left[\dot{\phi}^a p_a - H(\phi^a, p_a)\right]}, \tag{8}$$

where

$$H(\phi^a, p_a) = \frac{1}{2i\tau_0}\delta^{ab}\left(p_a + \frac{1}{2}\epsilon_{ac}\phi^c\right)\left(p_b + \frac{1}{2}\epsilon_{bd}\phi^d\right) + h(\phi^a) - E_0 \tag{9}$$

(the absolute normalisation of the functional integration measures is left unspecified). In particular note that the gaussian integrals over p_a are real and well defined precisely because the mass parameter, $m_0 = i\tau_0$, is pure positive imaginary.

Clearly it is this latter form of the path integral which defines the canonical formulation of the extended system, with as real canonically conjugate phase space variables (ϕ^a, p_a) and canonical Hamiltonian the function $H(\phi^a, p_a)$. As is well known, such a path integral is associated to an operator realisation over some Hilbert space providing a representation of the following extended Heisenberg algebra,

$$\left[\hat{\phi}^a, \hat{p}_b\right] = i\hbar\delta^a_b\mathbb{I}, \qquad \hat{\phi}^{a\dagger} = \hat{\phi}^a, \qquad \hat{p}^\dagger_a = \hat{p}_a, \tag{10}$$

with indeed hermitian operators, and note well, also commuting $\hat{\phi}^a$, namely $\hat{q}$ and $\hat{p}$ operators. Hence rather than consider the path integral in (1), an equivalent realisation of the same extended quantum system for a finite τ_0 value is defined by this operator algebra and the quantum Hamiltonian

$$\hat{H} = \frac{1}{2i\tau_0}\delta^{ab}\left(\hat{p}_a + \frac{1}{2}\epsilon_{ac}\hat{\phi}^c\right)\left(\hat{p}_b + \frac{1}{2}\epsilon_{bd}\hat{\phi}^d\right) + h(\hat{\phi}^a) - E_0. \tag{11}$$

It is thus the eigenspectrum of this operator that needs to be understood as a function of τ_0, as well as its behaviour in the limit $\tau_0 = 0$. Note well however that because of the pure imaginary mass parameter, this operator is not hermitian, $\hat{H}^\dagger \neq \hat{H}$, hence the quantum dynamics of the extended quantum system is not unitary, for any finite $\tau_0 > 0$. In particular, its eigenspectrum proves to be complex but with a dependence on τ_0 such that for those states that survive the limit $\tau_0 = 0$, their limiting energy eigenvalues are real once again and coincide with the eigenspectrum of the original unitary quantised system. This very point may thus be studied explicitly for a function $h(\phi^a)$ which, for example, is purely quadratic (and linear) in the variables ϕ^a, namely essentially the case of the one dimensional harmonic oscillator.

Let us henceforth consider a harmonic oscillator of mass m and angular frequency $\omega_0 > 0$, with then the choice $\lambda_0 = m\omega_0$. The Hamiltonian then reads

$$H_0(q,p) = \frac{1}{2m}p^2 + \frac{1}{2}m\omega_0^2 q^2 = \frac{1}{2}\omega_0\delta_{ab}\phi^a\phi^b. \tag{12}$$

Except for an additive constant proportional to $\hbar$ which may be absorbed in the choice for E_0, in this case the symbol $h(q,p)$ for the quantum Hamiltonian $\hat{H}_0$ coincides with $H_0(q,p)$. Consequently, the operator quantisation of the extended system in the case of the harmonic oscillator is defined by the Heisenberg algebra in (10) as well as the following quantum Hamiltonian,

$$\hat{H} = \frac{1}{2i\tau_0}\delta^{ab}\left(\hat{p}_a + \frac{1}{2}\epsilon_{ac}\hat{\phi}^c\right)\left(\hat{p}_b + \frac{1}{2}\epsilon_{bd}\hat{\phi}^d\right) + \frac{1}{2}\omega_0\delta_{ab}\hat{\phi}^a\hat{\phi}^b - E_0, \tag{13}$$

the diagonalisation of which we now address.

3 The Ordinary Harmonic Oscillator

3.1 A bi-module of Fock-like algebras

Given the hermitian operators $\hat{\phi}^a$ and $\hat{p}_a$, let us introduce first the following Fock operators,

$$a_a = \frac{1}{2\sqrt{\hbar}}\left(\hat{\phi}^a + 2i\hat{p}_a\right), \qquad a^\dagger_a = \frac{1}{2\sqrt{\hbar}}\left(\hat{\phi}^a - 2i\hat{p}_a\right), \tag{14}$$

which define the tensor product of two Fock algebras,

$$\left[a_a, a^\dagger_b\right] = \delta_{ab}\mathbb{I}. \tag{15}$$

Next, consider the following helicity Fock operators,

$$a_\pm = \frac{1}{\sqrt{2}}\left(a_1 \mp ia_2\right), \qquad a^\dagger_\pm = \frac{1}{\sqrt{2}}\left(a^\dagger_1 \pm ia^\dagger_2\right), \tag{16}$$

such that,

$$\left[a_\pm, a_\pm^\dagger\right] = \mathbb{I}, \qquad \left[a_\pm, a_\mp^\dagger\right] = 0. \tag{17}$$

The inverse relations are,

$$\begin{aligned}
\hat{\phi}^1 &= \sqrt{\frac{\hbar}{2}}\left(a_+ + a_- + a_+^\dagger + a_-^\dagger\right) \quad , \quad \hat{p}_1 = -\frac{i}{2}\sqrt{\frac{\hbar}{2}}\left(a_+ + a_- - a_+^\dagger - a_-^\dagger\right), \\
\hat{\phi}^2 &= i\sqrt{\frac{\hbar}{2}}\left(a_+ - a_- - a_+^\dagger + a_-^\dagger\right) \quad , \quad \hat{p}_2 = \frac{1}{2}\sqrt{\frac{\hbar}{2}}\left(a_+ - a_- + a_+^\dagger - a_-^\dagger\right),
\end{aligned} \tag{18}$$

with in particular,

$$\hat{p}_1 + \frac{1}{2}\hat{\phi}^2 = -i\sqrt{\frac{\hbar}{2}}\left(a_- - a_-^\dagger\right), \qquad \hat{p}_2 - \frac{1}{2}\hat{\phi}^1 = -\sqrt{\frac{\hbar}{2}}\left(a_- + a_-^\dagger\right). \tag{19}$$

To construct an abstract representation of these algebraic structures, consider now a normalised Fock vacuum, $|\Omega\rangle$, for the helicity Fock operators,

$$a_\pm|\Omega\rangle = 0, \qquad \langle\Omega|\Omega\rangle = 1, \tag{20}$$

with the following orthonormalised states spanning the Hilbert space of the extended quantum system,

$$|n_+, n_-;\Omega\rangle = \frac{1}{\sqrt{n_+!\,n_-!}}\left(a_+^\dagger\right)^{n_+}\left(a_-^\dagger\right)^{n_-}|\Omega\rangle, \qquad \langle n_+, n_-;\Omega|m_+, m_-;\Omega\rangle = \delta_{n_+,m_+}\,\delta_{n_-,m_-}, \tag{21}$$

where $n_+, n_- = 0, 1, 2, \ldots$, hence with the resolution of the unit operator,

$$\sum_{n_+,n_-=0}^{\infty} |n_+, n_-;\Omega\rangle\langle n_+, n_-;\Omega| = \mathbb{I}. \tag{22}$$

Incidentally these are the states that diagonalise the quantum operator $\hat{H}$ in the absence of the interaction coupling $h(\hat{\phi}^a)$, leading to Landau levels labelled by $n_- = 0, 1, \ldots$ and infinitely degenerate in $n_+ = 0, 1, \ldots$ In particular, the lowest Landau level, $|n_+, n_- = 0;\Omega\rangle$ $(n_+ = 0, 1, \ldots)$, will turn out to define the subspace of the Hilbert space of the extended quantum system which coincides with the Hilbert space of the original quantum system, namely the lowest Landau sector in presence of the interaction energy $h(\hat{\phi}^a)$ in the limit when $\tau_0 \to 0^+$. For that reason it is useful to already introduce the projector onto that subspace of quantum states of the extended system,

$$\mathbb{P}_0 = \sum_{n_+=0}^{\infty} |n_+, n_- = 0;\Omega\rangle\langle n_+, n_- = 0;\Omega|, \qquad \mathbb{P}_0^2 = \mathbb{P}_0, \qquad \mathbb{P}_0^\dagger = \mathbb{P}_0. \tag{23}$$

Note that we then have for the projected operators generating the Heisenberg algebra in the extended Hilbert space,

$$\mathbb{P}_0\left(\hat{p}_1 + \frac{1}{2}\hat{\phi}^2\right)\mathbb{P}_0 = 0, \qquad \mathbb{P}_0\left(\hat{p}_2 - \frac{1}{2}\hat{\phi}^1\right)\mathbb{P}_0 = 0, \tag{24}$$

showing that after projection only the projected coordinates $\mathbb{P}_0\hat{\phi}^a\mathbb{P}_0 = \bar{\phi}^a$ are independent operators, with as commutation relations,

$$\left[\bar{\phi}^a, \bar{\phi}^b\right] = -i\hbar\epsilon^{ab}\mathbb{P}_0. \tag{25}$$

Hence indeed on the projected subspace one recovers the Heisenberg algebra of the original quantum system, even though within the extended Hilbert space the phase space position operators $\hat{\phi}^a$ commute with each other.

