# Changing sign solutions for the CR-Yamabe equation

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**Abstract** In this paper we prove that the CR-Yamabe equation on the Heisenberg group has infinitely many changing sign solutions. By mean of the Cayley transform we will set the problem on the sphere $S^{2n+1}$; since the functional $I$ associated with the equation does not satisfies the Palais-Smale compactness condition, we will find a suitable closed subspace $X$ on which we can apply the minmax argument for $I_{|X}$. We generalize the result to any compact contact manifold of $K$-contact type.

## 1 Introduction

In this paper we prove that the CR-Yamabe equation on the Heisenberg group $\mathbb{H}^n$

$$-\Delta_{\mathbb{H}} u = |u|^{\frac{4}{q-2}} u, \qquad u \in S_0^1(\mathbb{H}^n) \tag{1}$$

has infinitely many changing sign solutions. Here $\Delta_{\mathbb{H}}$ denotes the sub-Laplacian of the group, $q = 2n + 2$ is the homogenous dimension of $\mathbb{H}^n$, and $S_0^1(\mathbb{H}^n)$ is the Folland-Stein Sobolev type space on $\mathbb{H}^n$.

We recall that for the positive solutions of (1), Jerison and Lee in [12] gave a complete classification. The problem is variational but, as in the Riemannian case, the functional associated with the equation (1) fails to satisfy the Palais-Smale compactness condition.

For the classical Yamabe equation on $\mathbb{R}^n$, the first result in this direction

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was proved by Ding in [8]: following the analysis by Ambrosetti and Rabinowitz [1], he found a suitable subspace $X$ of the space of the variations for the related functional, on which he performed the minmax argument. The same argument was then used by Saintier in [14] for the Yamabe equation on $\mathbb{R}^n$ involving the bi-Laplacian operator.
Later on, many authors proved the existence of infinitely many changing sign solutions using other kinds of variational methods (see [2], [3] and the references therein). Finally in a couple of recent works [6], [7], M. del Pino, M.Musso, F.Pacard and A.Pistoia found changing sign solutions, different from those of Ding, by using a superposition of positive and negative *bubbles* arranged on some special sets.
We are going to use the approach of Ding. Using the Cayley transform we will set the problem on the sphere $S^{2n+1}$ and with the help of the group of the isometries generated by the Reeb vector field of the standard contact form on $S^{2n+1}$, we will be able to exhibit a suitable closed subspace on which we can apply the minmax argument for the restriction of the functional associated with the equation (1). At the end we will generalize the result to any compact contact manifold of $K$-contact type.

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## 2 Preliminaries

Let $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R} \simeq \mathbb{R}^{2n+1}$ be the Heisenberg group. If we denote by $\xi = (z,t) = (x+iy,t) \simeq (x,y,t) \in (\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R})$ then the group law is given by

$$\xi_0 \centerdot \xi = (x+x_0, y+y_0, t+t_0+2(x\cdot y_0 - x_0 \cdot y)),\ \forall \xi, \xi_0 \in \mathbb{H}^n$$

where $\cdot$ denotes the inner product in $\mathbb{R}^n$. The left translations are defined by

$$\tau_{\xi_0}(\xi) := \xi_0 \centerdot \xi$$

Finally the dilations of the group are

$$\delta_\lambda : \mathbb{H}^n \to \mathbb{H}^n, \qquad \delta_\lambda(\xi) = (\lambda x, \lambda y, \lambda^2 t)$$

for any $\lambda > 0$. Moreover we will denote by $q = 2n + 2$ the homogeneous dimension of the group. The canonical left-invariant vector fields are

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad j = 1, \dots, n$$

The horizontal (or intrinsic) gradient of the group is

$$D_H = (X_1, \dots, X_n, Y_1, \dots, Y_n)$$

The Kohn Laplacian (or sub-Laplacian) on $\mathbb{H}^n$ is the following second order operator invariant with respect to the left translations and homogeneous of degree two with respect to the dilations:

$$\Delta_{\mathbb{H}} = \sum_{j=1}^{n} X_j^2 + Y_j^2$$

Let us now consider the following Yamabe type problem on the Heisenberg group $\mathbb{H}^n$

$$-\Delta_{\mathbb{H}} u = |u|^{\frac{4}{q-2}} u, \qquad u \in S_0^1(\mathbb{H}^n) \tag{2}$$

