Numerical solution of a singularly perturbed two-point boundary value problem using equidistribution: analysis of convergence

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Abstract

Adaptive grid methods are becoming established as valuable computational techniques for the numerical solution of differential equations with near-singular solutions. Adaptive methods are equally effective in approximating solutions of problems with boundary layers or interior layers (see, for example, [11]). Much is now being done in developing error analyses for methods that are based on adaptivity. In this paper, we present a rigorous error analysis for the solution of a singularly perturbed two-point boundary value problem on a grid that is constructed adaptively from a knowledge of the exact solution. The discrete solutions are generated by an upwind finite difference scheme and the grid is formed by equidistributing a monitor function based on arc-length. An error analysis shows that the discrete solutions are uniformly convergent with respect to the perturbation parameter, epsilon. The epsilon-uniform convergence is confirmed by numerical computations.

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1 Introduction

This paper deals with the error analysis of adaptive finite difference methods for solving differential equations. To gain insight into the nature of the convergence when there is a region in which the solution of the differential equation is steep, we consider the numerical approximation of singularly perturbed two-point boundary value problems. In particular, we consider the model problem

$$(\mathcal{L}u)(x) \equiv -\epsilon u_{xx} - p(x)u_x = 0, \qquad x \in (0,1), \tag{1.1}$$

with boundary conditions

$$u(0) = 0, \qquad u(1) = 1, \tag{1.2}$$

where ϵ is a constant satisfying $0 < \epsilon \leq 1$. It is also assumed that $p \in C[0,1]$ and that there are constants a and b such that

$$0 < a \leq p(x) \leq b \quad \text{and} \quad |p'(x)| \leq b, \qquad \forall x \in [0,1]. \tag{1.3}$$

The exact solution of the problem is

$$u(x) = \frac{G(x)}{G(1)}, \tag{1.4}$$

where

$$G(x) = \int_0^x \exp\,[-\frac{1}{\epsilon}\int_0^t p(s)\,ds]\,dt. \tag{1.5}$$

For $\epsilon \ll 1$ the solution has a boundary layer of thickness $O(\epsilon)$ near the boundary $x = 0$ and it is well known that a central or upwind difference scheme on an even mesh will not give a satisfactory numerical solution in this case. To obtain a reliable numerical solution for (1.1)-(1.2) when $\epsilon \ll 1$, it is advantageous to use a mesh that concentrates nodes in the boundary layer. Ideally, the mesh should be generated by adapting it to the features of the computed solution, and this is usually done by equidistributing a monitor function over the domain of the problem. The proper choice of monitor function is still an open question [2], but it is normally some measure of computational error or solution variation. There has been a great deal of work done recently on the use of adaptive methods for steady and unsteady solutions of partial differential equations. The reader is referred to [3], [6], [7], [12] and [21] for an overview of some of the recent work in this area. A study of the published work on adaptivity will show that although methods based on equidistribution are being used extensively to solve differential problems with steep solutions, much has still to be done on the error analysis of these methods. Many adaptive methods are based on error indicators and robust results using this approach are now being produced for singular perturbation problems (see, for example, [1], [4] and [20]).

A paper by Qiu and Sloan [13] presented an analysis of convergence for an adaptive finite difference solution of problem (1.1)-(1.2). In this earlier work the solution is obtained on a mesh that is close to the adaptive mesh arising from the equidistribution of a monitor function based on a power of the solution gradient. The monitor function adopted in [13] is impractical in the sense that it may assume zero values, whereas a useful monitor function should be bounded below by a positive constant [18]. The objective of the current work is to extend the analysis of Qiu and Sloan [13] to deal with convergence of the approximate solution of (1.1)-(1.2) on a mesh that is based on the equidistribution of an arc-length monitor function (see [6], [11] and [18]).

Discrete methods whose solutions converge independently of ϵ are said to be ϵ-uniform. In particular, a method of solving (1.1)-(1.2) is ϵ-uniform of order γ on the mesh

$$\Omega_N \equiv \{x_j : x_j = x_{j-1} + h_j,\ 1 \le j \le N,\ x_0 = 0, x_N = 1\} \tag{1.6}$$

if there exists a positive integer N_0 such that for $N \ge N_0$,

$$\max_{0 \le j \le N} |u(x_j) - u_j| \le C(\gamma) N^{-\gamma} \tag{1.7}$$

holds, where u is the solution of (1.1)-(1.2), $\{u_j\}_{j=0}^{N}$ is the numerical approximation to u, and γ and $C(\gamma)$ are positive constants that are independent of ϵ and N. If a method is ϵ-uniform, mesh refinement causes the error to decrease in a manner that is independent of the perturbation parameter.

Kellogg and Tsan [8] have analysed the error behaviour of three difference schemes for solving a general linear, singular perturbation problem on an even mesh. They show that the standard first-order upwind scheme is not ϵ-uniform. Two schemes are considered in [8] that have truncation errors $O(N^{-2})$ for fixed $\epsilon > 0$, with a loss of accuracy to $O(N^{-1})$ as $\epsilon \to 0$. Miller et al. [9] have analysed the performance of the standard first-order upwind scheme on a piecewise even mesh proposed by Shishkin [16] — fine in the boundary and coarse in the rest of the domain. They have demonstrated ϵ-uniform convergence on the Shishkin mesh (see also the texts [10] and [15]). Stynes and Roos [17] have recently analysed a midpoint upwind scheme for the solution of a singular perturbation problem on an arbitrary mesh. They have shown that the scheme is uniformly convergent in ϵ outside the boundary layer and they have pointed out that uniform convergence cannot be obtained at all interior mesh points unless the mesh is specially tailored to the solution of the problem. Gartland [5] has considered an exponentially graded mesh, and he has shown how to construct schemes that have arbitrarily high uniform order of convergence. Roos [14] has recently given an extremely useful survey of results on layer-adapted meshes.

Highly accurate computational solutions of singularly perturbed two-point boundary value problems have been obtained on adapted meshes [11]. Unlike the schemes proposed in [5, 8, 9], adaptive methods can handle not only boundary layer problems but also interior layer problems.

In this work, we consider an uneven grid that is generated by equidistributing the arc-length monitor function [6]. This approach is now commonly used in practical computations. We shall show that for any given $\gamma \in (0, 1)$ there exists a positive constant $C(\gamma)$ depending on γ but independent of ϵ and N , and an integer N_0 such that for $N \ge N_0$,

$$|u(x_j) - u_j| \le C(\gamma) N^{-\gamma}, \qquad 0 \le j \le N, \tag{1.8}$$

where $\{u_j\}_{j=0}^{N}$ is the numerical solution obtained using an upwind difference scheme (see (2.1)-(2.2) below) with an uneven grid generated by equidistributing the arc-length monitor function (see (2.8)). The steps involved in the proof of (1.8) are outlined in the following two paragraphs.

We divide the domain $[0, 1]$ into three regions: a boundary layer region, a transition region and a regular solution region. In the boundary layer the exact solution is very steep and the derivatives of the exact solution are very large. On the other hand, the solution is smooth in the regular solution region, and in this region any derivatives can be bounded by a constant that is independent of ϵ. The mesh spacing in the boundary layer region is very small and is of order $O(\epsilon N^{-1})$ as $\epsilon \to 0$ and $N \to \infty$. In the regular solution region the mesh spacing is of order $O(N^{-1})$, and it is observed that the boundary layer and the regular solution region each contain $O(N)$ mesh points. Moreover, the ratio of the mesh spacing in the boundary layer to that in the regular solution region is of order $O(\epsilon)$ as $\epsilon \to 0$. Finally, the number of mesh points in the transition region — between the boundary layer and the regular solution region — is independent of both N and ϵ: here, the mesh spacing is between $O(\epsilon N^{-1})$ and $O(N^{-1})$, where we select N such that $\epsilon \ll N^{-1}$.

The variation in mesh spacing across the three regions prevents us from analysing the error behaviour in the regions in a uniform manner. We first use M-matrix theory [19] to show that the error in the regular solution region can be bounded by $O(N^{-1})$. Secondly, we use the fact that there are $O(1)$ mesh points in the transition region to show that the error in this region can be bounded by $O(N^{-\gamma})$, where $\gamma \in (0,1)$. In particular, we establish that the error at the outermost mesh point of the boundary layer is bounded by $O(N^{-\gamma})$. Consequently, we obtain a two-point boundary value problem for the boundary layer region. Repeating the technique used in dealing with the regular solution region, we can obtain an error bound for mesh points in the boundary layer.