However the above states do not diagonalise the total Hamiltonian $\hat{H}$ in presence of the interaction $h(\hat{\phi}^a)$, even for the harmonic oscillator. In the latter case, other linear combinations of the basic operators $\hat{\phi}^a$ and $\hat{p}_a$ are required. For that purpose, let us introduce two specific real quantities, R_0 and a phase φ_0, defined by the following relation,

$$R_0^2\, e^{2i\varphi_0} \equiv 1 + 4i\omega_0\tau_0, \qquad R_0 > 0, \qquad 0 \le \varphi_0 < \frac{\pi}{4}, \tag{26}$$

as well as the complex variable, ρ, and its complex conjugate, $\bar{\rho}$,

$$\rho = \sqrt{R_0}\, e^{\frac{1}{2}i\varphi_0}, \qquad \bar{\rho} = \sqrt{R_0}\, e^{-\frac{1}{2}i\varphi_0}. \tag{27}$$

Note that in the limit $\tau_0 \to 0^+$, or in the absence of the coupling ω_0, R_0 and ρ both go to unity while φ_0 vanishes.

Consider then the operators,

$$A_a = \frac{1}{2\sqrt{\hbar}}\left(\rho\hat{\phi}^a + \frac{2i}{\rho}\hat{p}_a\right), \qquad B_a = \frac{1}{2\sqrt{\hbar}}\left(\rho\hat{\phi}^a - \frac{2i}{\rho}\hat{p}_a\right), \tag{28}$$

as well as their adjoints,

$$A^\dagger_a = \frac{1}{2\sqrt{\hbar}}\left(\bar{\rho}\hat{\phi}^a - \frac{2i}{\bar{\rho}}\hat{p}_a\right), \qquad B^\dagger_a = \frac{1}{2\sqrt{\hbar}}\left(\bar{\rho}\hat{\phi}^a + \frac{2i}{\bar{\rho}}\hat{p}_a\right), \tag{29}$$

which are such that,

$$[A_a, B_b] = \delta_{ab}\mathbb{I} = \left[B^\dagger_a, A^\dagger_b\right]. \tag{30}$$

Had it not been for the fact that ρ is a complex quantity, the operators A_a and B_a would have been adjoints of one another. We have, for instance,

$$A^\dagger_a = \cos\varphi_0\, B_a \,-\, i\sin\varphi_0\, A_a, \qquad B^\dagger_a = \cos\varphi_0\, A_a \,-\, i\sin\varphi_0\, B_a. \tag{31}$$

Furthermore the operators A_a and B_a almost coincide with a_a and $a^\dagger_a$ above, respectively, but only if $\rho = 1$, namely whenever $\omega_0\tau_0 = 0$. Clearly, the operators A_a, B_a and their adjoints may be expressed as linear combinations of a_a and $a^\dagger_a$.

Finally let us introduce the helicity combinations,

$$\begin{aligned} A_\pm &= \frac{1}{\sqrt{2}}\left(A_1 \mp iA_2\right) \quad , \quad B_\pm = \frac{1}{\sqrt{2}}\left(B_1 \pm iB_2\right), \\ A^\dagger_\pm &= \frac{1}{\sqrt{2}}\left(A^\dagger_1 \pm iA^\dagger_2\right) \quad , \quad B^\dagger_\pm = \frac{1}{\sqrt{2}}\left(B^\dagger_1 \mp iB^\dagger_2\right), \end{aligned} \tag{32}$$

such that

$$[A_\pm, B_\pm] = \mathbb{I} = \left[B^\dagger_\pm, A^\dagger_\pm\right], \tag{33}$$

as well as,

$$\begin{aligned} A^\dagger_\pm &= \cos\varphi_0\, B_\pm \,-\, i\sin\varphi_0\, A_\mp \quad , \quad A_\pm = \cos\varphi_0\, B^\dagger_\pm \,+\, i\sin\varphi_0\, A^\dagger_\mp, \\ B^\dagger_\pm &= \cos\varphi_0\, A_\pm \,-\, i\sin\varphi_0\, B_\mp \quad , \quad B_\pm = \cos\varphi_0\, A^\dagger_\pm \,+\, i\sin\varphi_0\, B^\dagger_\mp. \end{aligned} \tag{34}$$

Expressing these operators in terms of $a_\pm$ and $a^\dagger_\pm$, one finds,

$$\begin{aligned} A_\pm &= \frac{\rho+\rho^{-1}}{2}\, a_\pm \,+\, \frac{\rho-\rho^{-1}}{2}\, a^\dagger_\mp \quad , \quad A^\dagger_\pm = \frac{\bar{\rho}+\bar{\rho}^{-1}}{2}\, a^\dagger_\pm \,+\, \frac{\bar{\rho}-\bar{\rho}^{-1}}{2}\, a_\mp, \\ B_\pm &= \frac{\rho+\rho^{-1}}{2}\, a^\dagger_\pm \,+\, \frac{\rho-\rho^{-1}}{2}\, a_\mp \quad , \quad B^\dagger_\pm = \frac{\bar{\rho}+\bar{\rho}^{-1}}{2}\, a_\pm \,+\, \frac{\bar{\rho}-\bar{\rho}^{-1}}{2}\, a^\dagger_\mp. \end{aligned} \tag{35}$$

The Fock-like algebraic relations in (33) are very much similar to those of ordinary Fock algebras, except for the fact that the operators $B_\pm$ and $A_\pm$ (on the one hand, or their adjoints on the other hand) are not adjoints of one another. Yet, a representation theory may be constructed in very much the same way, leading to dual states we shall refer to as A- and B-Fock states. This representation is built on A- and B-Fock vacua, $|\Omega_A\rangle$ and $|\Omega_B\rangle$, respectively, such that

$$A_\pm|\Omega_A\rangle = 0, \qquad B^\dagger_\pm|\Omega_B\rangle = 0. \tag{36}$$

By an appropriate choice of phases and normalisations, it is always possible to assume that the inner product of these two states is set to unity,

$$\langle\Omega_A|\Omega_B\rangle = 1 = \langle\Omega_B|\Omega_A\rangle. \tag{37}$$

The A-Fock states are then defined by

$$|N_+, N_-; \Omega_A\rangle = \frac{1}{\sqrt{N_+!\, N_-!}} B_+^{N_+} B_-^{N_-} |\Omega_A\rangle, \tag{38}$$

while for the B-Fock states,

$$|N_+, N_-; \Omega_B\rangle = \frac{1}{\sqrt{N_+!\, N_-!}} \left(A_+^\dagger\right)^{N_+} \left(A_-^\dagger\right)^{N_-} |\Omega_B\rangle, \tag{39}$$

where $N_+, N_- = 0, 1, 2, \ldots$ As a matter of fact, since the operators $A_\pm$, $B_\pm$ and their adjoints are linear combinations of the Fock operators $a_\pm$ and $a_\pm^\dagger$, it is clear that either set of states, $|N_+, N_-; \Omega_A\rangle$ or $|N_+, N_-; \Omega_B\rangle$, spans the entire Hilbert space of the quantum extended system. More specifically, each of these two sets provides a basis of that space, these two bases being in fact dual to one another,

$$\langle N_+, N_-; \Omega_A | M_+, M_-; \Omega_B\rangle = \delta_{N_+,M_+}\delta_{N_-,M_-} = \langle N_+, N_-; \Omega_B | M_+, M_-; \Omega_A\rangle. \tag{40}$$

Consequently, one also has the following resolutions of the unit operator,

$$\sum_{N_+,N_-=0}^{\infty} |N_+, N_-; \Omega_A\rangle\langle N_+, N_-; \Omega_B| = \mathbb{I} = \sum_{N_+,N_-=0}^{\infty} |N_+, N_-; \Omega_B\rangle\langle N_+, N_-; \Omega_A|. \tag{41}$$

In other words, the three sets of states, $|n_+, n_-; \Omega\rangle$, $|N_+, N_-; \Omega_A\rangle$ and $|N_+, N_-; \Omega_B\rangle$, define three different bases of the same extended Hilbert space, with the basis $|n_+, n_-; \Omega\rangle$ being self-dual since orthonormalised, while the other two bases are dual to one another.

