CHARACTERIZING WHICH POWERS OF HYPERCUBES AND FOLDED HYPERCUBES ARE DIVISOR GRAPHS

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Abstract

In this paper, we show that $Q_n^k$ is a divisor graph, for $n = 2$, 3. For $n \geq 4$, we show that $Q_n^k$ is a divisor graph iff $k \geq n - 1$. For folded-hypercube, we get $FQ_n$ is a divisor graph when $n$ is odd. But, if $n \geq 4$ is even integer, then $FQ_n$ is not a divisor graph. For $n \geq 5$, we show that $(FQ_n)^k$ is not a divisor graph, where $2 \leq k \leq \lceil \frac{n}{2} \rceil - 1$.

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1. Introduction

A graph $G$ is called a divisor graph if there is a bijection $f : V(G) \rightarrow S$, for some finite nonempty set $S$ of the positive integers, such that $uv \in E(G)$ if and only if $\gcd(f(u), f(v)) = \min\{f(u), f(v)\}$ (this means $uv \in E(G)$ iff $f(u) \mid f(v)$ or $f(v) \mid f(u)$). The function $f$ is called a divisor labeling of $G$. 
Moreover, for a finite nonempty set $S$ of the positive integers, the divisor graph $G(S)$ of $S$ has $S$ as its vertex set and two distinct vertices $i$ and $j$ are adjacent if $i \mid j$ or $j \mid i$. A graph $G$ is a divisor graph if $G$ is isomorphic to $G(S)$. While the divisor digraph $D(S)$ of $S$ has a vertex set $S$ and $(i, j)$ is an arc of $D(S)$ iff $i \mid j$. In a digraph $D$, a transmitter is a vertex having indegree 0, a receiver is a vertex having outdegree 0, while a vertex $v$ is a transitive vertex if it has both positive outdegree and positive indegree, and $(u, w) \in E(D)$ whenever $(u, v) \in E(D)$ and $(v, w) \in E(D)$. An orientation $D$ of a graph $G$ in which every vertex is a transmitter, a receiver, or a transitive vertex is called a divisor orientation of $G$.

The length $g(n)$ of a longest path in the divisor graph whose divisor labeling has range $\{1, 2, \ldots, n\}$ was studied in [8], [12], and [13]. The concept of a divisor graph involving finite nonempty sets of integers rather than positive integers was introduced in [14]. It was shown in [14] that odd cycles of length greater than three are not divisor graphs, while even cycles and caterpillars are. Indeed, not only caterpillars, but also all bipartite graphs are divisor graphs, as shown in [7]. Divisor graphs do not contain induced odd cycles of length greater than three, but they may contain triangles, see [7]. For instance, complete graphs are divisor graphs, see [7].

The distance between any two vertices $x$ and $y$ is the length of a shortest path between them. We denote this distance by $d_G(x, y)$. The diameter of a graph $G$ is equal to $\sup\{d_G(x, y) : x, y \in V(G)\}$, denoted by $d_G$ or $\text{diam}(G)$. The power graph of $G$ is denoted by $G^k$, where the vertex set of $G^k$ is $V(G)$ and two vertices $x$, $y$ are adjacent iff $d_G(x, y) \leq k$. A complete characterization of powers of paths, cycles and caterpillars that are divisor graphs were given in [1], [2], and [3].

More results on divisor graphs can be found in [4], [5], and [7]. For undefined notions and terminology, the reader is referred to [6].

2. When a Power Graph of a Hypercube is a Divisor Graph

In this section we characterize which powers of hypercubes are divisor graphs.

**Definition.** A hypercube $Q_n$ is a graph whose vertex set $V(Q_n)$ is the set of $n$-bit binary strings. (Any vertex of $Q_n$ may be labeled as $x = x_1x_2\cdots x_n$, where $x_1x_2\ldots x_n$ is the binary representation of $x$.) Two vertices $x = x_1x_2\cdots x_n$ and $y = y_1y_2\cdots y_n$ are adjacent if they differ in exactly one bit.