where $S_0^1(\mathbb{H}^n)$ is the Folland-Stein Sobolev space on $\mathbb{H}^n$ (see [9]).
We will prove that the equation (2) has infinitely many changing sign solutions.
We will follow the scheme of Ding [8]. The idea is to consider the problem, after the Cayley transform, on the sphere $S^{2n+1}$ and there we will be able to find a suitable closed subspace $X \subseteq S^1(S^{2n+1})$, compactly embedded in $L^{q^*}(S^{2n+1})$, on which we can apply the minmax argument for the functional associated with the equation.
Here $q^* = \frac{2q}{q-2}$ denotes the critical exponent for the Sobolev embedding. We recall that a solution of the problem (2) on $\mathbb{H}^n$ can be found as critical point of the following functional

$$J : S_0^1(\mathbb{H}^n) \to \mathbb{R}, \qquad J(u) = \frac{1}{2} \int_{\mathbb{H}^n} |D_H u|^2 - \frac{1}{q^*} \int_{\mathbb{H}^n} |u|^{q^*}$$

Moreover any variational solution is actually a classical solution ([9], [10]).
We will prove the following

**Theorem 2.1.** *There exists a sequence of solutions $\{u_k\}$ of (2), with*

$$\int_{\mathbb{H}^n} |D_{\mathbb{H}} u_k|^2 \longrightarrow \infty, \qquad as \quad k \to \infty$$

The Theorem (2.1) will imply then that equation (2) has infinitely many changing sign solutions: in fact by a classification result by Jerison and Lee [12] all the positive solutions of the equation (2) are in the form

$$u = \omega_{\lambda,\xi} = \lambda^{\frac{2-q}{2}} \omega \circ \delta_{\frac{1}{\lambda}} \circ \tau_{\xi^{-1}}$$

for some $\lambda > 0$ and $\xi \in \mathbb{H}^n$, where

$$\omega(x, y, t) = \frac{c_0}{\left((1 + |x|^2 + |y|^2)^2 + t^2\right)^{\frac{q-2}{4}}}$$

with $c_0$ a positive constant; in particular all the solutions $\omega_{\lambda,\xi}$ have the same energy.

# 3 Proof of the Theorem (2.1)

Let us consider the sphere $S^{2n+1} \subseteq \mathbb{C}^{n+1}$ defined by

$$S^{2n+1} = \{(z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} \ s.t. \quad |z_1|^2 + \ldots + |z_{n+1}|^2 = 1\}$$

As in the Riemannian case, we will use an analogous of the stereographic projection. The Cayley transform is the CR-diffeomorphism between the sphere minus a point and the Heisenberg group

$$F : S^{2n+1} \setminus \{0, \ldots, 0, -1\} \to \mathbb{H}^n$$

$$F(z_1, \ldots, z_{n+1}) = \Big(\frac{z_1}{1 + z_{n+1}}, \ldots, \frac{z_n}{1 + z_{n+1}}, Re\big(i\frac{1 - z_{n+1}}{1 + z_{n+1}}\big)\Big)$$

Denoting by $\theta_0$ the standard contact form on $S^{2n+1}$ and by $\Delta_{\theta_0}$ the related sub-Laplacian, a direct computation show that equation (2) becomes

$$-\Delta_{\theta_0} v + c(n)v = |v|^{\frac{4}{q-2}} v, \qquad v \in S^1(S^{2n+1}) \tag{3}$$

with $c(n)$ a suitable positive constant related to the (constant) Webster curvature of the sphere (see [11] for a full detailed exposition); in particular by setting $u = v\varphi$ (where $\varphi$ is the function that gives the conformal factor in the change of the contact form) we have that every solution $u$ of (2) corresponds to a solution $v$ of (3) and it holds

$$\int_{\mathbb{H}^n} |D_H u|^2 = \int_{S^{2n+1}} |v|^{q^*}$$

At this point we can consider the variational problem on the sphere

$$I : S^1(S^{2n+1}) \to \mathbb{R}, \qquad I(v) = \frac{1}{2}\int_{S^{2n+1}} \left(|D_H u|^2 + c(n)v^2\right) - \frac{1}{q^*}\int_{S^{2n+1}} |v|^{q^*}$$

Since the embedding

$$S^1(S^{2n+1}) \hookrightarrow L^{q^*}(S^{2n+1})$$

is not compact, the functional $I$ does not satisfy the Palais-Smale condition. The following lemma by Ambrosetti and Rabinowitz gives a condition on some particular subspaces of the space of variations on which it is allowed to perform the minmax argument; we will omit the proof (see Theorems 3.13 and 3.14 in [1])