Note that the action of the $A_\pm$ and $B_\pm$ operators on the A-Fock states, on the one hand, and of the $B_\pm^\dagger$ and $A_\pm^\dagger$ operators on the B-Fock states, on the other hand, is precisely like that of ordinary annihilation and creation Fock operators, respectively, on ordinary Fock states. In particular, the A-Fock states $|N_+, N_-; \Omega_A\rangle$ (resp., B-Fock states $|N_+, N_-; \Omega_B\rangle$) are eigenstates of the operators $B_\pm A_\pm$ (resp., $A_\pm^\dagger B_\pm^\dagger$) with eigenvalues $N_\pm$.

Given the identities (35) relating the different Fock-like operators, it should be clear that the relations between these three different bases are obtained as Bogoliubov transformations. Introducing the complex parameter

$$\lambda = \frac{\rho - \rho^{-1}}{\rho + \rho^{-1}}, \qquad \bar{\lambda} = \frac{\bar{\rho} - \bar{\rho}^{-1}}{\bar{\rho} + \bar{\rho}^{-1}}, \tag{42}$$

a little analysis shows that the A- and B-Fock vacua are given as,

$$|\Omega_A\rangle = \left(\frac{2}{\rho + \rho^{-1}}\right) e^{-\lambda a_+^\dagger a_-^\dagger} |\Omega\rangle, \qquad |\Omega_B\rangle = \left(\frac{2}{\bar{\rho} + \bar{\rho}^{-1}}\right) e^{-\bar{\lambda} a_+^\dagger a_-^\dagger} |\Omega\rangle, \tag{43}$$

and similarly,

$$|\Omega_B\rangle = N_B(\varphi_0)\, e^{i \tan\varphi_0\, B_+ B_-} |\Omega_A\rangle, \qquad |\Omega_A\rangle = N_A(\varphi_0)\, e^{-i\tan\varphi_0\, A_+^\dagger A_-^\dagger} |\Omega_B\rangle, \tag{44}$$

$N_A(\varphi_0)$ and $N_B(\varphi_0)$ being two normalisation factors whose evaluation is not required here,

$$N_A^{-1}(\varphi_0) = \langle\Omega_B| e^{-i\tan\varphi_0\, A_+^\dagger A_-^\dagger} |\Omega_B\rangle, \qquad N_B^{-1}(\varphi_0) = \langle\Omega_A| e^{i\tan\varphi_0\, B_+ B_-} |\Omega_A\rangle. \tag{45}$$

These different representations relating the different Fock vacua as coherent helicity pairing excitations of one another, thus establish that indeed all three sets of Fock states provide complete bases of the same extended Hilbert space in which to diagonalise the total quantum Hamiltonian $\hat{H}$.

Finally, note that in the limit where $\tau_0 \to 0^+$, all three sets of Fock states then coalesce into a single set, namely the states $|n_+, n_-; \Omega\rangle$ $(n_+, n_- = 0, 1, \ldots)$, since then all three Fock vacua become identical to $|\Omega\rangle$ while we have the following correspondences for the creation and annihilation operators,

$$A_\pm \to a_\pm, \qquad B_\pm \to a_\pm^\dagger, \qquad A_\pm^\dagger \to a_\pm^\dagger, \qquad B_\pm^\dagger \to a_\pm. \tag{46}$$

3.2 The energy spectrum

With the previous representation theory of the extended Hilbert space at hand, diagonalisation of the total Hamiltonian (13) of the extended system is readily achieved. In terms of the operators introduced above, a little substitution easily finds,

$$\hat{H}=\hbar\frac{R_0e^{i\varphi_0}+1}{2i\tau_0}\,B_-A_-\,+\hbar\frac{R_0e^{i\varphi_0}-1}{2i\tau_0}\left(B_+A_++\frac{1}{2}\right)+\left(\hbar\frac{R_0e^{i\varphi_0}+1}{4i\tau_0}-E_0\right). \tag{47}$$

Obviously, the A-Fock states, $|N_+,N_-;\Omega_A\rangle$, are the eigenstates of that operator, while those of its adjoint, $\hat{H}^\dagger\neq\hat{H}$, are the B-Fock states, $|N_+,N_-;\Omega_B\rangle$. Furthermore the subtraction constant E_0 needs to be adjusted as follows,

$$E_0=\hbar\frac{R_0e^{i\varphi_0}+1}{4i\tau_0}\,-\,\Delta E_0(\omega_0,\tau_0),\qquad \lim_{\tau_0\to 0^+}\Delta E_0(\omega_0,\tau_0)=0, \tag{48}$$

where the function $\Delta E_0(\omega_0,\tau_0)$ is *a priori* otherwise arbitrary (it may even be complex for a finite value of τ_0), and in fact is of the form,

$$\Delta E_0(\omega_0,\tau_0)=\hbar\omega_0\Delta\mathcal{E}_0(\omega_0\tau_0), \tag{49}$$

$\Delta\mathcal{E}_0(\omega_0\tau_0)$ being a function only of the product $(\omega_0\tau_0)$ which vanishes when that argument vanishes. In the limit $\tau_0\to 0^+$, clearly then only the lowest Landau sector with $N_-=0$ retains finite energy values, namely the states $|N_+,N_-=0;\Omega_A\rangle\to|n_+=N_+,n_-=0;\Omega\rangle$ for $\hat{H}$ and $|N_+,N_-=0;\Omega_B\rangle\to|n_+=N_+,n_-=0;\Omega\rangle$ for $\hat{H}^\dagger$, with $N_+=0,1,\dots$

Given that choice for the subtraction constant E_0, the complex energy spectrum of the system, for a finite value of $\tau_0>0$, is given as,

$$\hat{H}|N_+,N_-;\Omega_A\rangle=E(N_+,N_-)|N_+,N_-;\Omega_A\rangle,\qquad \hat{H}^\dagger|N_+,N_-;\Omega_B\rangle=\bar{E}(N_+,N_-)|N_+,N_-;\Omega_B\rangle, \tag{50}$$

with

$$E(N_+,N_-)=\hbar\frac{R_0e^{i\varphi_0}+1}{2i\tau_0}\,N_-\,+\hbar\frac{R_0e^{i\varphi_0}-1}{2i\tau_0}\left(N_++\frac{1}{2}\right)+\Delta E_0(\omega_0,\tau_0), \tag{51}$$

while $\bar{E}(N_+,N_-)$ stands for the complex conjugate of $E(N_+,N_-)$. In particular, the lowest Landau sector energy eigenvalues are

$$E(N_+,N_-=0)=\hbar\frac{R_0e^{i\varphi_0}-1}{2i\tau_0}\left(N_++\frac{1}{2}\right)+\Delta E_0(\omega_0,\tau_0). \tag{52}$$

3.3 The $\tau_0\to 0^+$ limit

In the absence of the interaction energy $h(\hat{\phi}^a)$, namely when $\omega_0=0$, the energy spectrum reduces to,

$$\omega_0=0:\qquad E(N_+,N_-)=\frac{\hbar}{i\tau_0}\,N_-, \tag{53}$$

displaying the infinite degeneracy in $N_+=0,1,2,\dots$ of the Landau levels labelled by $N_-=0,1,2,\dots$ and separated by a gap $\hbar/(i\tau_0)$ as expected, then corresponding to the states $|n_+=N_+,n_-=N_-;\Omega\rangle$. In the limit that $\tau_0=0$, only the lowest Landau level retains a finite (vanishing) energy.