Alternatively, a hypercube can be defined recursively by $Q_n = Q_{n-1} \times K_2$, for $n \geq 2$ and $Q_1 = K_2$. In Figure 1, we have the graphs $Q_1$, $Q_2$, and $Q_3$.

It is well known that the diameter of $Q_n$ is $n$, hence $Q_n^k$ is a complete graph. Moreover, two vertices are adjacent in $Q_n^k$, if they differ in $k$ or less bits of their
binary strings.

The following important characterization of divisor graphs can be found in [7].

**Theorem 1.** Let $G$ be a graph. Then $G$ is a divisor graph if and only if $G$ has a divisor orientation. In particular, every bipartite graph is a divisor graph.

The following easy observation can be found in [7].

**Proposition 2.** Every induced subgraph of a divisor graph is a divisor graph.

We start with the following result for characterizing which powers of hypercubes are divisor graphs.

**Theorem 3.** Suppose that $Q_n$ is a hypercube. Then $Q_n^{n-1}$ is a divisor graph.

**Proof.** At first, two vertices in $Q_n^{n-1}$ are adjacent if they differ in $n - 1$ or less bits of their binary strings. Then according to the labeling of vertices in $Q_n$, each vertex of $Q_n^{n-1}$ is adjacent to every other vertex of $Q_n^{n-1}$ except its complement in labeling. Hence, we can partition the vertices in $Q_n$ into $\{a_i : 1 \leq i \leq 2^{n-1}, i \in \mathbb{N}\} \cup \{b_i : 1 \leq i \leq 2^{n-1}, i \in \mathbb{N}\}$ such that, for each $i$, $a_i$ is the complement of $b_i$. Now, a divisor labeling $f$ of $Q_n^{n-1}$ is defined as follows:

$$
\begin{align*}
  f(a_i) &= p^i q^{i-1} \quad \text{for } 1 \leq i \leq 2^{n-1}, i \in \mathbb{N}, \\
  f(b_i) &= p^{i-1} q^i \quad \text{for } 1 \leq i \leq 2^{n-1}, i \in \mathbb{N},
\end{align*}
$$

where $p$ and $q$ are distinct primes. Hence, $Q_n^{n-1}$ is a divisor graph.

**Remark 4.** This is an example of a graph for which both it and its complement graph ($Q_n^{n-1}$ being bipartite) are divisor graphs.

Note that $Q_n$ is a bipartite graph and hence a divisor graph. So, when $n = 2$ or 3, $Q_n^k$ is a divisor graph for any positive integer $k$. Now, we look at $n \geq 4$.

**Lemma 5.** $Q_4^2$ is not a divisor graph.

**Proof.** The induced subgraph on $\{0001, 0010, 0110, 1111, 1101\}$ in $Q_4^2$ is isomorphic to $C_5$. Hence, $Q_4^2$ is not a divisor graph by Proposition 2.
We use Theorem 3 and Lemma 5 to get the following corollary.

**Corollary 6.** $Q_k^4$ is not a divisor graph if and only if $k = 2$.

**Lemma 7.** $Q_5^3$ is not a divisor graph.

**Proof.** The induced subgraph on \{00001, 10000, 10110, 11111, 01011\} in $Q_5^3$ is isomorphic to $C_5$. Hence, $Q_5^3$ is not a divisor graph by Proposition 2.

By Theorem 3, Lemma 5, Proposition 2, and Lemma 7, we get the following corollary.

**Corollary 8.** $Q_k^5$ is not a divisor graph if and only if $k = 2$ or 3.

For any positive integer $n$ greater than three, we have the following lemma.