**Lemma 3.1.** *Let $X$ be a closed subspace of $S^1(S^{2n+1})$. Suppose that the embedding $X \hookrightarrow L^{q^*}(S^{2n+1})$ is compact. Then $I_{|X}$, the restriction of $I$ on $X$, satisfies the Palais-Smale condition. Furthermore, if $X$ is infinite-dimensional, then $I_{|X}$ has a sequence of critical points $\{v_k\}$ in $X$, such that*

$$\int_{S^{2n+1}} |v_k|^{q^*} \longrightarrow \infty, \qquad as \quad k \to \infty$$

Now let us suppose that we can find a closed and compact group $G$ such that the functional $I$ is invariant under the action of $G$, namely:

$$I(v) = I(v \circ g), \qquad \forall\, g \in G$$

Let us set

$$X_G = \{v \in S^1(S^{2n+1}) \ s.t. \quad v = v \circ g,\ \forall\, g \in G\}$$

Then if $X_G$ satisfies the condition of Lemma (3.1), by the Principle of Symmetric Criticality [13], any critical point of the restriction $I_{|X_G}$ is also a critical point of $I$ on the whole space of variations. We are going to prove that such a $X_G$ exists. First we observe the following fact:

**Lemma 3.2.** *The Reeb vector field $T$ related to the standard contact form $\theta_0$ on $S^{2n+1}$ is a Killing vector field.*

*Proof.* The proof is straightforward. The vector field $T$ is Killing if

$$g_0(\nabla_V T, W) + g_0(V, \nabla_W T) = 0, \qquad \forall \quad V, W \in T(S^{2n+1}) \tag{4}$$

where $g_0$ is the metric induced by $\mathbb{C}^{n+1}$ and $\nabla$ is the Levi-Civita connection (we will call $g_0$ and $\nabla$ also the standard metric and the related Levi-Civita connection in $\mathbb{C}^{n+1}$). Since we are on the sphere we can consider an orthonormal basis on $T(S^{2n+1})$ of eigenvectors of the Weingarten operator $A$ (we recall that for every $V \in T(S^{2n+1})$, then $A(V) := -\nabla_V N$) with eigenvalues all equal to 1:

$$E = \{X_1, \dots, X_n, Y_1, \dots, Y_n, T\}, \qquad Y_j = JX_j, \quad j = 1, \dots, n$$

where $J$ is the standard complex structure on $\mathbb{C}^{n+1}$. Moreover $T = JN$, with $N$ the unit inner normal to $S^{2n+1}$. We have that

$$g_0(\cdot, \cdot) = g_0(J\cdot, J\cdot), \qquad J\nabla_{(\cdot)}\cdot = \nabla_{(\cdot)}J\cdot$$

We also recall that $T$ is a geodesic vector field, namely

$$g_0(\nabla_T T, V) = 0, \qquad \forall \quad V \in T(S^{2n+1})$$

Since the formula (4) is linear in $V$ and $W$, we can check it on the basis $E$. Let us first fix $W = T$. Then $g_0(\nabla_V T, T) = 0$ since $T$ is unitary, and $g_0(V, \nabla_T T) = 0$ since $T$ is geodesic. Now suppose $(V, W) = (X_j, X_k)$ or $(V, W) = (Y_j, Y_k)$. Then

$$g_0(\nabla_{X_j} T, X_k) = -g_0(\nabla_{X_j} N, Y_k) = 0$$

$$g_0(\nabla_{Y_j} T, Y_k) = g_0(\nabla_{Y_j} N, X_k) = 0$$

as $X_j$ and $Y_j$ are eigenvectors for $A$. The same holds for $(V, W) = (X_j, Y_k)$ with $j \neq k$. Finally let us consider $(V, W) = (X_j, Y_j)$. Then we have

$$g_0(\nabla_{X_j} T, Y_j) + g_0(X_j, \nabla_{Y_j} T) = g_0(\nabla_{X_j} N, X_j) - g_0(Y_j, \nabla_{Y_j} N) = 0$$

□

Then $T$ generates a one-parameter family of diffeomorphisms, namely a closed group $G$, and since $T$ is Killing these diffeomorphisms are isometries. Moreover on $S^{2n+1}$ we have the following crucial property of commutation

$$T\,\Delta_{\theta_0} = \Delta_{\theta_0}\,T \tag{5}$$

As we will see in the next section the sphere $(S^{2n+1}, \theta_0)$ is a particular case of $K$-contact manifold according the definition in [4]. The formula (5) is then true for every $K$-contact manifold (see Lemma 4.3. in[16]). In particular we can conclude that the the functional $I$ is invariant under the action of