When the interaction energy $h(\hat{\phi}^a)$ is included, the gap between Landau sectors is determined by the quantity

$$\hbar\frac{R_0e^{i\varphi_0}+1}{2i\tau_0}, \tag{54}$$

which in the limit $\tau_0\to 0^+$ behaves as,

$$\hbar\frac{R_0e^{i\varphi_0}+1}{2i\tau_0}\overset{\tau_0\to 0^+}{\simeq}\frac{\hbar}{i\tau_0}+\hbar\omega_0+\cdots. \tag{55}$$

Hence once again it is the scale $\hbar/(i\tau_0)$ which sets the leading contribution to that gap, which diverges in the limit $\tau_0 \to 0^+$. Consequently, only the Landau sector with $N_- = 0$ retains a finite energy in that limit. Furthermore within a given Landau sector, the spacing between states is determined by the second relevant quantity,

$$\hbar \frac{R_0 e^{i\varphi_0} - 1}{2i\tau_0}, \tag{56}$$

which in the limit $\tau_0 \to 0^+$ behaves as,

$$\hbar \frac{R_0 e^{i\varphi_0} - 1}{2i\tau_0} \stackrel{\tau_0 \to 0^+}{\simeq} \hbar\omega_0 + \cdots . \tag{57}$$

Hence, in that limit, the energy spectrum behaves as,

$$E(N_+, N_-) \stackrel{\tau_0 \to 0^+}{\simeq} \frac{\hbar}{i\tau_0} \left(1 + i\omega_0\tau_0 + \cdots\right) N_- + \left(\hbar\omega_0 + \cdots\right) \left(N_+ + \frac{1}{2}\right) + \cdots . \tag{58}$$

Those states retaining a finite energy in that limit belong only to the lowest Landau sector with $N_- = 0$,

$$\lim_{\tau_0 \to 0^+} E(N_+, N_- = 0) = \hbar\omega_0 \left(N_+ + \frac{1}{2}\right). \tag{59}$$

In this expression one recognizes the real energy spectrum of the harmonic oscillator, including its quantum vacuum energy, the corresponding energy eigenstates being the Fock states $|n_+ = N_+, n_- = 0; \Omega\rangle$. Hence indeed the subspace of the extended Hilbert space of the extended system spanned by the lowest Landau sector in the limit $\tau_0 = 0$ determines the Hilbert space of the original quantum system, in the present case that of the harmonic oscillator.

To show that the remaining Landau sectors do decouple from the energy spectrum in the limit $\tau_0 = 0$, it suffices to consider the quantum evolution operator of the extended system. Given the spectral resolution of the unit operator in terms of the eigenstates of the Hamiltonian operator, $\hat{H}$, and its adjoint, one has for the evolution operator with $T = t_f - t_i > 0$,

$$e^{-\frac{i}{\hbar}T\hat{H}} = \sum_{N_+, N_- = 0}^{\infty} |N_+, N_-; \Omega_A\rangle \, e^{-\frac{i}{\hbar}TE(N_+, N_-)} \, \langle N_+, N_-; \Omega_B|. \tag{60}$$

Using the above expansion in τ_0 for $E(N_+, N_-)$, one thus finds that all the states with $N_- \geq 1$ decouple exponentially in the considered limit,

$$\lim_{\tau_0 \to 0^+} e^{-\frac{i}{\hbar}T\hat{H}} \stackrel{T \geq 0}{=} \sum_{N_+ = 0}^{\infty} |N_+, N_- = 0; \Omega\rangle \, e^{-i\omega_0 T(N_+ + 1/2)} \, \langle N_+, N_- = 0; \Omega|. \tag{61}$$

Hence indeed all but the states belonging to the lowest Landau sector have decoupled from the dynamics of the extended system in the limit $\tau_0 \to 0^+$, leaving over precisely the Hilbert space of the ordinary harmonic oscillator with the correct energy spectrum and quantum time evolution operator. The states $|n_+, n_- = 0; \Omega\rangle$ correspond exactly to the usual Fock states $|n_+\rangle$ of the harmonic oscillator with energy spectrum $E(n_+) = \hbar\omega_0(n_+ + 1/2)$, $|n_+, n_- = 0; \Omega\rangle \equiv |n_+\rangle$.

This conclusion is thus in full accord with the general discussion and results of Ref. [1] within the functional integral setting, but achieved in the specific case of the harmonic oscillator and using rather operator quantisation techniques. As a matter of fact, a similar analysis based on the operator quantisation of the extended system for whatever initial system and given (polynomial [1]) Hamiltonian $H_0(q, p)$ is possible, leading of course to the same general conclusion [18, 19].

The above has thus also established that the limit $\tau_0 \to 0^+$ enforces the projection effected by the operator $\mathbb{P}_0$ introduced previously, thereby leading back to noncommuting hermitian projected phase space operators, $Q = \mathbb{P}_0 \hat{q} \mathbb{P}_0$ and $P = \mathbb{P}_0 \hat{p} \mathbb{P}_0$, obeying the usual Heisenberg algebra as it should, $[Q, P] = i\hbar\mathbb{P}_0$, of which the projected Hilbert space spanned by the Fock states $|n\rangle \equiv |n, 0; \Omega\rangle$ provides the usual Fock space representation. However, before the projection is effected, as two dimensional configuration space operators, the unprojected coordinates $\hat{q}$ and $\hat{p}$ acting on the extended Hilbert space are commuting operators. This feature, unique to the Klauder–Daubechies construction of the phase space path integral which is covariant under canonical transformations of the original system, can be put to use to exploit at the quantum level all the advantages of classical action-angle transformations for systems which are integrable in the Liouville sense and which possess nonperturbative configurations [18, 19].

4 The Deformed Harmonic Oscillator

4.1 A deformed quantum dynamics

The previous discussion has thus established that one has for the $\mathbb{P}_0$ projected evolution operator of the extended system, when $T > 0$,

$$\lim_{\tau_0\to 0^+} \mathbb{P}_0\, e^{-\frac{i}{\hbar}T\hat{H}}\mathbb{P}_0 = \lim_{\tau_0\to 0^+} e^{-\frac{i}{\hbar}T\hat{H}}, \tag{62}$$

the latter quantity then reproducing the quantum evolution operator of the original system. However since $\mathbb{P}_0$ effects the projection onto the Hilbert space of the original quantum system, it may be worth considering the projected evolution operator also for a finite value of τ_0,

$$T > 0: \qquad \mathbb{U}(T) = \mathbb{P}_0\, e^{-\frac{i}{\hbar}T\hat{H}}\,\mathbb{P}_0 \neq \lim_{\tau_0\to 0^+} \mathbb{P}_0\, e^{-\frac{i}{\hbar}T\hat{H}}\mathbb{P}_0, \tag{63}$$

knowing that in the limit $\tau_0 = 0$ this operator reproduces the correct evolution operator of the original quantum system,

$$U(T) = \lim_{\tau_0\to 0^+} \mathbb{U}(T) = \lim_{\tau_0\to 0^+} e^{-\frac{i}{\hbar}T\hat{H}}. \tag{64}$$

Keeping τ_0 finite for $\mathbb{U}(T)$ thus induces a deformed quantum dynamics inside the Hilbert space of the original quantum system, as compared to the operator $U(T)$. Such a deformation may be of physical interest, in a spirit comparable to that which suggests to consider noncommutative deformations of the geometrical properties of spacetime in attempts towards formulations for a quantum theory of gravity through deformations of quantum algebras [15–17]. Nevertheless, it should be pointed out that for a finite value of τ_0, because of the irreversible character of its Brownian motion component, such a dynamics is no longer unitary,

$$\mathbb{U}^\dagger(T) \neq \mathbb{U}^{-1}(T), \qquad \mathbb{U}^\dagger(T)\,\mathbb{U}(T) \neq \mathbb{P}_0, \qquad \mathbb{U}(T)\,\mathbb{U}^\dagger(T) \neq \mathbb{P}_0, \tag{65}$$

and thus cannot preserve quantum probabilities, or more correctly in the present context, the total occupation number (the sum of the occupation densities over all quantum states of the system). Nor does it meet the usual convolution property under consecutive time evolution intervals,

$$\mathbb{U}(T_2)\cdot\mathbb{U}(T_1) \neq \mathbb{U}(T_2+T_1), \qquad \mathbb{P}_0\, e^{-\frac{i}{\hbar}T_2\hat{H}}\,\mathbb{P}_0\cdot\mathbb{P}_0\, e^{-\frac{i}{\hbar}T_1\hat{H}}\,\mathbb{P}_0 \neq \mathbb{P}_0\, e^{-\frac{i}{\hbar}(T_2+T_1)\hat{H}}\,\mathbb{P}_0. \tag{66}$$