It should be noted that the analysis presented here deals only with semi-discretisation of the adaptive method. We use the term semi-discretisation in this context to indicate that the exact solution (1.4) is used in the equidistribution principle to generate the mesh, and the solution for $\{u_j\}_{j=0}^{N}$ is then considered on this known mesh. A fully discretised scheme is one in which a discrete approximation of the equidistribution principle is conjoined with the finite difference equation to give a nonlinear algebraic system for the set of unknowns $\{x_j, u_j\}_{j=1}^{N-1}$. Error analysis for the fully discretised schemes is more complicated and will be investigated in a future project. We regard this work as an initial step in a convergence analysis for adaptive methods based on arc-length equidistribution.

2 Difference scheme and main theorem

The upwind difference approximation to (1.1)-(1.2) that we wish to analyse is

$$(\mathcal{L}_\Delta u_\Delta)(j) \equiv -\epsilon(D_+D_-u_\Delta)(j) - p_j(D_+u_\Delta)(j) = 0, \qquad 1 \le j \le N-1, \tag{2.1}$$

$$(u_\Delta)(0) = 0, \qquad (u_\Delta)(N) = 1, \tag{2.2}$$

where $p_j = p(x_j)$ and u_Δ is the mesh function with $u_\Delta(j)$ denoting the approximation, u_j, to $u(x_j)$. The operators used above are given by

$$(D_+u_\Delta)(j) = \frac{u_{j+1} - u_j}{h_{j+1}}, \qquad (D_-u_\Delta)(j) = (D_+u_\Delta)(j-1),$$

$$(D_+D_-u_\Delta)(j) = \frac{(D_+u_\Delta)(j) - (D_-u_\Delta)(j)}{\tilde{h}_j}, \quad \tilde{h}_j = \frac{h_j + h_{j+1}}{2}.$$

The scheme (2.1)-(2.2) is conveniently expressed as

$$-C_j u_{j-1} + A_j u_j - B_j u_{j+1} = 0, \qquad 1 \le j \le N-1, \tag{2.3}$$

$$u_0 = 0, \qquad u_N = 1, \tag{2.4}$$

where

$$A_j = \frac{2\epsilon}{h_j h_{j+1}} + \frac{p_j}{h_{j+1}},$$
$$B_j = \frac{\epsilon}{h_{j+1}\tilde{h}_j} + \frac{p_j}{h_{j+1}},$$
$$C_j = \frac{\epsilon}{h_j \tilde{h}_j}.$$

It is easy to verify that

$$A_j > 0, \quad B_j > 0, \quad C_j > 0, \qquad 1 \le j \le N-1,$$
$$A_j = B_j + C_j, \qquad 1 \le j \le N-1.$$

We construct a mesh by equidistributing the monitor function

$$M(u(x), x) = \sqrt{1 + u_x^2}$$

over the domain $[0,1]$. This gives rise to a mapping $x = x(\xi)$, relating the computational coordinate $\xi \in [0,1]$ to the physical coordinate $x \in [0,1]$, defined by

$$\int_0^x M(u(s), s)ds = \xi \int_0^1 M(u(s), s)ds = \xi L, \tag{2.5}$$

where L is the arc length of u over $(0,1)$. If (2.5) defines a mapping $x = x(\xi)$, then

$$\frac{dx}{d\xi} = \frac{L}{\sqrt{1 + u_x^2}}. \tag{2.6}$$

2.1 Semi-discretised scheme

The evenly-spaced grid for the coordinate ξ will be given by

$$\xi_j = \frac{j}{N}, \qquad 0 \le j \le N. \tag{2.7}$$

We use (2.6) to calculate the grid points in the physical domain. More precisely, we use (2.6) to obtain

$$x_j = \int_0^{\xi_j} \frac{L}{\sqrt{1 + u_x^2}} d\xi, \qquad 0 \le j \le N, \tag{2.8}$$

where, inside the integrand, $u_x = u'(x(\xi))$. The mesh size is given by

$$h_j = x_j - x_{j-1}, \qquad 1 \le j \le N. \tag{2.9}$$

Using (2.8) and (2.9) an equivalent form of the above formula is given by

$$h_j = \int_{\xi_{j-1}}^{\xi_j} \frac{L}{\sqrt{1 + u_x^2}} d\xi, \qquad j = 1, 2, \ldots, N. \tag{2.10}$$

The problem (1.1)-(1.2) will be solved numerically by (2.3)-(2.4) and (2.10). This approach is called semi-discretisation, since in (2.10) the exact solution (1.4) is used in u_x and in the evaluation of L. In practice, the grid is computed using a numerical solution of (2.6) (see Section 8).

2.2 Fully-discretised scheme

In practical computation, we can avoid using L and we can replace u_x in (2.10) by suitable approximations. This approach is called full discretisation and is used in real computations. From (2.6) we have

$$\left(1+u_x^2\right)(dx)^2=(Ld\xi)^2,$$

which gives, with first-order difference approximation,

$$\left[1+\left(\frac{u_{j+1}-u_j}{x_{j+1}-x_j}\right)^2\right](x_{j+1}-x_j)^2=\left(\frac{L}{N}\right)^2,\quad 0\le j\le N-1.$$

It follows from the above equations that

$$(x_{j+1}-x_j)^2+(u_{j+1}-u_j)^2=(x_j-x_{j-1})^2+(u_j-u_{j-1})^2,\quad 1\le j\le N-1,\quad (2.11)$$

$$x_0=0,\qquad x_N=1.\quad (2.12)$$

The simultaneous solution of (2.3)-(2.4) and (2.11)-(2.12) produces the numerical approximation to the solution of (1.1)-(1.2) and also the numerical grid.

2.3 Main theorem

As mentioned before, we will only consider the semi-discretisation scheme (2.3)-(2.4) and (2.10) in this work. The main result that we shall prove is given in the following theorem.

Theorem 2.1 *Let $u(x)$ be the exact solution to (1.1)-(1.2) and let $u_j, 0\le j\le N$ be obtained by (2.3)-(2.4) on the grid defined by (2.10) and (1.4). For any given $\gamma\in(0,1)$, there exists a positive constant $C(\gamma)$ depending on γ but independent of ϵ and N such that*

$$|u(x_j)-u_j|\le C(\gamma)N^{-\gamma},\qquad 0\le j\le N,\quad (2.13)$$

provided that N satisfies

$$N\ge\frac{4p(0)L}{a}\gamma|\ln\gamma|^{-1}+\frac{L}{D},\quad (2.14)$$

where L is the arc length of u over $(0,1)$, and $D:=\epsilon u'(0)/p(0)$.

In the following section we shall show that D can be bounded by a constant that is independent of ϵ. Also, in subsequent sections, we let $O(1), c, c_1, ...$ denote positive constants that may take different values in different occurrences, but always being independent of N and ϵ. Similarly, $C(\gamma), C_1(\gamma), ...$ are positive constants depending only on γ but independent of ϵ and N.

3 Properties of the exact solution

¿From (1.4) and (1.5), we obtain

$$u'(x)=\frac{G'(x)}{G(1)}$$

and

$$u(x) = u'(0)G(x).$$

Furthermore, from

$$G(1) = \int_0^1 \exp\left[-\frac{P(x)}{\epsilon}\right] dx,$$

where

$$P(x) := \int_0^x p(s)\, ds,$$

it follows

$$\frac{\epsilon}{b}\left(1 - e^{-\frac{b}{\epsilon}}\right) \leq G(1) \leq \frac{\epsilon}{a},$$

or,

$$\frac{a}{\epsilon} < u'(0) \leq \frac{2b}{\epsilon}, \tag{3.1}$$

provided $\frac{1}{2} \leq 1 - e^{-\frac{b}{\epsilon}}$, and this gives $D = \epsilon u'(0)/p(0) = O(1)$. It is also clear that with this constraint on ϵ we have

$$\frac{a}{\epsilon}e^{-\frac{bx}{\epsilon}} < u'(x) \leq \frac{2b}{\epsilon}e^{-\frac{ax}{\epsilon}}, \qquad \forall x \in [0,1]. \tag{3.2}$$

We shall see in the subsequent analysis that it is convenient to have solution gradient bounds within and outwith the boundary layer. To this end we partition the dependent variable in the form

$$u(x) = A(x) + Z(x), \tag{3.3}$$

where

$$A(x) = D\left[1 - e^{-\frac{p(0)x}{\epsilon}}\right]. \tag{3.4}$$

It is readily seen that $A(x)$ and $Z(x)$ satisfy the conditions

- i $\quad A(0) = 0 \quad$ and $\quad Z(0) = 0$;
- ii $\quad A'(x) = u'(0)e^{-\frac{p(0)x}{\epsilon}}$;
- iii $\quad Z'(x) = u'(0)\left[e^{-\frac{1}{\epsilon}\int_0^x p(s)\, ds} - e^{-\frac{1}{\epsilon}p(0)x}\right]$.