**Lemma 9.** For $n \geq 4$, $Q_n^{(n-2)}$ is not a divisor graph.

**Proof.** The result follows from Lemmas 5 and 7 when $n = 4$ and 5. For $n \geq 6$, we look at the induced subgraph, say $G_1$, on $S = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ in $Q_n^{(n-2)}$, where the $a_i$'s are:

$\begin{align*}
a_1 &= 0 \ldots 00, \\
a_2 &= 0 \ldots 011, \\
a_3 &= 0 \ldots 101, \\
a_4 &= 1 \ldots 100, \\
a_5 &= 1 \ldots 101, \\
a_6 &= 1 \ldots 11.
\end{align*}$

The graph $G_1$ is represented in Figure 2. Assume to the contrary that $G_1$ is a divisor graph. Hence, it has a divisor orientation, say $D$. Suppose that $(a_1, a_2) \in E(D)$. Since $a_1 a_5 \notin E(G_1)$, we must have $(a_5, a_2) \in E(D)$. We get $(a_3, a_2) \in E(D)$, because $a_3 a_5 \notin E(G_1)$. We must have $(a_1, a_4)$, $(a_3, a_6) \in E(D)$, because $a_4 a_2, a_6 a_2 \notin E(G_1)$.

Now, assume that $(a_4, a_6) \in E(D)$, and since $(a_1, a_4) \in E(D)$, we get $(a_1, a_6) \in E(D)$, which is a contradiction, since $a_1 a_6 \notin E(G_1)$. Otherwise, if $(a_6, a_4) \in E(D)$, and since $(a_3, a_6) \in E(D)$, then we get $(a_3, a_4) \in E(D)$. This leads to a contradiction, since $a_3 a_5 \notin E(G_1)$. Similarly, we argue the case $(a_2, a_1) \in E(D)$.
Theorem 10. If \( n \geq 5 \) and \( 2 \leq k \leq n - 3 \), then \( Q^k_n \) is not a divisor graph.

**Proof.** For any positive integer \( k \) with \( 2 \leq k \leq n - 3 \), we have the following nested induced subgraphs: \( Q^n_k \supseteq \cdots \supseteq Q^k_{k+3} \supseteq Q^k_{k+2} \). But, \( Q^k_{k+2} \) is not a divisor graph, by previous lemma. Hence, \( Q^k_n \) is not a divisor graph, by Proposition 2.

Using the previous theorem, Lemma 9, and Theorem 3, we get the following corollary.

**Corollary 11.** Suppose \( n \geq 4 \) and \( k \geq 2 \). Then \( Q^k_n \) is a divisor graph if and only if \( k \geq n - 1 \).

3. Characterizing when Powers of Folded-Hypercubes Are Divisor Graphs

In this section we characterize when powers of folded-hypercubes are divisor graphs. First we give the definition of the folded-hypercube.

**Definition.** A *folded-hypercube*, denoted by \( FQ_n \), is a graph whose vertex set is \( V(Q_n) \) and two vertices \( x \) and \( y \) are adjacent if \( d_{Q_n}(x, y) = 1 \) or \( n \), i.e., \( x \) and \( y \) differ in exactly one or \( n \) bits.

A folded-hypercube is a standard hypercube with some extra edges between the vertices with complementary binary strings. In addition, if \( n \) is odd then \( FQ_n \) is bipartite, see [9]. Hence, \( FQ_n \) is a divisor graph if \( n \) is odd or \( n = 2 \). While for \( n \geq 4 \) is even, we get the following lemma.

**Lemma 12.** If \( n \) is an even integer with \( n \geq 4 \), then \( FQ_n \) is not a divisor graph.

**Proof.** Consider the set \( A = \{a_1, a_2, \ldots, a_{n+1}\} \) in \( V(FQ_n) \), where

\[
\begin{align*}
a_1 &= 01 \underbrace{\ldots}_{n-1} 1, \\
a_2 &= 001 \underbrace{\ldots}_{n-2} 11,
\end{align*}
\]
The induced subgraph on \( A \) in \( FQ_n \) is isomorphic to \( C_{n+1} \), which is an odd cycle. Hence, \( FQ_n \) is not a divisor graph by Proposition 2.