Hence such a proposal raises a series of interpretational issues, which we shall not attempt to address here. However let us point out that when extrapolated to a quantum field theory context [20], a finite τ_0 value provides in effect a regularisation of short-distance singularities, akin to a soft exponential cut-off in the momentum of quantum states, indeed so efficient that all quantum amplitudes for whatever field theory in a perturbative expansion, even including general relativity, are ultra-violet finite (the only potential source of trouble being some tadpole contributions, which may always be dealt with by a proper choice of quantum Hamiltonian). The combination of the time scale τ_0—expected to be extremely small as well if non vanishing in the physical world—and of the Planck time in a quantum gravitational context, $\tau_{\rm Planck} = \sqrt{\hbar G_N/c^5} \simeq 10^{-43}$ s—irrespective of whether these two time scales should prove to be unrelated or not—, may thus offer some tantalising prospects for strongly gravitationally interacting quantum systems [15], a physical situation in which perhaps the requirements of unitarity and Lorentz invariance may be relaxed to some slight degree for what concerns experimentally unexplored extreme regimes. Whatever the case may be, at least a nonvanishing time scale τ_0 provides yet another regularisation of short-distance quantum dynamics for local field theories whose usefulness is worth exploring.

As a matter of fact the projected operator $\mathbb{U}(T)$ has already been computed [15, 16] directly from the KD-PI in (1) using a saddle point approach for what is indeed a purely gaussian functional integral in the case of the harmonic oscillator. Here rather, we shall exploit the operator solution constructed above to reproduce the same result, making it readily explicit that the deformed quantum dynamics remains diagonal in the Fock state basis of the harmonic oscillator.

4.2 The projected evolution operator

Since the operator of interest is of the form

$$\mathbb{U}(T) \stackrel{T\geq 0}{=} \sum_{n_+,m_+=0}^{\infty} |n_+,0;\Omega\rangle\langle n_+,0;\Omega|e^{-\frac{i}{\hbar}T\hat{H}}|m_+,0;\Omega\rangle\langle m_+,0;\Omega|, \tag{67}$$

while the eigenstates of $\hat{H}$ (resp., $\hat{H}^\dagger$) are $|N_+,N_-;\Omega_A\rangle$ (resp., $|N_+,N_-;\Omega_B\rangle$), one first needs to consider the following change of basis matrix elements,

$$\langle n_+,0;\Omega|N_+,N_-;\Omega_A\rangle, \qquad \langle m_+,0;\Omega|N_+,N_-;\Omega_B\rangle. \tag{68}$$

Using the definition of the A-Fock states $|N_+,N_-;\Omega_A\rangle$ and the representation of $|\Omega_A\rangle$ as a coherent helicity pairing excitation of $|\Omega\rangle$, a detailed evaluation of the first matrix element finds the following result,

$$\langle n_+,0;\Omega|N_+,N_-;\Omega_A\rangle = \left(\frac{2}{\rho+\rho^{-1}}\right)\left(\frac{2}{\rho+\rho^{-1}}\right)^{N_+}\left(\frac{\rho-\rho^{-1}}{2}\right)^{N_-}\sqrt{\frac{N_+!}{n_+!\,N_-!}}\,\delta_{N_+,N_-+n_+}. \tag{69}$$

In a similar fashion,

$$\langle m_+,0;\Omega|N_+,N_-;\Omega_B\rangle = \left(\frac{2}{\bar\rho+\bar\rho^{-1}}\right)\left(\frac{2}{\bar\rho+\bar\rho^{-1}}\right)^{N_+}\left(\frac{\bar\rho-\bar\rho^{-1}}{2}\right)^{N_-}\sqrt{\frac{N_+!}{m_+!\,N_-!}}\,\delta_{N_+,N_-+m_+}. \tag{70}$$

It then readily follows that the matrix elements of the deformed evolution operator $\mathbb{U}(T)$ in the Fock state basis of the harmonic oscillator are diagonal in that basis,

$$\langle n|\mathbb{U}(T)|\ell\rangle \equiv \langle n,0;\Omega|\mathbb{U}(T)|\ell,0;\Omega\rangle = \delta_{n,\ell}\langle n|\mathbb{U}(T)|n\rangle. \tag{71}$$

Using the above results, a direct evaluation of the diagonal matrix element then leads to,

$$\langle n|\mathbb{U}(T)|n\rangle = e^{-\frac{i}{\hbar}T\Delta E_0}\,e^{-i(n+\frac{1}{2})\alpha_+}\,F^{n+1}(T), \tag{72}$$

where,

$$\alpha_+ = \omega_0 T\,\frac{R_0 e^{i\varphi_0}-1}{2i\omega_0\tau_0}, \qquad \alpha_- = \omega_0 T\,\frac{R_0 e^{i\varphi_0}+1}{2i\omega_0\tau_0}, \tag{73}$$

and,

$$\frac{1}{F(T)} = \left(\frac{\rho+\rho^{-1}}{2}\right)^2 - \left(\frac{\rho-\rho^{-1}}{2}\right)^2 e^{-i(\alpha_++\alpha_-)}. \tag{74}$$

In order to bring this matrix element to a more amenable form, in terms of the two quantities R_0 and φ_0 defined previously already through the identification (26) let us introduce the following further notations,

$$R = \sqrt{\frac{1}{2}\,(R_0^2+1)}, \qquad S = \frac{1}{2}\,(R+1)\,, \tag{75}$$

which are such that

$$R-1 = 2\frac{\omega_0^2\tau_0^2}{R^2 S}, \qquad \frac{\omega_0\tau_0}{RS} = \sqrt{1-\frac{1}{S}}, \qquad \cos\varphi_0 = \frac{R}{R_0}, \qquad \sin\varphi_0 = \frac{2\omega_0\tau_0}{R_0 R}. \tag{76}$$

It then follows that,

$$i\alpha_+ = T\frac{R-1}{2\tau_0} + i\frac{\omega_0 T}{R}, \qquad i\,(\alpha_+ + \alpha_-) = T\frac{R}{\tau_0} + 2i\frac{\omega_0 T}{R}, \tag{77}$$

as well as,

$$\rho^2 = R + 2i\frac{\omega_0\tau_0}{R}, \qquad \rho^{-2} = \frac{R^2 - 2i\omega_0\tau_0}{R_0^2 R}, \tag{78}$$

and finally,

$$\frac{1}{F(T)} = e^{-\frac{R}{\tau_0}T}\, e^{-2i\frac{\omega_0}{R}T} \,+\, S\frac{R+2i\omega_0\tau_0}{R^2+2i\omega_0\tau_0}\left(1 - e^{-\frac{R}{\tau_0}T}\, e^{-2i\frac{\omega_0}{R}T}\right). \tag{79}$$

Hence we have so far,

$$\langle n|\mathbb{U}(T)|n\rangle = e^{-\frac{i}{\hbar}T\Delta E_0(\omega_0,\tau_0)}\, e^{-i\frac{\omega_0 T}{R}\left(n+\frac{1}{2}\right)}\, e^{-\frac{R-1}{2\tau_0}T\left(n+\frac{1}{2}\right)}\, F^{n+1}(T). \tag{80}$$

Since

$$\lim_{T\to+\infty} F(T) = \frac{1}{S}\frac{R^2+2i\omega_0\tau_0}{R+2i\omega_0\tau_0}, \tag{81}$$

in order that the asymptotic time limit $T \to +\infty$ leaves over at least one of the matrix elements $\langle n|\mathbb{U}(T)|n\rangle$ with a finite and non vanishing occupation, given that $R > 1$ this can only be the case for the Fock vacuum $|n=0\rangle = |\Omega\rangle$, which requires then to specify the choice for the arbitrary function $\Delta E_0(\omega_0,\tau_0)$ as follows,

$$\Delta E_0(\omega_0,\tau_0) = -i\hbar\frac{R-1}{4\tau_0} \overset{\tau_0\to 0^+}{\simeq} -\frac{1}{2}i\hbar\omega_0^2\tau_0 + \ldots, \tag{82}$$

indeed a pure imaginary quantity but such that it vanishes in the limit $\tau_0 = 0$, as it should. Correspondingly, we have for the energy subtraction constant E_0,

$$E_0 = \hbar\frac{R_0 e^{i\varphi_0} - R + 2}{4i\tau_0} = \hbar\frac{1}{2i\tau_0} + \hbar\frac{\omega_0}{2R}. \tag{83}$$

Incidentally, the exact same choice had to be made in Refs. [15, 16] for precisely the same reason. Note also that in the absence of the interaction energy $h(\hat{\phi}^a)$, namely when $\omega_0 = 0$, the value $E_0 = \hbar/(2i\tau_0)$ coincides precisely with the one specified in Ref. [1] for the factor $e^{C_0 T/\tau_0}$ in (1).