Since we may write $p(s) = p(0) + sp'(\theta)$, where $\theta \in (0, s)$, we have

$$-\frac{1}{\epsilon}\int_0^x p(s)\, ds = -\frac{p(0)x}{\epsilon} - \frac{1}{\epsilon}\int_0^x sp'(\theta)\, ds$$

and

$$Z'(x) = A'(x)\left[e^{-\frac{1}{\epsilon}\int_0^x sp'(\theta)\, ds} - 1\right].$$

This yields

$$|Z'(x)| \leq A'(x) \max\left|e^{\pm\frac{1}{\epsilon}\int_0^x bsds} - 1\right|.$$

If we restrict x to the region $x \leq \frac{2\epsilon}{a}|\ln\epsilon| := x^*$, it follows that

$$\begin{aligned} e^{\pm\frac{1}{\epsilon}\int_0^x bsds} &= e^{\pm\frac{bx^2}{2\epsilon}} \\ &= 1 \pm \frac{bx^2}{2\epsilon} + O\left(\frac{x_\theta^4}{\epsilon^2}\right), \quad 0 < x_\theta < x. \end{aligned}$$

Whence

$$|Z'(x)| \leq \frac{bx^2}{\epsilon} A'(x) \quad \text{if} \quad x \leq x^*.$$

Since $A'(x) = u'(0)e^{-\frac{p(0)x}{\epsilon}} \leq \frac{2b}{\epsilon}e^{-\frac{ax}{\epsilon}}$, then

$$|Z'(x)| \leq \frac{2b^2x^2}{\epsilon^2}e^{-\frac{ax}{\epsilon}} \quad \text{for } \ x \leq x^*.$$

On the other hand, if $x > x^*$ then

$$|Z'(x)| \leq 2u'(0)e^{-\frac{ax}{\epsilon}} \leq 4b\epsilon.$$

The bound on $|Z'(x)|$ may be written as

$$|Z'(x)| \leq \begin{cases} \frac{2b^2x^2}{\epsilon^2}e^{-\frac{ax}{\epsilon}}, & x \leq x^* \\ 4b\epsilon, & x > x^* \end{cases} \tag{3.5}$$

Finally, we seek a bound on $|Z(x)|$, $\forall x \in [0,1]$. To this end, we use (3.5) for $x \leq x^*$ to obtain

$$\begin{aligned} |Z(x)| &\leq \int_0^{x^*} |Z'(s)|\, ds \\ &\leq \frac{2b^2}{\epsilon^2}\int_0^{x^*} s^2\, e^{-\frac{as}{\epsilon}}\, ds \\ &= \frac{2b^2}{\epsilon^2} I, \quad \text{say.} \end{aligned}$$

Integration by parts yields

$$\begin{aligned} I &= -\frac{4\epsilon^5}{a^3}|\ln\epsilon|^2 - \frac{4\epsilon^5}{a^3}|\ln\epsilon| + \frac{2\epsilon^3}{a^3}(1-\epsilon^2) \\ &\leq \frac{2\epsilon^3}{a^3}, \end{aligned}$$

and it follows that

$$|Z(x)| \leq \frac{4b^2}{a^3}\epsilon \quad \text{for } \ x \leq x^*.$$

If $x > x^*$, we may write

$$\begin{aligned} |Z(x)| &\leq \int_0^{x^*} |Z'(s)|\, ds + \int_{x^*}^1 |Z'(s)|\, ds \\ &\leq \frac{4b^2}{a^3}\epsilon + 4b\epsilon \\ &= \frac{4b(a^3+b)}{a^3}\epsilon = \Phi\epsilon, \quad \text{say.} \end{aligned}$$

These inequalities may be combined to show that for any $x \in [0, 1]$,

$$|Z(x)| \leq \Phi\epsilon, \tag{3.6}$$

where Φ is the constant defined above.

In the following section we shall see that the régime of interest is $\epsilon N \ll 1$. For $\epsilon \ll 1$ it is therefore safe to assume that N may be chosen such that

$$\epsilon N \leq \frac{D}{2\Phi\beta},$$

where

$$\beta := \frac{a}{4p(0)L}\frac{1}{\gamma}\ln\left(\frac{1}{\gamma}\right) \tag{3.7}$$

and γ is a fixed number in the interval $(0, 1)$. This condition on ϵN enables us to write a bound on $|Z(x)|$, $\forall x \in [0, 1]$, in the form

$$|Z(x)| \leq \frac{D}{2N\beta}. \tag{3.8}$$

The form of this bound is chosen to suit subsequent analysis.

4 Truncation error and mesh structure

4.1 The local truncation error of (2.1)

The local truncation error of (2.1) at node x_j is, for $j = 1, 2, ..., N-1$,

$$\tau_j = (\mathcal{L}_\Delta u)(j) - (\mathcal{L}u)(x_j),$$

where u in the first term denotes the set of exact solution values at the nodes. It is readily shown that this reduces to

$$\begin{aligned} \tau_j \;=\; & -\frac{\epsilon}{2\tilde{h}_j}\Big\{\frac{1}{h_{j+1}}\int_{x_j}^{x_{j+1}}(s - x_{j+1})^2 u'''(s)ds \\ & -\frac{1}{h_j}\int_{x_{j-1}}^{x_j}(s - x_{j-1})^2 u'''(s)ds\Big\} + \frac{p_j}{h_{j+1}}\int_{x_j}^{x_{j+1}}(s - x_{j+1})u''(s)ds, \end{aligned}$$

from which we obtain the bound

$$|\tau_j| \leq \epsilon\int_{x_{j-1}}^{x_{j+1}}|u'''(s)|ds + b\int_{x_j}^{x_{j+1}}|u''(s)|ds.$$

If we use the equation (1.1) this may be simplified to

$$|\tau_j| \leq c\int_{x_{j-1}}^{x_{j+1}}|u''(s)|\,ds, \quad j = 1, 2, \ldots, N-1, \tag{4.1}$$

where c is a constant that is independent of ϵ and N.

4.2 Mesh structure

We are interested in the case that $\epsilon \ll N^{-1}$. If $\epsilon N = O(1)$, then the error analysis is straightforward using the techniques in Section 5. In the case $\epsilon N \ll 1$, we note that $x = \epsilon \ln N$ is within the region of steep solution gradient near the boundary $x = 0$, since $u_x \geq (\epsilon N)^{-1} \gg 1$ at that point. In particular, we assume

$$\epsilon \ln N \leq \frac{1}{N}. \tag{4.2}$$

In the case that the above assumptions do not hold, the error analysis in Section 5 can be applied to obtain a global error estimate.

Let K be a positive integer satisfying

$$1 - \frac{LK}{DN} \geq \frac{1}{N\beta} \qquad \text{and} \qquad 1 - \frac{L(K+1)}{DN} < \frac{1}{N\beta}. \tag{4.3}$$

In other words, K is characterised by the requirement

$$\frac{D}{L}(N - \beta^{-1}) - 1 < K \leq \frac{D}{L}(N - \beta^{-1}). \tag{4.4}$$

If we assume (2.14), which is

$$N \geq \beta^{-1} + \frac{L}{D}, \tag{4.5}$$

then (4.4) and (4.5) ensure the existence of a positive K.

We first show that $u_x(x_K) \gg 1$: that is, x_K is within the region of steep variation at the boundary. It follows from (2.6) that

$$\begin{aligned} \frac{LK}{N} &= \int_0^{x_K} \sqrt{1 + u_x^2}\, dx \\ &> \int_0^{x_K} u_x\, dx \\ &= u(x_K) = A(x_K) + Z(x_K) \\ &\geq A(x_K) - |Z(x_K)|. \end{aligned}$$

This, together with (3.4), (3.8) and (4.3), shows that

$$e^{-\frac{p(0)x_K}{\epsilon}} > \frac{1}{2N\beta}, \qquad \text{or} \qquad x_K < \frac{\epsilon}{p(0)} \ln(2N\beta). \tag{4.6}$$

It is readily seen that

$$x_K \leq \frac{\epsilon}{a} |\ln \epsilon| = \frac{1}{2} x^*, \qquad \text{if } \epsilon N \ll 1, \tag{4.7}$$

which implies that x_K is inside the steep boundary region.