We denote the complement of \( x \) by \( \bar{x} \). According to the definition of folded-hypercube, we get \( d_{FQ_n}(x, y) = \min\{d_{Q_n}(x, y), d_{Q_n}(x, \bar{y}) + 1\} \). This gives that \( (FQ_n)^{\lceil \frac{n}{2} \rceil} = K_{2^n} \), i.e., \( (FQ_n)^{\lceil \frac{n}{2} \rceil} \) is a divisor graph. Hence, for \( n = 2, 3, 4 \) and \( k \geq 2 \), we get \( (FQ_n)^k \) is a divisor graph. For \( n \geq 5 \), we get the following results.

**Lemma 13.** \( (FQ_n)^2 \) is not a divisor graph, for \( n \geq 5 \).

**Proof.** Consider the induced subgraph on the set \( S = \{a_1, a_2, a_3, a_4, a_5, a_6\} \) in \( (FQ_n)^2 \), where the \( a_i \)'s are:

\[
\begin{align*}
a_1 &= 0 \ldots 00000, \\
a_2 &= 0 \ldots 01111, \\
a_3 &= 0 \ldots 01100, \\
a_4 &= 0 \ldots 00110, \\
a_5 &= 0 \ldots 00110, \\
a_6 &= 0 \ldots 01100.
\end{align*}
\]

The induced subgraph on the set \( S \) in \( (FQ_5)^2 \) is given in Figure 3. This induced subgraph is isomorphic to \( G_1 \) and the result follows by Lemma 9.

For \( n \geq 6 \), the induced subgraph on the set \( S_1 = \{a_1, a_3, a_4, a_5, a_6\} \) in \( (FQ_n)^2 \) is isomorphic to \( C_5 \). The result follows by Proposition 2.

**Lemma 14.** \( (FQ_n)^3 \) is not a divisor graph, for \( n \geq 7 \).
Proof. Consider the induced subgraph on the set $S = \{a_1, a_2, a_3, a_4, a_5\}$ in $(FQ_n)^3$, where the $a_i$'s are:

$$
\begin{align*}
    a_1 &= 0_{\frac{n-7}{n-7}} \ldots 0100001, \\
    a_2 &= 0_{\frac{n-7}{n-7}} \ldots 0100001, \\
    a_3 &= 0_{\frac{n-7}{n-7}} \ldots 0100010, \\
    a_4 &= 0_{\frac{n-7}{n-7}} \ldots 0010010, \\
    a_5 &= 0_{\frac{n-7}{n-7}} \ldots 0101000.
\end{align*}
$$

For $n \geq 7$, the induced subgraph on the set $S$ in $(FQ_n)^3$ is isomorphic to $C_5$. The result follows by Proposition 2.

For $k = 4, 5$ with $n \geq 2k + 1$, we get that $(FQ_n)^k$ is not a divisor graph. The proof is similar to that of Lemma 13. For $k \geq 6$, we proceed as follows.

Lemma 15. Suppose $i = 3$ or $4$, $k = 2i$, $n_i = 4i + 1$, and $n \geq n_i$. Then $(FQ_n)^k$ is not a divisor graph.

Proof. Consider the induced subgraph on the set $S = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ in $(FQ_n)^k$, where the $a_i$'s are:

$$
\begin{align*}
    a_1 &= 0_{\frac{n-n_i}{n-n_i}} \ldots 010_{\frac{i}{i}} \ldots 001_{\frac{i}{i}} \ldots 110_{\frac{i-1}{i-1}} \ldots 00, \\
    a_2 &= 0_{\frac{n-n_i}{n-n_i}} \ldots 01_{\frac{i+2}{i+2}} \ldots 010_{\frac{i}{i}} \ldots 11_{\frac{2i-1}{2i-1}}, \\
    a_3 &= 0_{\frac{n-n_i}{n-n_i}} \ldots 01_{\frac{i-1}{i-1}}01_{\frac{i+1}{i+1}}01_{\frac{i-2}{i-2}}01_{\frac{i-3}{i-3}}01, \\
    a_4 &= 0_{\frac{n-n_i}{n-n_i}} \ldots 01_{\frac{i-2}{i-2}}00_{\frac{i-2}{i-2}}011_{\frac{2i-4}{2i-4}}01101_{\frac{i-2}{i-2}}10, \text{ for } i = 3,
\end{align*}
$$
Figure 3. This induced subgraph is isomorphic to \((FQ_n)^k\) isomorphic to the graph in Figure 3. This induced subgraph is isomorphic to \(G_1\) and the result follows by Lemma 9. For \(n \geq 1 + n_i\), the induced subgraph on the set \(S_1 = \{a_2, a_3, a_4, a_5, a_6\}\) in \((FQ_n)^k\) is isomorphic to \(C_5\). The result follows by Proposition 2.

**Lemma 16.** Suppose \(i = 3\) or 4, \(k = 2i + 1\), \(n_i = 4i + 3\), and \(n \geq n_i\). Then \((FQ_n)^k\) is not a divisor graph.

**Proof.** Consider the induced subgraph on the set \(S = \{a_1, a_2, a_3, a_4, a_5\}\) in \((FQ_n)^k\), where the \(a_i\)'s are:

\[
\begin{align*}
a_1 &= 0 \ldots 0 1 0 \ldots 0 1 0 \ldots 0 1 1 0 \ldots 1 1 0 \ldots 0 1 1, \\
a_2 &= 0 \ldots 0 1 0 \ldots 0 1 0 \ldots 0 1 1 0 \ldots 1 1 0 \ldots 1 1 0, \\
a_3 &= 0 \ldots 0 0 0 \ldots 0 1 0 \ldots 0 1 1 0 \ldots 0 1 1 0 \ldots 0 1, \\
a_4 &= 0 \ldots 0 0 0 1 \ldots 0 1 0 \ldots 0 0 1 0 \ldots 0 1 1, \\
a_5 &= 0 \ldots 0 0 0 0 0 \ldots 0 1 1 0 0 \ldots 1 1 0 0 \ldots 1 1 0 0 1.
\end{align*}
\]

For \(n \geq n_i\), the induced subgraph on the set \(S\) in \((FQ_n)^k\) is isomorphic to \(C_5\). The result follows by Proposition 2.

**Theorem 17.** Suppose \(i \geq 5\), \(k = 2i\), \(n_i = 4i + 1\), and \(n \geq n_i\). Then \((FQ_n)^k\) is not a divisor graph.

**Proof.** Consider the induced subgraph on the set \(S = \{a_1, a_2, a_3, a_4, a_5, a_6\}\) in \((FQ_n)^k\), where the \(a_i\)'s are:

\[
\begin{align*}
a_1 &= 0 \ldots 0 1 0 \ldots 0 1 0 \ldots 0 1 1 0 \ldots 1 1 0 \ldots 1 0 0 0 0 0 0 0, \\
a_2 &= 0 \ldots 0 1 1 \ldots 1 1 0 \ldots 0 1 1 0 \ldots 1 1 0 \ldots 1 1 0, \\
a_3 &= 0 \ldots 0 0 0 1 0 \ldots 0 0 1 0 \ldots 0 1 0 \ldots 0 1 1 0 0 \ldots 0 0 1, \\
a_4 &= 0 \ldots 0 0 0 0 0 1 0 \ldots 0 0 1 0 \ldots 0 1 1 0 \ldots 0 1 1 0 0 0 0 0 1, \\
a_5 &= 0 \ldots 0 0 0 0 0 0 0 0 0 1 1 1 1 0 \ldots 1 1 0 0 \ldots 1 1 0 0 1 1 0 0 1.
\end{align*}
\]
for $i$ odd,
\[
a_3 = 0 \ldots 000 \ldots 01 \ldots 111 \ldots 10 \ldots 10 \ldots 10 \ldots 0,\]
and for $i$ even,
\[
a_4 = 0 \ldots 010 \ldots 01 \ldots 110 \ldots 01 \ldots 10 \ldots 10 \ldots 0,\]