In conclusion, the final expression for the relevant matrix elements, which agrees with the result obtained through a functional integral calculation [15, 16], is,

$$\langle n|\mathbb{U}(T)|\ell\rangle = \delta_{n,\ell}\cdot e^{-i\frac{\omega_0}{R}T\left(n+\frac{1}{2}\right)}\, e^{-n\frac{R-1}{2\tau_0}T}\, F^{n+1}(T) \equiv \delta_{n,\ell}\cdot \mathbb{U}_n(T), \tag{84}$$

hence,

$$\mathbb{U}(T) = \sum_{n=0}^{\infty} |n\rangle\, e^{-i\frac{\omega_0}{R}T\left(n+\frac{1}{2}\right)}\, e^{-n\frac{R-1}{2\tau_0}T}\, F^{n+1}(T)\,\langle n| = \sum_{n=0}^{\infty}|n\rangle\,\mathbb{U}_n(T)\,\langle n|. \tag{85}$$

Note that we have

$$\lim_{T\to 0^+}\mathbb{U}(T) = \sum_{n=0}^{\infty}|n\rangle\langle n| = \mathbb{P}_0, \qquad \lim_{\tau_0\to 0^+}\mathbb{U}(T) = \sum_{n=0}^{\infty}|n\rangle\, e^{-i\omega_0 T(n+\frac{1}{2})}\,\langle n| = U(T), \tag{86}$$

as it should, while,

$$\lim_{T\to+\infty}\mathbb{U}(T) = e^{-\frac{1}{2}i\frac{\omega_0}{R}T}\,\frac{1}{S}\frac{R^2+2i\omega_0\tau_0}{R+2i\omega_0\tau_0}\,|\Omega\rangle\langle\Omega|, \tag{87}$$

thus displaying how because of the Brownian motion contribution to the quantum dynamics when $\tau_0 \neq 0$, whatever the initial state of the system it eventually decays to the Fock vacuum with a specific factor rescaling the initial occupation of that particular state.

Before commenting on the significance of these results, let us consider how the original Heisenberg algebra of phase space operators is deformed in the time evolved picture of the system, because of a non vanishing value for $\tau_0 > 0$. Defining quantum operators, $A(t_f)$, in the Heisenberg picture in the usual way but in terms of the projected evolution operator, $\mathbb{U}(T)$ with $T = t_f - t_i > 0$, as,

$$A(t_f) = \mathbb{U}^\dagger(T)\, A(t_i)\, \mathbb{U}(T), \tag{88}$$

a direct calculation in the case of the (projected) position and momentum operators, $Q(t_i) = \mathbb{P}_0 \hat{q}(t_i) \mathbb{P}_0$ and $P(t_i) = \mathbb{P}_0 \hat{p}(t_i) \mathbb{P}_0$ with $[Q(t_i), P(t_i)] = i\hbar \mathbb{P}_0$, finds indeed a deformed Heisenberg algebra,

$$[Q(t_f), P(t_f)] = i\hbar\, |0\rangle\, F_0^3\, D^3(T)\, e^{-\frac{R-1}{\tau_0}T}\, \langle 0| \; + \tag{89}$$
$$+\, i\hbar \sum_{n=1}^{\infty} |n\rangle\, F_0^{2n+1}\, D^{2n+1}(T)\, e^{-(2n-1)\frac{R-1}{\tau_0}T} \left[(n+1)\, F_0^2\, D^2(T)\, e^{-2\frac{R-1}{\tau_0}T} - n \right] \langle n|.$$

In this expression the quantities F_0 and $D(T)$ are defined according to the relation

$$|F(T)|^2 = F_0 \cdot D(T), \tag{90}$$

where

$$F_0 = \frac{1}{S^2}\, \frac{R^4 + 4\omega_0^2 \tau_0^2}{R^2 + 4\omega_0^2 \tau_0^2}, \tag{91}$$

and,

$$\frac{1}{D(T)} = 1 + 2\left(\frac{R-1}{R+1}\right) e^{-\frac{R}{\tau_0}T} \cos 2\left(\frac{\omega_0}{R} T + \varphi_0\right) + \left(\frac{R-1}{R+1}\right)^2 e^{-\frac{2R}{\tau_0}T}. \tag{92}$$

Note that we have,

$$\lim_{T\to 0^+} [Q(t_f), P(t_f)] = i\hbar \mathbb{P}_0, \qquad \lim_{\tau_0 \to 0^+} [Q(t_f), P(t_f)] = i\hbar \mathbb{P}_0, \tag{93}$$

as it should, while,

$$\lim_{T\to +\infty} [Q(t_f), P(t_f)] = 0. \tag{94}$$

The reason why in the asymptotic time limit, $T \to +\infty$, the two phase space operators $Q(t_f)$ and $P(t_f)$ end up commuting with one another as in the classical system, is that in that limit all quantum states of the harmonic oscillator except for its Fock vacuum have exponentially decayed to zero, as shown explicitly by (87).

4.3 Physical implications

More precisely, given an initial quantum state

$$|\psi, t_i\rangle = \sum_{n=0}^{\infty} |n\rangle\, \psi_n(t_i), \qquad \psi_n(t_i) \in \mathbb{C}, \qquad \sum_{n=0}^{\infty} |\psi_n(t_i)|^2 < \infty, \tag{95}$$

its configuration at time t_f with $T = t_f - t_i > 0$ is

$$|\psi, t_f\rangle = \mathbb{U}(T)|\psi, t_i\rangle = \sum_{n=0}^{\infty} |n\rangle\, \mathbb{U}_n(T)\, \psi_n(t_i) = \sum_{n=0}^{\infty} |n\rangle\, \psi_n(t_f), \qquad \psi_n(t_f) = \mathbb{U}_n(t_f - t_i)\, \psi_n(t_i). \tag{96}$$

Consequently, the time evolution of the occupation densities of the Fock eigenstates of the harmonic oscillator is determined by,

$$|\psi_n(t_f)|^2 = |\mathbb{U}_n(t_f - t_i)|^2 \cdot |\psi_n(t_i)|^2. \tag{97}$$

Based on the expressions above, one has

$$|\mathbb{U}_n(T)|^2 = |F(T)|^{2(n+1)}\, e^{-n\frac{R-1}{\tau_0}T}, \tag{98}$$

namely,

$$|\mathbb{U}_n(T)|^2 = F_0^{n+1}\, D^{n+1}(T)\, e^{-n\frac{R-1}{\tau_0}T}, \tag{99}$$

where the quantities F_0 and $D(T)$ are given in (91) and (92), respectively.

Stochastic Brownian motion leads to so efficient a statistical decoherence of the quantum system that whatever dynamics there is to begin with, it totally decays away. All that remains in a rescaled occupation of the initial ground state occupation of the system. Given the asymptotic values,

$$\lim_{T\rightarrow+\infty} |\mathbb{U}_{n=0}(T)|^2 = \frac{1}{S^2}\frac{R^4+4\omega_0^2\tau_0^2}{R^2+4\omega_0^2\tau_0^2}, \qquad \lim_{T\rightarrow+\infty} |\mathbb{U}_{n\geq 1}(T)|^2 = 0, \tag{100}$$

the time asymptotics of the Fock state occupations is such that,

$$\lim_{t_f\rightarrow+\infty} |\psi_{n=0}(t_f)|^2 = \frac{1}{S^2}\frac{R^4+4\omega_0^2\tau_0^2}{R^2+4\omega_0^2\tau_0^2}\ |\psi_{n=0}(t_i)|^2, \qquad \lim_{t_f\rightarrow+\infty} |\psi_{n\geq 1}(t_f)|^2 = 0. \tag{101}$$

In its large time behaviour, the dynamics of the (non interacting closed) system which is irreversible provided τ_0 is non vanishing however small its value, is such that the Hilbert space of the quantum system thus becomes effectively one dimensional, being aligned along the direction only of the oscillator Fock vacuum $|\Omega\rangle = |0\rangle$. All other excited Fock states $|n\rangle$ decouple by decay (without being coupled to some external environment or interaction) with a hierarchy of lifetimes determined by $\tau_+^{(n)} = \tau_0/(n(R-1))$, $n = 1, 2, \ldots$