The second objective in this section is to show that

$$e^{-\frac{p(0)x_K}{\epsilon}} \leq \frac{c(\gamma)}{2N\beta} \tag{4.8}$$

and

$$x_K \geq \frac{\epsilon}{p(0)} \ln(2N\beta) - c(\gamma)\epsilon. \tag{4.9}$$

To establish these inequalities we see from (2.6) that

$$\begin{aligned}\frac{LK}{N} &\leq \int_0^{x_K} (1+u_x)dx \\ &= x_K + u(x_K) \\ &\leq \frac{\epsilon}{p(0)} \ln(2N\beta) + D\left[1 - e^{-\frac{p(0)x_K}{\epsilon}}\right] + \frac{D}{2N\beta},\end{aligned}$$

where we have made use of (4.6), (3.3) and (3.8). Use of (4.2) gives

$$\frac{LK}{DN} \leq \frac{c}{N} + \frac{1}{2N\beta} + \left[1 - e^{-\frac{p(0)x_K}{\epsilon}}\right],$$

and it now follows from the second inequality in (4.3) that

$$1 - \frac{L}{DN} - \frac{1}{N\beta} \leq \frac{c_1}{N} + \frac{1}{2N\beta} + \left[1 - e^{-\frac{p(0)x_K}{\epsilon}}\right].$$

Conditions (4.8) and (4.9) are given directly by this result.

In the course of this discussion of the mesh structure it is convenient to give one further condition on the solution in the steep boundary region that we make use of later. For $x \leq x^*$, it is clear from (3.5) that ϵ may be taken to be sufficiently small to ensure that $|Z'(x)| < \frac{1}{2}A'(x)$. From this it follows that for $x \leq x^*$

$$\frac{1}{2}A_x(x) < u_x(x) < \frac{3}{2}A_x(x). \tag{4.10}$$

We are now in a position to perform an error analysis in the three subregions

- I: Boundary layer region $(0, x_K)$;
- II: Transition region (x_K, x_J);
- III: Regular solution region $(x_J, 1)$,

where $|u_x| \gg 1$ if $x < x_J$ and $|u_x| = O(1)$ if $x > x_J$.

Lemma 4.1 *There are $O(N)$ grid points inside the boundary layer $(0, x_K)$. Moreover, we have*

$$(i) \qquad h_j \leq \frac{\epsilon}{p(0)\gamma} \ln\left(\frac{1}{\gamma}\right) \qquad \text{for} \qquad j \leq K; \tag{4.11}$$

$$(ii) \qquad h_j \geq C(\gamma)\epsilon \qquad \text{for} \qquad j > K, \tag{4.12}$$

where $C(\gamma)$ is a constant depending on γ, but independent of ϵ and N.

Proof. It follows from (4.4) that $K = O(N)$. Therefore, there are $O(N)$ grid points inside the boundary layer region. Making use of (3.1), (4.6) and (4.10), it is clear that for $x \leq x_K$ we have

$$u_x(x) \geq u_x(x_K) \geq \frac{a}{4N\beta\epsilon}.$$

Consequently, for $j \leq K$,

$$h_j = \int_{\xi_{j-1}}^{\xi_j} \frac{L}{\sqrt{1+u_x^2}} d\xi < \int_{\xi_{j-1}}^{\xi_j} \frac{L}{u_x(x_K)} d\xi \leq \frac{4}{a} L\beta\epsilon = \frac{\epsilon}{p(0)\gamma} \ln\left(\frac{1}{\gamma}\right), \qquad \text{using (3.7).}$$

It remains to prove (4.12).

For $j > K$, if $x \in (x_{j-1}, x_j)$ then $u_x(x) < u_x(x_K)$. Therefore, since $u_x(x_K) > 1$,

$$h_j = \int_{\xi_{j-1}}^{\xi_j} \frac{L}{\sqrt{1+u_x^2}} d\xi > \frac{L}{N} \frac{1}{2u_x(x_K)}.$$

Since

$$u_x(x_K) \leq \frac{3}{2} A_x(x_K) \leq \frac{c(\gamma)}{\epsilon N \beta},$$

it follows that

$$h_j \geq C(\gamma)\epsilon, \qquad j > K.$$

This completes the proof of Lemma 4.1.

Lemma 4.2 *There are $O(1)$ grid points inside the transition region (x_K, x_J): here $O(1)$ indicates a number independent of N and ϵ.*

Proof. It is easy to show that $x_{J-1} \leq \frac{\epsilon}{a}|\ln \epsilon|$, since $u_x = O(1)$ if $x = \frac{\epsilon}{a}\ln(1/\epsilon)$. In other words, $(\frac{\epsilon}{a}|\ln \epsilon|, 1)$ belongs to the regular solution region. It follows from (2.6) that

$$\begin{aligned} L\frac{(J-1-K)}{N} &= \int_{x_K}^{x_{J-1}} \sqrt{1+u_x^2} dx \\ &\leq \int_{x_K}^{x_{J-1}} (1+u_x) dx \\ &\leq x_{J-1} - x_K + \frac{3}{2}\int_{x_K}^{x_{J-1}} A_x(x) dx \\ &< x_{J-1} + \frac{3b}{p(0)} e^{-p(0)x_K/\epsilon}, \qquad \text{using(3.1)} \\ &\leq \frac{\epsilon}{a}|\ln \epsilon| + \frac{c(\gamma)}{2N\beta}, \qquad \text{using(4.8)}. \end{aligned}$$

Using the assumptions (4.2) we conclude that $J - K$ can be bounded by a number independent of N and ϵ.

Lemma 4.3 *There are $O(N)$ grid points inside the regular solution region $(x_J, 1)$. Moreover, for $j \geq J+1$, we have $h_j = O(N^{-1})$.*

Proof. The results are obvious.

5 Error in the regular solution region

In this section, we shall investigate the maximum pointwise error in the regular solution region $(x_J, 1)$. It is recalled that $u_x = O(1)$ and $h_j = O(N^{-1})$ in this region.

Lemma 5.1

$$|\tau_j| \leq \frac{c}{N\epsilon^{1+\lambda}} \exp\left(-\frac{\lambda a x_{j-1}}{\epsilon}\right), \quad j = 1, 2, \ldots, N-1, \tag{5.1}$$

for any $0 < \lambda < 1$, with λ independent of ϵ and N.

Proof. It follows from (4.1) that

$$\begin{aligned}|\tau_j| &\leq c_1\int_{x_{j-1}}^{x_{j+1}}|u_{xx}|dx \leq c_1 L\int_{\xi_{j-1}}^{\xi_{j+1}}\frac{|u_{xx}|}{\sqrt{1+u_x^2}}d\xi \\ &\leq \frac{c}{\epsilon}\int_{\xi_{j-1}}^{\xi_{j+1}}\frac{|u_x|}{\sqrt{1+u_x^2}}d\xi,\end{aligned}$$

where in the last step we have used the equation (1.1). Utilising the expression $u_x(x) = u_x(0)G_x(x) = u_x(0)e^{-\frac{P(x)}{\epsilon}}$, and (3.1), it follows that

$$\frac{c_1}{\epsilon}e^{-\frac{P(x)}{\epsilon}} \leq u_x(x) \leq \frac{c_2}{\epsilon}e^{-\frac{P(x)}{\epsilon}},$$

where c_1 and c_2 are constants independent of ϵ and N. Since $f(y) = y/\sqrt{1+y^2}$ is an increasing function, we may show, using (1.4), that

$$\begin{aligned}|\tau_j| &\leq \frac{c}{\epsilon}\int_{\xi_{j-1}}^{\xi_{j+1}}\frac{\frac{c_2}{\epsilon}e^{-\frac{P(x)}{\epsilon}}}{\sqrt{1+\left(\frac{c_1}{\epsilon}\right)^2 e^{-\frac{2P(x)}{\epsilon}}}}d\xi \\ &\leq \frac{c}{N\epsilon}\frac{\frac{c_1}{\epsilon}e^{-\frac{P(x_{j-1})}{\epsilon}}}{\sqrt{1+\left(\frac{c_1}{\epsilon}\right)^2 e^{-\frac{2P(x_{j-1})}{\epsilon}}}} \\ &= J_j e^{-\frac{\lambda P(x_{j-1})}{\epsilon}}, \end{aligned} \tag{5.2}$$

where J_j is defined by

$$J_j = \frac{c}{N\epsilon^{1+\lambda}}\frac{\left(\frac{c_1}{\epsilon}\right)^{(1-\lambda)}\exp\left(-\frac{(1-\lambda)P(x_{j-1})}{\epsilon}\right)}{\sqrt{1+\left(\frac{c_1}{\epsilon}\right)^2\exp\left(-\frac{2P(x_{j-1})}{\epsilon}\right)}},$$

with $0<\lambda<1$ and $\lambda = O(1)$ in the sense that λ is independent of ϵ and N. Let

$$y_j = \frac{c_1}{\epsilon}\exp\left(\frac{-P(x_{j-1})}{\epsilon}\right), \qquad g(y) = \frac{y^{1-\lambda}}{\sqrt{1+y^2}}, \quad y>0.$$