for $i$ odd,
\[
a_4 = 0 \ldots 010 \ldots 01 \ldots 110 \ldots 01 \ldots 10 \ldots 10 \ldots 001,\]
and for $i$ even,
\[
a_5 = 0 \ldots 000 \ldots 01 \ldots 110 \ldots 01 \ldots 10 \ldots 111 \ldots 100 \ldots 00,\]
\[
a_6 = 0 \ldots 000 \ldots 01 \ldots 10 \ldots 01 \ldots 10 \ldots 01 \ldots 10 \ldots 001.
\]

The induced subgraph on the set $S$ in $(FQ_n)^k$ is isomorphic to the graph in Figure 3. This induced subgraph is isomorphic to $G_1$ and the result follows by Lemma 9. For $n \geq 1 + n_i$, the induced subgraph on the set $S_1 = \{a_2, a_3, a_4, a_5, a_6\}$ in $(FQ_n)^k$ is isomorphic to $C_5$. The result follows by Proposition 2.

**Theorem 18.** Suppose $i \geq 5$, $k = 2i + 1$, $n_i = 4i + 3$, and $n \geq n_i$. Then $(FQ_n)^k$ is not a divisor graph.

**Proof.** Consider the induced subgraph on the set $S = \{a_1, a_2, a_3, a_4, a_5\}$ in $(FQ_n)^k$, where the $a_i$'s are:

For $i$ odd,
\[
a_3 = 0 \ldots 000 \ldots 01 \ldots 111 \ldots 10 \ldots 1 \ldots 0 \ldots 001,\]
and for $i$ even,
\[
a_4 = 0 \ldots 010 \ldots 01 \ldots 110 \ldots 01 \ldots 10 \ldots 10 \ldots 0,\]
\[
a_5 = 0 \ldots 000 \ldots 01 \ldots 110 \ldots 01 \ldots 10 \ldots 111 \ldots 100 \ldots 00,\]
\[
a_6 = 0 \ldots 000 \ldots 01 \ldots 10 \ldots 01 \ldots 10 \ldots 01 \ldots 10 \ldots 001.
\]
for $i$ even,

\[
\begin{align*}
  a_4 &= 0 \ldots 0 0 \ldots 0 \begin{array}{c}
  \lfloor \frac{i}{2} \rfloor \quad \lfloor \frac{i}{2} \rfloor + 1 \\
  \lfloor \frac{i}{2} \rfloor - 1
\end{array} 10 \ldots 10 \ldots 0 1 \ldots 10 \ldots 0 1 \ldots 10 \ldots 0 0 111, \\
  a_5 &= 0 \ldots 0 0 \ldots 0 \begin{array}{c}
  \lfloor \frac{i}{2} \rfloor \quad \lfloor \frac{i}{2} \rfloor \\
  \lfloor \frac{i}{2} \rfloor - 1 \quad \lfloor \frac{i}{2} \rfloor + 1 \quad \lfloor \frac{i}{2} \rfloor + 1
\end{array} 0 1 \ldots 0 1 \ldots 1 0 \ldots 1 0 \ldots 0 1 \ldots 10 \ldots 0 0 11.
\end{align*}
\]

For $n \geq n_i$, the induced subgraph on the set $S$ in $F(Q_n)^k$ is isomorphic to $C_5$. The result follows by Proposition 2.

Previous results characterize which powers of folded-hypercubes are divisor graphs. We collect these results in the following theorem.

**Theorem 19.** $(FQ_n)^k$ is not a divisor graph iff $2 \leq k \leq \left\lceil \frac{n}{2} \right\rceil - 1$, where $n \geq 5$.

**References**


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