More specifically, first one observes an oscillatory pattern contributing both to the overall phase factor proportional to $(n+1/2)$ in $\mathbb{U}_n(T)$ and to the function $F(T)$, and thus to its modulus squared $|F(T)|^2 = F_0 D(T)$ in $|\mathbb{U}_n(T)|^2$. The periodicity of this pattern is set by a rescaling of the proper time scale of the oscillator by the factor R, namely by the following effective angular frequency,

$$\omega_{\rm effective} = \frac{\omega_0}{R} < \omega_0. \tag{102}$$

Besides this oscillatory pattern, the time dependence of the Fock state occupations, modulated by $|\mathbb{U}_n(T)|^2$, is furthermore governed by two more real exponential time scales, the first of which modulates the factor $|F(T)|^2$ and the second which modulates the exponential in time normalisation of $|\mathbb{U}_n(T)|^2$ for $n \geq 1$,

$$\tau_- = \frac{\tau_0}{R} \stackrel{\tau_0\rightarrow 0^+}{\simeq} \tau_0 + \cdots, \qquad \tau_+^{(n)} = \frac{1}{n}\frac{\tau_0}{R-1} = \frac{1}{n}\frac{R^2 S}{2\omega_0^2\tau_0} \stackrel{\tau_0\rightarrow 0^+}{\simeq} \frac{1}{n}\frac{1}{2\omega_0^2\tau_0} + \cdots \tag{103}$$

or, when measured in units either of the characteristic time scale of the oscillator, $1/\omega_0$, or the intrinsic time scale τ_0 of the quantum deformation of its dynamics,

$$\begin{aligned} \omega_0\tau_- &= \tfrac{\omega_0\tau_0}{R} \stackrel{\tau_0\rightarrow 0^+}{\simeq} \omega_0\tau_0 + \cdots \quad , \quad \omega_0\tau_+^{(n)} = \tfrac{1}{n}\tfrac{\omega_0\tau_0}{R-1} = \tfrac{1}{n}\tfrac{R^2 S}{2\omega_0\tau_0} \stackrel{\tau_0\rightarrow 0^+}{\simeq} \tfrac{1}{n}\tfrac{1}{2\omega_0\tau_0} + \cdots, \\ \tfrac{\tau_-}{\tau_0} &= \tfrac{1}{R} \stackrel{\tau_0\rightarrow 0^+}{\simeq} 1 + \cdots \quad , \quad \tfrac{\tau_+^{(n)}}{\tau_0} = \tfrac{1}{n}\tfrac{1}{R-1} = \tfrac{1}{n}\tfrac{R^2 S}{2(\omega_0\tau_0)^2} \stackrel{\tau_0\rightarrow 0^+}{\simeq} \tfrac{1}{n}\tfrac{1}{2(\omega_0\tau_0)^2} + \cdots. \end{aligned} \tag{104}$$

For a given value of $\omega_0\tau_0$, and provided n is small enough such that $\tau_- < \tau_+^{(n)}$ (which is always the case for $n = 1$ at least), for any given Fock state $|n\rangle$ there are then effectively three time windows characteristic of different regimes for the deformed quantum dynamics, namely $0 \leq T \leq \tau_-$, $\tau_- \leq T \leq \tau_+^{(n)}$ and $\tau_+^{(n)} \leq T < \infty$ [15, 16]. To describe these windows it is relevant to consider the value of the characteristic time scale of the system, $1/\omega_0$, relative to the time scale of the deformation, namely the quantity $1/(\omega_0\tau_0)$ (note that if the physical system under consideration does not carry any characteristic time scale, for instance a free particle, no deviation from ordinary unitary quantum dynamics is present even when $\tau_0 \neq 0$). Since ordinary quantum behaviour is recovered in the limit $\tau_0 \rightarrow 0^+$, when the quantity $1/(\omega_0\tau_0)$ is extremely large, for all practical purposes the quantum behaviour of the system does not significantly differ from that of ordinary quantum mechanics, at least up to the time scale $\tau_+^{(n)}$ for each of those Fock states $|n\rangle$ such that $\tau_+^{(n)} > \tau_-$. In the time window $\tau_- \leq T \leq \tau_+^{(n)}$, only a very small time dependent rescaling of the Fock state occupation is occurring which is the less perceptible the larger is the value of $1/(\omega_0\tau_0)$. Since if indeed non vanishing in the physical world the actual value of τ_0 is expected to be on the order of the Planck time, some 10^{-43} s, while in comparison experimental conditions have not yet observed extremely high intensity excitations of modes of large enough frequencies for particle and interaction fields, it seems fair to assume that until now all experiments conducted in laboratories have remained inside this "ordinary quantum physics window" (this does not include violent astrophysical phenomena in strong gravitational quantum regimes that may be observed). It is only by moving into time scales $1/\omega_0$ becoming comparable to τ_0, that the time window for ordinary quantum mechanics begins to grow narrow enough that the deformed quantum dynamics of the system may start display deviations from ordinary unitary quantum behaviour, and thereby enable at least experimental upper bounds to be set on the deformation parameter τ_0.

When reaching such a regime, which is then essentially also the situation for those Fock states $|n\rangle$ with n sufficiently large such that now $\tau_+^{(n)} < \tau_-$, as well as for the time window $\tau_+^{(n)} \le T < \infty$ even in the discussion above, the telltale signs for the lack of a unitary quantum dynamics are, first, the total decoherence of the dynamics decaying ultimately to its ground state (on a time scale which is the smaller the larger is $\omega_0\tau_0$), and second, the time dependent rescaling or renormalisation of the occupation density of that ground state and of the excited states at intermediate times, with in particular for the ground state an asymptotic in time rescaling of its occupation given by the quantity

$$F_0 = \frac{1}{S^2}\,\frac{R^4 + 4\omega_0^2\tau_0^2}{R^2 + 4\omega_0^2\tau_0^2}. \tag{105}$$

The behaviour of the latter factor as a function of $1/(\omega_0\tau_0)$ is noteworthy [15, 16],

$$F_0 \stackrel{1/(\omega_0\tau_0)\to 0}{\longrightarrow} 4\,\frac{1}{\omega_0\tau_0} + \dots, \qquad F_0 \stackrel{1/(\omega_0\tau_0)\to +\infty}{\longrightarrow} 1 + \frac{2}{\left(1/(\omega_0\tau_0)\right)^2} + \dots \tag{106}$$

Hence, as $\tau_0 \to 0^+$, the population rescaling factor F_0 keeps on approaching the unit value it has when $\tau_0 = 0$ *but from above*, which means that as $1/(\omega_0\tau_0)$ decreases F_0 keeps on growing ever larger than unity, until it reaches a maximal value lying above unity ($F_0^{\rm max} \simeq 1.079$ for $1/(\omega_0\tau_0) \simeq 2.591$) and from which further on, as $1/(\omega_0\tau_0)$ still keeps decreasing, F_0 starts decreasing as well, then passes the unit value, to finally reach a vanishing value in the limit that $1/(\omega_0\tau_0)$ also vanishes. Consequently given a value for τ_0, for an angular frequency larger than a certain threshold, $\omega_{\rm threshold}(\tau_0)$, the survival occupation density of even the Fock vacuum is always less than its initial value, while for ω_0 values less than $\omega_{\rm threshold}(\tau_0)$, the survival occupation density is always larger than its initial value. Nonetheless in all circumstances all excited Fock states end up not being populated at all at asymptotic times. Within a quantum field theory context, especially for the gravitational field, clearly such behaviour implies some tantalising prospects for dynamics at the smallest spacetime scales, leading to an effective coarse-graining of spacetime geometry since this geometry may only be probed through interacting quantum fields.

5 Conclusions

This paper considered the canonical operator quantisation formalism corresponding to the functional integral of the Klauder–Daubechies construction of the phase space path integral [1]. The latter formulation introduces a regularisation parameter, equivalent to a new time scale $\tau_0 > 0$, such that in the limit where it vanishes the construction reproduces the correct quantum dynamics of the system. This result was demonstrated explicitly from the operator representation of the same construction, in the specific case of the harmonic oscillator, thereby highlithing from a different and complementary point of view the inner workings of the Klauder–Daubechies approach to quantum dynamics.