The function $g(y)$ is increasing for $y \in [0, y^*]$ and decreasing for $y \in [y^*, +\infty)$, where $y^* = \sqrt{(1-\lambda)/\lambda}$. Since $\lambda = O(1)$, we have $y^* = O(1)$. Further,

$$J_j = \frac{c}{N\epsilon^{1+\lambda}}g(y_j) \leq \frac{c}{N\epsilon^{1+\lambda}}g(y^*) \leq \frac{c}{N\epsilon^{1+\lambda}}.$$

Thus, for $j = 1, ..., N-1$,

$$\begin{aligned}|\tau_j| &\leq \frac{c}{N\epsilon^{1+\lambda}}\exp\left(-\frac{\lambda P(x_{j-1})}{\epsilon}\right) \\ &\leq \frac{c}{N\epsilon^{1+\lambda}}\exp\left(-\frac{\lambda a x_{j-1}}{\epsilon}\right).\end{aligned}$$

This completes the proof.

We are now able to construct an error bound for the regular solution region using an approach similar to that in [13] (see also [8]). To this end we introduce the quantities

$$S_0 = 1, \qquad S_j = \prod_{k=1}^{j}\frac{1}{1+\lambda a h_k/\epsilon}, \qquad 1 \leq j \leq N.$$

Lemma 5.2 *The error function, $e_j = u(x_j) - u_j$, satisfies the inequality*

$$(\mathcal{L}_\Delta e)(j) < \frac{c}{N\epsilon^{1+\lambda}} S_{j-1}, \quad j = 1, 2, \ldots, N-1. \tag{5.3}$$

Proof. It is easy to verify that $\tau_j = (\mathcal{L}_\Delta e)(j)$. Noting that x_{j-1} may be written as $\sum_{k=1}^{j-1} h_k$, we obtain from (5.1) that

$$|\tau_j| \leq \frac{c}{N\epsilon^{1+\lambda}} \exp\left(-\sum_{k=1}^{j-1} \frac{\lambda a h_k}{\epsilon}\right) = \frac{c}{N\epsilon^{1+\lambda}} \prod_{k=1}^{j-1} e^{-\lambda a h_k/\epsilon}. \tag{5.4}$$

Using (5.4) and the fact $e^{-\phi} < (1+\phi)^{-1}$ for any $\phi > 0$, we obtain (5.3).

The following lemma is important for our error analysis. Its proof can be found in [8] and [19]. The proof is based on the theory of M-matrices.

Lemma 5.3 *The system $(\mathcal{L}_\Delta u)(j) = f_j$, $1 \leq j \leq N-1$, with $u(0)$ and $u(N)$ specified, has a solution. If $(\mathcal{L}_\Delta u)(j) < (\mathcal{L}_\Delta v)(j), 1 \leq j \leq N-1$, and if $u(0) < v(0)$, $u(N) < v(N)$, then $u(j) < v(j)$ for all $1 \leq j \leq N-1$.*

We are now able to proceed with the construction of an error bound in the regular solution region. It is readily shown that

$$\frac{S_j - S_{j-1}}{h_j} = -\frac{\lambda a}{\epsilon} S_j, \qquad 1 \leq j \leq N. \tag{5.5}$$

Using (2.3), we see that for $j = 1, 2, \cdots, N-1$,

$$\begin{aligned}
(\mathcal{L}_\Delta S)(j) &= -C_j S_{j-1} + A_j S_j - B_j S_{j+1} \\
&= -\frac{p_j}{h_{j+1}} (S_{j+1} - S_j) + \frac{\lambda a}{\tilde{h}_j} (S_{j+1} - S_j) \\
&= \frac{\lambda a}{\epsilon} \left(p_j - \frac{\lambda a h_{j+1}}{\tilde{h}_j}\right) S_{j+1},
\end{aligned}$$

where in the last step we have used (5.5). Since $h_{j+1}/\tilde{h}_j \leq 2$, it follows that

$$(\mathcal{L}_\Delta S)(j) \geq \frac{\lambda a}{\epsilon} (a - 2\lambda a) S_{j+1} \geq \frac{\lambda a^2}{2\epsilon} S_{j+1}, \tag{5.6}$$

provided $\lambda \leq 1/4$. Combining this requirement and that in Lemma 5.1, we choose

$$0 < \lambda \leq \frac{1}{4}, \qquad \lambda = O(1). \tag{5.7}$$

By the definition of S_j, we obtain from (5.6) that

$$(\mathcal{L}_\Delta S)(j) \geq \frac{c\lambda}{\epsilon} S_{j-1} \frac{1}{(1 + \lambda a h_j/\epsilon)(1 + \lambda a h_{j+1}/\epsilon)} \qquad 1 \leq j \leq N-1.$$

This, together with (5.3), yields

$$(\mathcal{L}_\Delta e)(j) < \frac{c}{N(\lambda\epsilon^\lambda)} \left(1 + \frac{\lambda a h_j}{\epsilon}\right) \left(1 + \frac{\lambda a h_{j+1}}{\epsilon}\right) (\mathcal{L}_\Delta S)(j), \qquad 1 \leq j \leq N-1, \tag{5.8}$$

provided that λ satisfies the requirements in (5.7). Since $h_j \le c_1/N$ for all j, we have

$$(\mathcal{L}_\Delta e)(j) < \frac{c}{N(\lambda\epsilon^\lambda)}\left(1+\frac{c_1\lambda}{N\epsilon}\right)^2 (\mathcal{L}_\Delta S)(j), \qquad 1 \le j \le N-1.$$

Since $e_0 = e_N = 0$, it follows from Lemma 5.3 that

$$e_j < \frac{c}{N(\lambda\epsilon^\lambda)}\left(1+\frac{c_1\lambda}{N\epsilon}\right)^2 S_j, \qquad 1 \le j \le N-1.$$

We may follow the same procedure to obtain a similar estimate for $-e_j$. Therefore, we have

$$|e_j| < \frac{c}{N(\lambda\epsilon^\lambda)}\left(1+\frac{c_1\lambda}{N\epsilon}\right)^2 S_j, \qquad 1 \le j \le N-1.$$

Lemma 4.3 indicates that $h_{J+1} = O(N^{-1})$ and $h_{J+2} = O(N^{-1})$, and from this we can show that

$$S_j \le \left(1+\frac{c_1\lambda}{N\epsilon}\right)^{-2}, \qquad j \ge J+2.$$

Therefore, we have

$$|e_j| \le \frac{c}{N(\lambda\epsilon^\lambda)}, \qquad J+2 \le j \le N. \tag{5.9}$$

We may assume that the parturbation parameter ϵ may be written in the form $\epsilon = 10^{-m}$, where m is real and positive. We now choose $\lambda = 1/m_0$, with $m_0 = \max\{4, m\}$, and it follows that

$$\frac{1}{\lambda\epsilon^\lambda} \le 10m_0.$$

Equation (5.9) can now be replaced by

$$|e_j| \le \frac{10m_0 c}{N}, \quad J+2 \le j \le N,$$

and this may be written as

$$|e_j| \le \frac{c}{N^\gamma}, \quad J+2 \le j \le N, \tag{5.10}$$

provided that $10m_0 \le N^{(1-\gamma)}$. This final inequality is equivalent to $N \ge N_0 := 10m_0^{\frac{1}{1-\gamma}}$. Since $\epsilon N_0 \le 1$ when $0 < \gamma \le 1 - \frac{1}{m_0}\log(10m_0)$, it follows that an integer N_0 exists for which (5.10) holds for $N \ge N_0$.