$G$. Moreover $G$ is a closed subgroup of the compact Lie group $O(2n+2)$ of the isometries on $S^{2n+1}$, in particular $G$ is compact. Now we will call basic a function that belongs to the following set:

$$\{v \in S^1(S^{2n+1}) \; s.t. \quad Tv = 0\}$$

In other words, a basic function is a function on $S^{2n+1}$ invariant for $T$. There is an explicit way to characterize the basic functions on $S^{2n+1}$: since the orbits of $T$ are circles, we can consider the following Hopf fibration:

$$S^1 \hookrightarrow S^{2n+1} \xrightarrow{p} \mathbb{C}P^n$$

where the fibers are exactly the orbits of $T$. Therefore a function $v$ is basic if and only if can be written as

$$v = w \circ p$$

for some function $w : \mathbb{C}P^n \to \mathbb{R}$. Let us consider now the subspace of functions $X_G \subseteq S^1(S^{2n+1})$. We have

**Lemma 3.3.** *The embedding*

$$X_G \hookrightarrow L^{q^*}(S^{2n+1})$$

*is compact.*

*Proof.* The proof is based on the following simple observation. First note that the *Riemannian* critical exponent

$$p^* = \frac{2m}{m-2}, \qquad m = 2n+1$$

is always greater than the *sub-Riemannian* critical exponent $q^*$. In addition for every basic function, the horizontal gradient (and the sub-Laplacian as well) coincides with the usual Riemannian gradient (the Laplacian, respectively). Therefore, denoting by $H^1(S^{2n+1})$ the usual Sobolev space on $S^{2n+1}$, we have the following chain of embeddings

$$X_G \hookrightarrow H^1(S^{2n+1}) \hookrightarrow L^{q^*}(S^{2n+1})$$

and the second one is compact since $q^*$ is subcritical for the (Riemannian) Sobolev embedding. □

**Remark 3.4.** *We explicitly note that our group is different from that used by Ding in [8]. Moreover he needed to prove the compact embedding of the subspace $X_G$ in a direct way; in our sub-Riemannian setting and with our choice of the group $G$ we can "lower" the dimension and switch from one critical exponent to another.*

## 4 The case of $K$-contact manifolds

Let us consider the case of the Yamabe type equation on a general metric contact manifold with the additional condition of being $K$-contact. Here we give the basic definitions and properties, for a comprehensive presentation concerning the subject we refer the reader to [4]. Let $(M^{2n+1}, \theta)$ be a compact contact manifold and let $\xi$ the Reeb vector field related to $\theta$. A Riemannian metric $g$ associated with the contact structure defined by $\theta$ is a metric with the following properties:

*i)* $\theta(X) = g(X, \xi), \qquad \forall\, X \in TM$
*ii)* there exists a tensor field $J$ of type $(1,1)$ such that
$J^2 = -I + \theta \otimes \xi, \qquad d\theta(\cdot,\cdot) = g(\cdot, J\cdot)$

A contact manifold $M$ with an associated metric is said *contact metric manifold*. The sub-Laplacian $\Delta_\theta$ with respect to $\theta$ is defined by the following relation:

$$\Delta_g = \Delta_\theta + \xi^2$$

where $\Delta_g$ is the standard Laplacian with respect to the metric $g$. The contact Yamabe type equation is given by (see [15])

$$-\Delta_\theta u + c(n) S_\theta u = |u|^{\frac{4}{q-2}} u, \qquad q = 2n+2, \quad u \in S^1(M) \tag{6}$$

where $S_\theta$ is the Webster scalar curvature of $M$ with respect to $\theta$, $c(n)$ is a suitable dimensional constant and $S^1(M)$ is the Follan-Stein Sobolev type space defined on $M$. We want to show that equation (6) has infinitely many changing sign solutions. We need the following definition:

**Definition 4.1.** *Let $M$ a contact metric manifold. If the Reeb vector field $\xi$ is a Killing vector field then $M$ is said of $K$-contact type.*

We have then:

**Theorem 4.2.** *Let $M$ be a compact contact metric manifold of $K$-contact type. Then there exists a sequence of solutions $\{u_k\}$ of (6), with*

$$\int_M |u_k|^{q^*} \longrightarrow \infty, \qquad as \quad k \to \infty$$

The scheme of the proof is the same of that in the case of the sphere $S^{2n+1}$. Now we observe that on a general manifold there is no classification of positive solutions of the Yamabe type equation. However we can prove the following fact