In effect, this approach promotes the original system to the dynamics of an extended one of which the configuration space is the phase space of the original system, equipped not only with that phase space's symplectic geometry but also a Riemannian metric with identical volume form. The latter structure is related to a Brownian motion component added to the quantum dynamics of the original system, such that when the Brownian motion regularisation is taken away again, only the original quantum system survives. This formulation offers a number of advantages, not least of which is its manifest covariance under general canonical transformations of the phase space parametrisation, which may be put to efficient use to develop new nonpertubative quantisation techniques [18, 19]. Furthermore the extended regularising dynamics is of the form of a generalised Landau problem in phase space, with a pure positive imaginary mass set by the time scale parameter τ_0. In this respect, the Klauder–Daubechies construction comes in close resonance with present day developments in noncommutative geometry and quantum mechanics, most of which are inspired precisely by the Landau problem in the plane in which the mass parameter is taken to vanish [17].

The operator formulation of the Klauder–Daubechies construction should also make it possible to extend it to systems with more than a single degree of freedom, one first case of interest being precisely the Landau problem itself and its associated noncommutative geometry of the Moyal plane. But beyond that, relativistic quantum field theories with their short-distance divergences in perturbation theory are another case in point. Indeed, the operator technique is well adapted to keep the value of τ_0 finite throughout, which is possibly a choice of physical relevance in the spirit of deformations of quantum algebraic structures, which however then reveals some appealing as well as some not so appealing new features. If only for that purpose, a finite τ_0 provides a new type of short-distance regularisation in local quantum field theory taming all short-distance divergences. On the other hand, unitarity and Lorentz invariance are then lost at time scales less than τ_0, with however a suppression of dynamics precisely on those scales as well which is bound to induce an effective coarse-graining of spacetime geometry in strong gravitational quantum systems. In the latter context, the status of initial cosmological singularities, or the issue of trans-Planckian energies in black hole radiation are open issues that come to mind, which could be addressed within the Klauder–Daubechies framework for quantum dynamics.

Acknowledgements

The authors wish to address their warm words of appreciation to Prof. John R. Klauder for insightful discussions and his constant interest in the present work. Prof. Gerhard C. Hegerfeldt is also thanked for a constructive question having led to further clarification in the analysis of this paper.

The first part of this work was initiated in February-March 2008 while two of us were visiting the African Institute for Mathematical Sciences (AIMS, Muizenberg, South Africa), J.G. as invited lecturer and C.M.B. as the beneficiary of a two months Victor Rothschild Fellowship. We wish to thank Prof. Fritz Hahne for his interest and constant encouragements, and AIMS for its wonderful hospitality and the financial support which made our joint stay there possible. Laure Gouba, postdoctoral Fellow at AIMS at that time, also took part in the initial stages of the analysis, for which we acknowledge her collaboration. Over the three years of his PhD work, C.M.B. benefited from a PhD Fellowship at the University of Kinshasa from the "Coopération Universitaire au Développement (CUD)" of the Universities of the French speaking Community of Belgium, which also made three visits of three months each at the Catholic University of Louvain (Belgium) possible during that period. C.M.B. is grateful to the CUD for this most essential support. J.G. acknowledges the Abdus Salam International Centre for Theoretical Physics (ICTP, Trieste, Italy) Visiting Scholar Programme in support of a Visiting Professorship at the ICMPA-UNESCO (Republic of Benin). O.M. acknowledges partial support by the Marie Curie programme RTN MRTN-CT-2006-035505 through the University of Roma Tre (Rome, Italy). The work of J.G. and O.M. is supported in part by the Institut Interuniversitaire des Sciences Nucléaires (I.I.S.N., Belgium), and by the Belgian Federal Office for Scientific, Technical and Cultural Affairs through the Interuniversity Attraction Poles (IAP) P6/11.

References

[1] J. R. Klauder, *Quantization IS Geometry after All*, Ann. Phys. **188**, 120–141 (1988).

[2] R. Alicki, J. R. Klauder and J. Lewandowski, *Landau-Level Ground-State Degeneracy and its Relevance for a Generalized Quantization Procedure*, Phys. Rev. **A48**, 2538 (1993).

[3] J. R. Klauder and P. Maraner, *Dynamics as Shadow of Phase Space Geometry*, Ann. Phys. **253**, 356–375 (1997).

[4] J. R. Klauder, *Is Quantization Geometry?*, Presented at the *International Conference on 70 Years of Quantum Mechanics and Recent Trends in Theoretical Physics*, 2–14 January 1996, Calcutta (India), e-print `arXiv:quant-ph/9604032` (April 1996).

[5] J. R. Klauder, *Metrical Quantization*, Presented at the *Workshop of the 10th Max Born Symposium: Quantum Future*, 24–27 September 1997, Wroclaw (Poland), e-Print `arXiv:quant-ph/9804009` (April 1998).

[6] J. R. Klauder, *Phase Space Geometry in Classical and Quantum Mechanics*, in the *Proceedings of the Second International Workshop on Contemporary Problems in Mathematical Physics*, eds. J. Govaerts, M. N. Hounkonnou and A. Z. Msezane (World Scientific, Singapore, 2002), pp. 395–408 [e-print `arXiv:quant-ph/0112010` (December 2001)].

[7] J. R. Klauder and I. Daubechies, *Measures for Path Integrals*, *Phys. Rev. Lett.* **48**, 117 (1982).

[8] I. Daubechies and J. R. Klauder, *Measures for More Quadratic Path Integrals*, *Lett. Math. Phys.* **7**, 229 (1983).

[9] I. Daubechies and J. R. Klauder, *Constructing Measures for Path Integrals*, *J. Math. Phys.* **23**, 1806–1822 (1982).

[10] J. R. Klauder and I. Daubechies, *Quantum Mechanical Path Integrals with Wiener Measures for all Polynomial Hamiltonians*, *Phys. Rev. Lett.* **52**, 1161 (1984).

[11] I. Daubechies and J. R. Klauder, *Quantum Mechanical Path Integrals with Wiener Measures for all Polynomial Hamiltonians. 2.*, *J. Math. Phys.* **26**, 2239 (1985).

[12] I. Daubechies, J. R. Klauder and T. Paul, *Wiener Measures for Path Integrals with Affine Kinematic Variables*, *J. Math. Phys.* **28**, 85 (1987).

[13] J. R. Klauder and E. C. G. Sudarshan, *Fundamentals of Quantum Optics* (Benjamin, New York, 1968, 1996, Dover Publications, New York, 2006).

[14] G. 't Hooft, *Emergent Quantum Mechanics and Emergent Symmetries*, Presented at the $13^{\rm th}$ International Symposium on Particles, Strings and Cosmology (PASCOS 07), 2–7 July 2007, London (England), *AIP Conf. Proc.* **957**, 154–163 (2007) [e-Print `arXiv:0707.4568 [hep-th]` (July 2007)].

[15] J. Govaerts and O. Mattelaer, *A Deformation of Quantum Dynamics through the Phase Space Path Integral*, in the *Proceedings of the Fifth International Workshop on Contemporary Problems in Mathematical Physics*, eds. J. Govaerts and M. N. Hounkonnou (International Chair in Mathematical Physics and Applications, UNESCO Chair, Cotonou, Republic of Benin, 2008), pp. 170–186 [e-print `arXiv:0812.0596 [hep-th]` (December 2008)].

[16] O. Mattelaer, *Étude d'une Déformation de la Dynamique Quantique dans l'Espace de Phase*, M.Sc. Thesis (Catholic University of Louvain, Louvain-la-Neuve, Belgium, September 2006), unpublished, available in electronic format on request.

[17] See for instance, and references therein,
F. G. Scholtz, L. Gouba, A. Hafver and C. M. Rohwer, *Formulation, Interpretation and Application of Non-Commutative Quantum Mechanics*, *J. Phys. A* **42**, 175303 (2009) [e-Print: `arXiv:0812.2803 [math-ph]` (December 2008)].

[18] C. Matondo Bwayi, *États Cohérents, Intégrale Fonctionnelle et Transformation Canonique Action-Angle: Une Approche Non Perturbative à la Dynamique Quantique*, PhD Thesis, University of Kinshasa (Kinshasa, Democratic Republic of Congo, 10 June 2009), unpublished.

[19] J. Govaerts and C. Matondo Bwayi, in preparation.

[20] J. Govaerts and O. Mattelaer, in preparation.