6 Error in the transition region

One important property of an adaptive mesh is that the exact solution of the differential problem should not experience an $O(1)$ jump between adjacent nodes as $N \to \infty$. More precisely, we know

$$|u(x_j) - u(x_{j-1})| < \frac{L}{N}, \qquad j = 1, 2, \ldots, N. \tag{6.1}$$

To prove this inequality we write

$$|u(x_j) - u(x_{j-1})| \le \int_{x_{j-1}}^{x_j} |u_x| dx = \int_{\xi_{j-1}}^{\xi_j} \frac{L|u_x|}{\sqrt{1+u_x^2}} d\xi < \frac{L}{N}.$$

In order to obtain error bounds in the transition region, we need to investigate the difference scheme (2.1)-(2.2). It follows from (2.1) that

$$(D_+u_\Delta)(j) = \frac{1}{1+\tilde{h}_j p_j/\epsilon}(D_+u_\Delta)(j-1), \quad j = 1, 2, \ldots, N-1. \tag{6.2}$$

This identity is used in the three lemmas that follow.

Lemma 6.1 *We have*

$$|(D_+u_\Delta)(0)| \le c\epsilon^{-1}.$$

Proof. Let $M := \left[\frac{a}{b}\frac{N}{2L}\right]$, where $[\bullet]$ denotes the integer part of $\bullet$. From (2.6) and (3.2) we have

$$\frac{Lj}{N} = \int_0^{x_j}\sqrt{1+u_x^2}dx > \int_0^{x_j} u_x dx > \frac{a}{b}\left(1-e^{-bx_j/\epsilon}\right), \qquad 1 \le j \le N.$$

Consequently,

$$e^{-bx_j/\epsilon} > 1 - \frac{b}{a}\frac{Lj}{N} \ge 1 - \frac{1}{2} = \frac{1}{2}, \qquad 1 \le j \le M, \tag{6.3}$$

which gives

$$e^{bx_j/\epsilon} < 2, \qquad \text{and } x_j < \epsilon \ln 2, \qquad 1 \le j \le M.$$

For $j \le M$, we may use (2.10) again to obtain

$$h_j < \frac{L}{N}\frac{1}{u_x(x_j)} < \frac{L\epsilon}{aN}e^{bx_j/\epsilon} < \frac{2L\epsilon}{aN}. \tag{6.4}$$

Similarly, since $u_x < c/\epsilon$, we can show using (6.3) that

$$h_j \ge \frac{c}{Nu_x(x_{j-1})} \ge \frac{c\epsilon}{N} \ge \frac{c\epsilon}{N}e^{bx_j/\epsilon}, \qquad j \le M. \tag{6.5}$$

It follows from (6.2) that

$$|(D_+u_\Delta)(j-1)| \ge e^{-b\tilde{h}_{j-1}/\epsilon}|(D_+u_\Delta)(j-2)| \ge \cdots \ge \exp\left(-b\sum_{k=1}^{j-1}\tilde{h}_k/\epsilon\right)|(D_+u_\Delta)(0)|. \tag{6.6}$$

Combining (6.4), (6.5) and (6.6) we find

$$\begin{aligned}
|u_j - u_{j-1}| \quad &\ge \quad \frac{h_j}{h_1}\exp\left(-b\sum_{k=1}^{j-1}\tilde{h}_k/\epsilon\right)|u_1-u_0|, \\
&\ge \quad c\exp\left(\frac{bh_1}{2\epsilon}+\frac{bh_j}{2\epsilon}\right)|u_1-u_0|, \\
&\ge \quad c|u_1-u_0|, \qquad j \le M.
\end{aligned}$$

It is easy to show from (2.1)-(2.2) that $\{u_j\}$ is an increasing sequence. Therefore,

$$1 \ge u_M - u_0 = \sum_{j=1}^{M}|u_j - u_{j-1}| \ge cM|u_1-u_0|. \tag{6.7}$$

Since $M = \left[\frac{a}{b}\frac{N}{2L}\right] = O(N)$, (6.7) implies $|u_1-u_0| \le cN^{-1}$. This, and the fact that $h_1 \ge c\epsilon N^{-1}$ (see (6.5)), lead to $|(D_+u_\Delta)(0)| \le c\epsilon^{-1}$. This completes the proof of the lemma.

Lemma 6.2 *Let $\gamma \in (0,1)$ be a fixed number and let K be defined by (4.4). Then*

$$|u_j - u_{j-1}| \le \frac{c}{N\gamma} e^{(1-\gamma)p(0)x_j/\epsilon}, \qquad j \le K.$$

Proof. We first show that

$$g(\phi) = e^{\gamma\phi} - 1 - \phi \le 0, \qquad \text{for} \quad 0 \le \phi \le \frac{1}{\gamma} \ln\left(\frac{1}{\gamma}\right). \tag{6.8}$$

This can be done by using the following results:

$$g(0) = 0, \quad g'(\phi) = \gamma e^{\gamma\phi} - 1 \le \gamma \exp\left(\gamma \cdot \frac{1}{\gamma} \ln\left(\frac{1}{\gamma}\right)\right) - 1 = 0.$$

For $j \le K$, we have $x_j \le x_K < \epsilon/p(0)\ \ln(2N\beta)$, using (4.6).

Hence,

$$p(x_j) = p(0) + x_j p'(\theta) \ge p(0) - bx_K,$$

and it follows that

$$\begin{aligned} 1 + p_j\tilde{h}_{j-1}/\epsilon &\ge 1 + p(0)\tilde{h}_{j-1}/\epsilon - bx_K\tilde{h}_{j-1}/\epsilon, \\ &> 1 + p(0)\tilde{h}_{j-1}/\epsilon - c\epsilon\ln(2N\beta), \quad \text{using (4.6) and (4.11)}, \\ &> 1 + p(0)\tilde{h}_{j-1}/\epsilon - c/N, \quad \text{using (4.2)}. \end{aligned}$$

¿From (4.11) we have

$$\frac{p(0)\tilde{h}_{j-1}}{\epsilon} - \frac{c}{N} \le \frac{1}{\gamma} \ln\left(\frac{1}{\gamma}\right), \qquad j \le K. \tag{6.9}$$

It follows from (6.8) and (6.9) that

$$\begin{aligned} \frac{1}{1 + p_j\tilde{h}_{j-1}/\epsilon} &\le \frac{1}{1 + p(0)\tilde{h}_{j-1}/\epsilon - c/N}, \\ &\le e^{c/N} e^{-\gamma p(0)\tilde{h}_{j-1}/\epsilon}, \qquad j \le K. \end{aligned}$$

Use of (6.2) and the above inequality gives

$$\begin{aligned} |(D_+u_\Delta)(j-1)| &\le e^{c/N} e^{-\gamma p(0)\tilde{h}_{j-1}/\epsilon} |(D_+u_\Delta)(j-2)| \\ &\le \cdots \\ &\le e^{c(K-1)/N} \exp\left(-\gamma p(0) \sum_{k=1}^{j-1} \tilde{h}_k/\epsilon\right) |(D_+u_\Delta)(0)|, \qquad j \le K. \end{aligned} \tag{6.10}$$

It follows from (2.10) that $h_j \le c\epsilon N^{-1} e^{p(0)x_j/\epsilon}$. This, together with (6.10), Lemma 6.1 and the condition $K = O(N)$, yields

$$\begin{aligned} |u_j - u_{j-1}| &\le \frac{c}{N} \exp\left(\frac{\gamma p(0)h_1}{2\epsilon} + \frac{\gamma p(0)h_j}{2\epsilon}\right) \exp\left((1-\gamma)p(0)\frac{x_j}{\epsilon}\right) \\ &\le \frac{c}{N}\left(\frac{1}{\gamma}\right) \exp\left((1-\gamma)p(0)\frac{x_j}{\epsilon}\right), \end{aligned}$$

where in the last step we have used (4.11).

Lemma 6.3 *Let $\gamma \in (0,1)$ be a fixed number and let K be defined by (4.4). Then*

$$|u_j - u_{j-1}| \le C(\gamma)N^{-\gamma}, \tag{6.11}$$

for $K \le j \le J+2$.

Proof. It is noted that there is only a finite number of grid points between x_K and x_{J+2}: that is, $J - K = O(1)$. We can use mathematical induction to prove (6.11). From Lemma 6.2 and (4.6), we know that (6.11) is true for $j = K$. Assume (6.11) is true for an index s with $K \le s \le J+1$; that is,

$$|u_s - u_{s-1}| \le C(\gamma)N^{-\gamma}. \tag{6.12}$$

We will show that (6.11) holds for the index $s+1$. It follows from (6.2) that

$$|u_{s+1} - u_s| < \frac{h_{s+1}}{h_s}\frac{\epsilon}{p_s\tilde{h}_s}|u_s - u_{s-1}| \le \frac{2\epsilon}{ah_s}|u_s - u_{s-1}|. \tag{6.13}$$

Since $s \ge K$, (4.12) implies that $h_s \ge C(\gamma)\epsilon$. This, together with (6.12) and (6.13), yields

$$|u_{s+1} - u_s| \le C(\gamma)N^{-\gamma}.$$

The proof of the lemma is complete.