**Proposition 4.3.** *Let $(M^m, g)$ be a compact Riemannian manifold. There exists a positive constant $C$ such that for every smooth positive solution $u$ of*

$$-\Delta_g u + hu = u^{p-1}, \qquad 2 < p < p^* = \frac{2m}{m-2} \tag{7}$$

*with $h$ any smooth bounded function on $M$, we have*

$$\|u\|_{L^p(M)} < C.$$

*Proof.* We will prove that

$$\|u\|_{L^\infty(M)} < C$$

uniformly. Then since $M$ is compact we have $L^\infty(M) \subseteq L^p(M)$. We argue by contradiction using a blow-up argument. Suppose that there exists a sequence of functions $\{u_k\}$ that are solutions of (7) and that blow-up as $k \to \infty$. Let us set

$$\mu_k := u_k(x_k) = \max_M u_k, \qquad \text{for some} \quad x_k \in M$$

We can use local normal coordinates centered at $x_k$, then we define

$$v_k(x) = \mu_k^{-1} u_k(\mu_k^{\frac{2-p}{2}} x)$$

We have $v_k(0) = 1$ for every $k$. Moreover equation (7) becomes

$$-\Delta_g v_k + \widetilde{h}\mu_k^{2-p} v_k = v_k^{p-1} \tag{8}$$

with $\widetilde{h}(x) = h(\mu_k^{\frac{2-p}{2}} x)$. Let $v$ be the limit of $v_k$ as $k \to \infty$. Then $v$ solves

$$-\Delta v = v^{p-1}, \qquad \text{on} \quad \mathbb{R}^m$$

By the classification result of Caffarelli, Gidas and Spruck [5], any nonnegative entire solution of the Yamabe equation with subcritical exponent must be zero. This is a contradiction since $v(0) = 1$. □

Now Let us consider the sequence $\{u_k\}$ that we found in Theorem (4.2). We recall that they are basic functions, in particular they solve a Yamabe type equation on $M$ with subcritical exponent since $q^* < p^*$. By Proposition (4.3) we get that there exist infinitely many changing sign solutions to (6).

## References


[1] A. Ambrosetti, P.H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Functional Analysis 14 (1973), 349-381.

[2] A. Bahri, S. Chanillo, *The difference of topology at infinity in changing-sign Yamabe problems on $S^3$S3 (the case of two masses)*, Comm. Pure Appl. Math. 54 (2001), no. 4, 450-478.

[3] A. Bahri, Y. Xu, *Recent progress in conformal geometry*, ICP Advanced Texts in Mathematics, 1. Imperial College Press, London, 2007.

[4] D.E. Blair, *Riemannian geometry of contact and symplectic manifolds. Second edition*, Progress in Mathematics, 203. Birkhäuser Boston, Inc., Boston, MA, 2010.

[5] L.A. Caffarelli, B. Gidas, J. Spruck, *Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth*, Comm. Pure Appl. Math. 42 (1989), no. 3, 271-297.

[6] M. del Pino, M. Musso, F. Pacard, A. Pistoia, *Torus action on $S^n$ and sign changing solutions for conformally invariant equations*, preprint.

[7] M. del Pino, M. Musso, F. Pacard, A. Pistoia, *Large energy entire solutions for the Yamabe equation*, to appear on Journal of Differential Equations.

[8] W.Y. Ding, *On a conformally invariant elliptic equation on $\mathbb{R}^n$*, Comm. Math. Phys. 107 (1986), no. 2, 331-335.

[9] G.B. Folland, E.M. Stein, *Estimates for the $\bar{\partial}_b$ complex and analysis on the Heisenberg group*, Comm. Pure Appl. Math. 27 (1974), 429-522.

[10] N. Garofalo, E. Lanconelli, *Existence and nonexistence results for semilinear equations on the Heisenberg group*, Indiana Univ. Math. J. 41 (1992), no. 1, 71-98.

[11] D. Jerison, J.M. Lee, *The Yamabe problem on CR manifolds*, J. Differential Geom. 25 (1987), no. 2, 167-197.

[12] D. Jerison, J.M. Lee, *Extremals for the Sobolev inequality on the Heisenberg group and the CR Yamabe problem*, J. Amer. Math. Soc. 1 (1988), no. 1, 1-13.

[13] R.S. Palais, *The principle of symmetric criticality*, Comm. Math. Phys. 69 (1979), no. 1, 19-30.

[14] N. Saintier, *Changing sign solutions of a conformally invariant fourth-order equation in the Euclidean space*, Comm. Anal. Geom. 14 (2006), no. 4, 613624.

[15] S. Tanno, *Variational problems on contact Riemannian manifolds*, Trans. Amer. Math. Soc. 314 (1989), no. 1, 349-379.

[16] Y.B. Zhang, *The contact Yamabe flow on K-contact manifolds*, Sci. China Ser. A 52 (2009), no. 8, 1723-1732.