Having the above two lemmas, we are ready to obtain the error bounds in the transition region. It follows from (6.11) that

$$|u_{J+2} - u_j| \le C(\gamma)N^{-\gamma}, \qquad K \le j \le J+1, \tag{6.14}$$

where we have used the fact $J - K = O(1)$ (see Lemma 4.2). For any $K \le j \le J+1$, we now have

$$\begin{aligned} & |u(x_j) - u_j| \le |u(x_j) - u(x_{J+2})| + |u(x_{J+2}) - u_{J+2}| + |u_{J+2} - u_j| \\ \le \quad & cN^{-1} + c_1N^{-\gamma} + C(\gamma)N^{-\gamma}, \end{aligned}$$

where in the last step, we have used (6.1), (5.10) and (6.14). The above estimate suggests that within the transition region the error bound may be written as

$$|u(x_j) - u_j| \le C(\gamma)N^{-\gamma}, \qquad K \le j \le J+1. \tag{6.15}$$

7 Error in boundary layer region

To obtain an error bound in the boundary layer we use inequality (5.8) over the restricted range $1 \le j \le K-1$. This inequality is conveniently re-written, in the notation of Section 5, as

$$(\mathcal{L}_\Delta e)(j) < \frac{C}{N(\lambda\epsilon^\lambda)}\left(1 + \frac{\lambda a h_j}{\epsilon}\right)\left(1 + \frac{\lambda a h_{j+1}}{\epsilon}\right)(\mathcal{L}_\Delta S)(j), \qquad 1 \le j \le K-1, \tag{7.1}$$

provided λ satisfies conditions (5.7). If (4.11) is employed in (7.1) it becomes

$$(\mathcal{L}_\Delta e)(j) < \frac{C_2(\gamma)}{N(\lambda\epsilon^\lambda)}(\mathcal{L}_\Delta S)(j), \qquad 1 \le j \le K-1. \tag{7.2}$$

A boundary condition on this set of inequalities is given by the results in the preceding section as

$$|e_K| = |u(x_K) - u_K| \leq C_1(\gamma)N^{-\gamma}, \tag{7.3}$$

where $C_1(\gamma)$ is a constant depending only on γ. Since $0 < \gamma < 1$, we may modify the constant on the right hand side of (7.2) to obtain

$$(\mathcal{L}_\Delta e)(j) < \frac{(1+\beta)C_3(\gamma)}{N^\gamma(\lambda\epsilon^\lambda)}(\mathcal{L}_\Delta S)(j), \qquad 1 \leq j \leq K-1, \tag{7.4}$$

where $C_3(\gamma) = \max\{C_1(\gamma), C_2(\gamma)\}$. Observe that

$$\begin{aligned} S_K &= \prod_{k=1}^{K} \frac{1}{1+\lambda a h_k/\epsilon} > \exp(-\lambda a \sum_{k=1}^{K} h_k/\epsilon) \\ &= \exp\left(-\frac{\lambda a x_K}{\epsilon}\right) \geq \exp\left(-\frac{\lambda p(0) x_K}{\epsilon}\right) > \frac{1}{(2N\beta)^\lambda}, \end{aligned}$$

where in the final step we have used (4.6). Moreover, since $0 < \lambda \leq 1/4$ and $N\epsilon < 1$, we have

$$\lambda(N\epsilon\beta)^\lambda < \beta^\lambda < 1+\beta.$$

A combination of the last two results gives

$$\frac{(1+\beta)C_3(\gamma)}{N^\gamma(\lambda\epsilon^\lambda)} S_K > \frac{(1+\beta)C_3(\gamma)}{2^\lambda \lambda N^\gamma (N\epsilon\beta)^\lambda} > \frac{C_3(\lambda)}{2^\lambda N^\gamma} \geq e_K, \tag{7.5}$$

where in the last step we have used (7.3). Since $e_0 = 0$, we also have

$$\frac{(1+\beta)C_3(\gamma)}{N^\gamma(\lambda\epsilon^\lambda)} S_0 > e_0. \tag{7.6}$$

Now, with (7.4), (7.5) and (7.6), an application of Lemma 5.3 gives

$$e_j < \frac{C(\gamma)}{N^\gamma(\lambda\epsilon^\lambda)} S_j, \qquad 0 \leq j \leq K,$$

where $C(\gamma) = (1+\beta)C_3(\gamma)$. Since a similar result holds for $-e_j$, we have

$$|e_j| < \frac{C(\gamma)}{N^\gamma(\lambda\epsilon^\lambda)} S_j, \qquad 0 \leq j \leq K. \tag{7.7}$$

Again, as in Section 5, we assume that $\epsilon = 10^{-m}$ and choose $\lambda = \frac{1}{m_0}$, then $\lambda\epsilon^\lambda = O(1)$, if we assume that m_0 is bounded above by a constant. This limitation implies that the convergence analysis holds for $0 < \epsilon_m \leq \epsilon < 1$, where ϵ_m can be smaller than machine accuracy. Since $S_j \leq 1$ for all j, the bound (7.7) may be written as

$$|e_j| \leq C(\gamma)N^{-\gamma}, \qquad 0 \leq j \leq K. \tag{7.8}$$

The error bounds given by (5.10), (6.15) and (7.8) may by combined to give the global result

$$|e_j| \leq C(\gamma)N^{-\gamma}, \qquad 0 \leq j \leq N. \tag{7.9}$$

This establishes the ϵ-uniform convergence result of Theorem 2.1.

8 Numerical experiments

To support the theoretical convergence analysis the approximate solution of (1.1)-(1.2) is obtained by solving (2.3) and (2.4) on a grid defined by (2.10). The grid was obtained by means of a numerical solution of (2.6) at the nodes (2.7), with u_x given by the exact solution (1.4). Table 1 shows the maximum pointwise error in the computed solution of (1.1)-(1.2), with $p(x) = 1/(1+x)$, and at various values of ϵ and N.

N	$\epsilon = 10^{-1}$	$\epsilon = 10^{-2}$	$\epsilon = 10^{-3}$	$\epsilon = 10^{-4}$
20	5.94×10^{-2}	1.26×10^{-1}	1.48×10^{-1}	1.55×10^{-1}
40	3.17×10^{-2}	7.37×10^{-2}	9.28×10^{-2}	9.87×10^{-2}
80	1.63×10^{-2}	3.98×10^{-2}	5.47×10^{-2}	5.92×10^{-2}
160	8.32×10^{-3}	2.06×10^{-2}	3.07×10^{-2}	3.39×10^{-2}
320	4.19×10^{-3}	1.05×10^{-2}	1.63×10^{-2}	1.89×10^{-2}

Table 1: L_∞ errors in solution of (2.3) and (2.4) on mesh given by (2.10) and (1.4), with $p(x) = 1/(1+x)$.

The results demonstrate an error behaviour that satisfies (7.9). The upwind difference scheme (2.1)-(2.2) produces approximate solutions that are ϵ-uniform convergent of order γ, where $\gamma \in (0.7, 1)$, as established by the theoretical analysis. For any of the values of ϵ specified in Table 1, the order γ increases within the range $(0.7, 1)$ as N increases. Also, for any fixed value of N, the value of γ increases as ϵ increases.

The distribution of the error is shown in Figure 1a for $N = 40$ and $\epsilon = 10^{-3}$. Note that approximately one half of the nodes are within the boundary layer, and that the maximum error occurs in the transition region at the outer edge of the boundary layer. The error behaviour, including the convergence properties as $N \to \infty$, is altered considerably if an element of smoothing is included in the equidistribution principle. The grid was recomputed using a discrete form of (2.5), with the arc-length monitor function $M = \sqrt{1 + \alpha u_x^2}$, $\alpha \geq 0$. The smoothing adopted was similar to that described in [11]. Since we are interested in a qualitative rather than a quantitative comparison, the parameter values in the smoothing are not of major interest: what is of interest is the change in the error behaviour when smoothing is included. Table 2 shows the maximum pointwise error in the computed solution of (1.1)-(1.2), with $p(x) = 1/(1+x)$, and at various values of ϵ and N. The grid employed in the computations that produced Table 2 was produced by an approximate solution of the equidistribution equation (2.5), with smoothing incorporated.

N	$\epsilon = 10^{-1}$	$\epsilon = 10^{-2}$	$\epsilon = 10^{-3}$	$\epsilon = 10^{-4}$
20	5.87×10^{-2}	1.19×10^{-1}	1.25×10^{-1}	1.29×10^{-1}
40	3.12×10^{-2}	6.31×10^{-2}	6.64×10^{-2}	6.73×10^{-2}
80	1.63×10^{-2}	3.23×10^{-2}	3.41×10^{-2}	3.48×10^{-2}
160	8.22×10^{-3}	1.65×10^{-2}	1.71×10^{-2}	1.76×10^{-2}
320	4.18×10^{-3}	8.26×10^{-3}	8.53×10^{-3}	8.61×10^{-3}

Table 2: L_∞ errors in solution of (2.3) and (2.4) on mesh given by a smoothed solution of (2.5), with $p(x) = 1/(1+x)$.

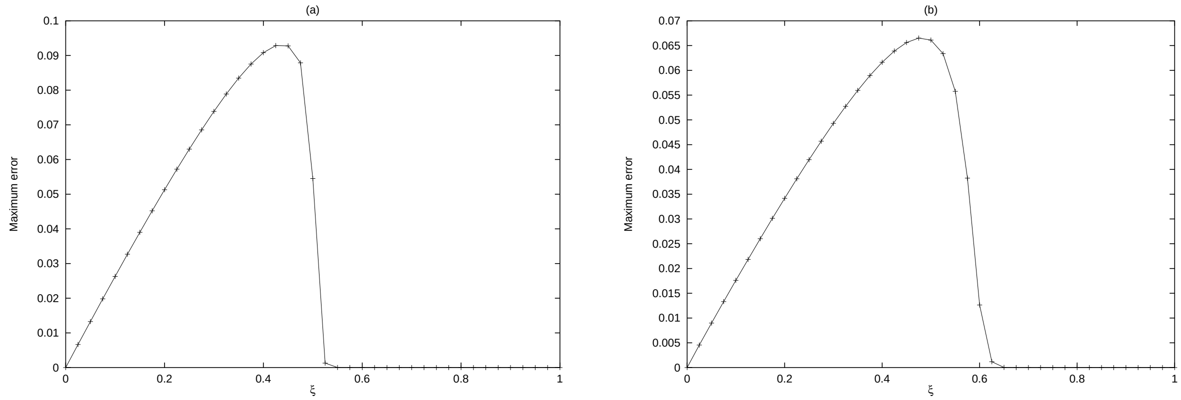

Figure 1: Distribution of the error in terms of the computational coordinate ξ, with $p(x) = 1/(1+x)$. In (a) the mesh is given by (2.10) and in (b) the mesh is given by a smoothed solution of (2.5).

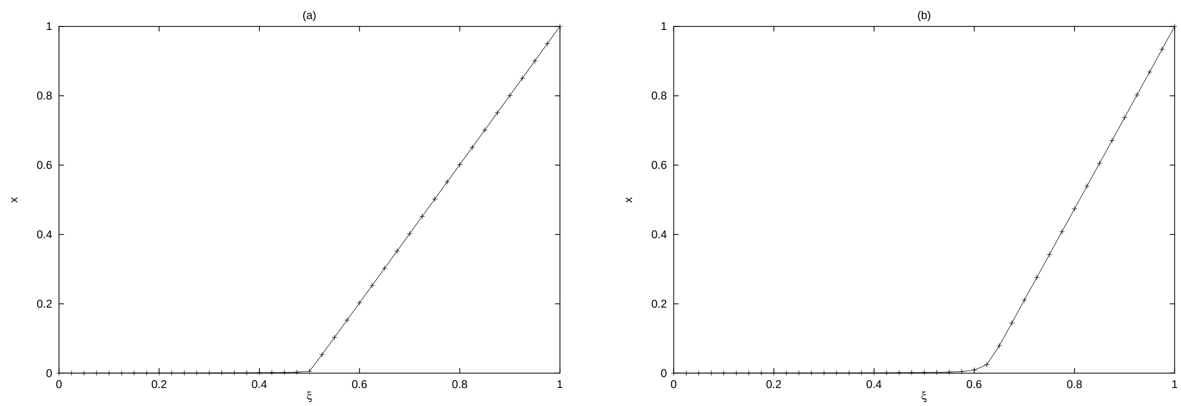

Figure 2: The map $x = x(\xi)$, with $p(x) = 1/(1 + x)$. (a) and (b) give the maps for the unsmoothed and smoothed grids, respectively.

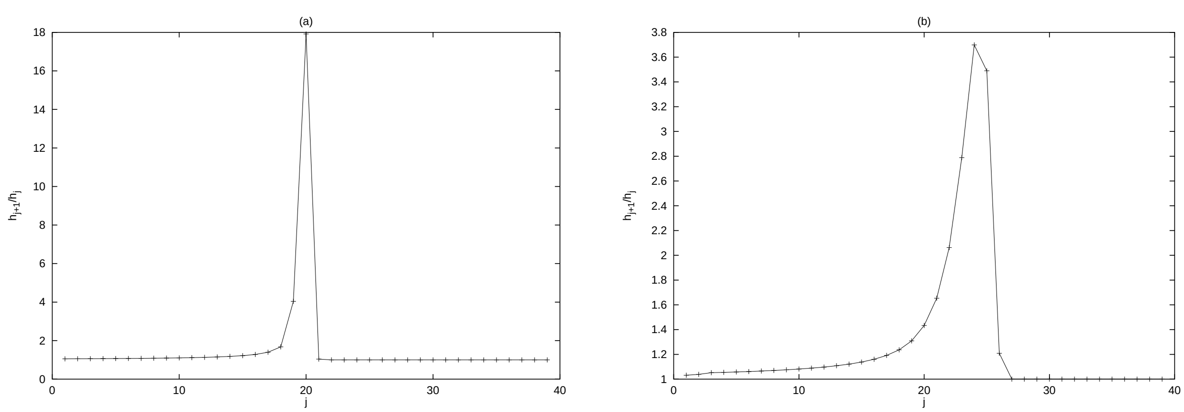

Figure 3: Distribution of the ratio h_{j+1}/h_j in terms of the integer j. (a) and (b) correspond to the unsmoothed and smoothed grids, respectively.

In the case described by Table 2, the approximate solutions are ϵ-uniformly convergent of order γ, where $\gamma \sim 1$. Figure 1b shows the error distribution on the smoothed grid with $N = 40$ and $\epsilon = 10^{-3}$. A comparison of Figures 1a and 1b shows that the smoothing and arc-length scaling have pulled more mesh points into the boundary layer and transition regions. Figures 2a and 2b show, respectively, the map $x = x(\xi)$ for the unsmoothed and smoothed grids. In each case we used the parameter values $N = 40$ and $\epsilon = 10^{-3}$. As might be expected, the inclusion of smoothing has the effect of producing a much smoother relation between x and ξ.

Further insight into the interplay between the grid and the error distribution is given in Figure 3. This display shows the ratio $r_j = h_{j+1}/h_j$ as a function of the node index, j, for the grids used in Figures 1 and 2. As before, (a) and (b) correspond to the unsmoothed and smoothed grids, respectively. Note that in the case of the unsmoothed grid the ratio is marginally larger than unity in the boundary layer, and extremely close to unity throughout the regular solution region. We may take $O(\epsilon/N)$ as a representative value of h_j in the boundary layer and $O(1/N)$ as a representation of the uniform value of h_j in the regular solution region. The value of r_j greatly exceeds unity in the narrow transition region within which the grid spacing changes from $O(\epsilon/N)$ to $O(1/N)$. If $r_j > 1$ for $K \leq j \leq J$ then, subject to the above spacing representations, $\prod_{j=K}^{J} r_j$ is $O(1/\epsilon)$ as $\epsilon \to 0$. Figure 3b shows that the maximum value of r_j is reduced considerably if smoothing is introduced, and this is accompanied by an enlargement of the width of the transition region. The monitor function has the effect of attracting nodes into the boundary layer, so the transition region is located at the outer edge of the layer where the two uniform grids have to be merged.

9 Concluding Remarks

The work presented in this paper is an extension of that presented in [13] to the case of an arc-length monitor function. The key result established here is that the discrete solutions computed on the adaptive grid are uniformly convergent with respect to the perturbation parameter.

It should be noted that the error analysis presented here is very much a first stage in the analysis of convergence of adaptive finite difference solutions of singular perturbation problems. Several extensions should be considered: for example, the general singular perturbation problems that were analysed by Kellogg and Tsan [8] become an obvious target. Fully discrete adaptive methods for simple model problems should be examined — even problems with constant coefficients. Problems with interior layers and higher order upwind scheme on adaptive grids offer alternative research challenges.

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