Stationary and transient solution of Markovian queues – an alternate approach

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Abstract: This paper aims at presenting an alternate approach to derive the exact transient solution of Markovian queues. The continued fraction (Jacobi fraction) is expressed as power series and the power series coefficients are connected by a two-dimensional recurrence relation associated with infinite Stieltjes matrix equation. The recurrence relation is solved by using generating functions. A novel state-dependent birth-death queueing model is taken and expressed as continued fractions by employing integral transforms. The stationary probabilities of general state-dependent Markovian queues are obtained from the continued fractions and its associated tridiagonal determinants. As a special case the time-dependent system size probabilities and busy period distribution of a classical single server queue are deduced using continued fraction and its power series. Numerical illustrations are also presented.

Keywords: continued fractions; CFs; two-dimensional recurrence relation; generating functions; power series; time-dependent system-size probabilities; stationary probabilities; numerical solution.


Biographical notes: R. Sudhesh is an Associate Professor of Mathematics, University College of Engineering Villupuram, Anna University Chennai. He received his PhD from Indian Institute of Technology Madras. He has published 20 papers relating to queueing and birth and death process in leading international journals and conferences. Currently, he is working in the area of queueing models for communication systems.
1 Introduction

Birth and death processes are powerful tools to describe stochastic models arising in population dynamics. It is used in various applied fields, such as adaptive queueing systems, neurophysiology and so on. In particular, transition probabilities and first-passage-time densities of birth-death process play a relevant role in many applied contexts. Hence, obtaining closed form expressions for these functions is an important task. Queueing theory provides a large number of alternative mathematical models for describing such waiting line situations. It provides the analyst with a powerful tool for designing and evaluating the performance of queueing systems. Discrete-time queueing model presented by Jain and Srinivasa Raghavan (2008) is used for evaluating the performance of a production-inventory system. The periods of random lengths of preventive-maintenance may be considered as periods of server vacation when the server is unavailable (Sengupta, 1990). Using quasi birth-death process and matrix-geometric solution method, Xu et al. (2009) have obtained the distributions for the stationary queue length and waiting time of a customer in the system of a single server queue with single working vacation and setup times. Single server retrial queueing models involving batch arrivals that are subject to server breakdown has been analysed for the reliability of the server and other related performance measures (Senthilkumar and Arumuganathan, 2010). Also, bulk queues with multiple types of server breakdown under $N$-policy were studied by Jain and Agarwal (2009) using matrix-geometric methods.

The continued fractions (CFs) and its variants have been studied by many distinguished mathematicians. The CFs play a fundamental role in many investigations due to its applications to diverse fields like number theory, special functions, approximations, moment problems, digital networks, statistics and signal processing. Approximations employing CFs often provide a good representation for transcendental functions and CFs arise naturally in calculations of random walk probabilities. Their importance has grown further with the advent of fast computing facilities.

Various CFs’ applications in statistics are described in Bowman and Shenton (1989) and its usage in numerical computation (see Baker and Graves-Morris, 1996; Jones and Thron, 1980). There is a close connection between birth-death queueing models and CFs through three-term recurrence relations. This association has been exploited in order to find the time-dependent solutions of certain Markovian systems in closed form with non-linear arrival and service rates (e.g., Parthasarathy and Lenin, 2004). Parthasarathy and Sudhesh (2006a) have used CF and its power series to analytically derive the queue size distribution of the state-dependent birth and death processes. Sudhesh (2010) has considered single server queue with system disasters and customer impatience and has obtained the transient solution analytically in closed form by employing an effective new CF methodology.
2 Background for the research

In the study of CFs, there arises two main problems in a rather natural way. First is the problem of obtaining a CF expansion making use of the coefficients of a given power series and the second is the converse of the first, that is, to evaluate the power series coefficients from a known CF expansion.

Converting a power series into CF is well-known. The coefficients of a CF can be determined from the coefficients of a given power series through quotient of Hankel determinants (see Vein and Dale, 1999). Jones and Thron (1980) describe a quotient-difference algorithm (QD algorithm) for computing the coefficients of CFs corresponding to a given power series. The J-fraction corresponding to a power series has been obtained from an addition formula by means of a decomposition [see Goulden and Jackson, (1983), p.295]. Euler’s connection describes an equivalence between a T-fraction and a power series (e.g., Gill, 1999). The problem of converting a CF into a power series has also been studied by many distinguished mathematicians and number of partial results are also known [see Rogers, 1907; Wall, 1948; Zajta and Pandikow, 1975; Flajolet, 1980; Goulden and Jackson, 1983; Berndt, (Entry 17, 1989)].

In the literature, analytical results for the transient analysis of queuing models have received less attention rather than easily obtainable steady state results. The assumptions required to derive the steady state solutions for queueing systems are seldom satisfied in the design and analysis of real systems. The steady state results are inappropriate in situations wherein the time horizon of operations is finite. Time-dependent analysis helps us to understand the behaviour of a system when the parameters involved are perturbed. The transient analysis is mainly based on a generating function technique. However, an alternative technique is used to derive explicit expressions in terms of power series for the transient state distribution of an infinite systems (see Tarabia and El-Baz, 2007; Tarabia et al., 2009) and for the numerical approach (see Hooghiemstra et al., 1988; Blanc, 1992).

In this paper, the CF is first expressed as a power series using the results established by Wall (1948). He has presented an infinite Stieltjes matrix equation to obtain the coefficients of the power series expansion of a J-fraction. The infinite Stieltjes matrix equation gives a two-dimensional recurrence relation which can be solved analytically using generating functions. Hence, the power series coefficients are obtained in closed form.

A birth-death queue is considered with general state-dependent arrival and service rates. In this analysis, the underlying forward Kolmogorov differential-difference equations are first transformed into a set of linear algebraic equations by employing Laplace transforms which in turn leads to a J-fraction. The steady state probabilities of finite capacity state-dependent queueing models are obtained using its CFs and its associated orthogonal polynomials. From this, the steady state probabilities of an infinite capacity state-dependent queue is deduced. Further, using this CFs and its power series, the transient system size probabilities and busy period distribution of M/M/1 queue are deduced in closed form. Also the numerical results are discussed through tables.
3 CF and its power series

In this section, we focus on the connection between CF and its power series coefficients and obtaining power series coefficients in closed form. It has been proved by Wall (1948) that, in general, a power series in \( x \), \( a_0 + a_1x + a_2x^2 + \ldots \), can be represented as a CF of the form

\[
\frac{1}{b_0 + z - b_1 + z - b_2 + z - b_3 + z - \ldots}
\]

then it can be expressed as a power series of the form

\[
P\left(\frac{1}{z}\right) = \sum_{r=0}^{\infty} \frac{(-1)^r c_r}{z^{r+1}}
\]

where the notation for CF is used as

\[
\frac{a_0}{b_0 - b_1} \frac{a_1}{b_1 - b_2} \frac{a_2}{b_2 - b_3} \ldots
\]

The coefficients of the CF can be connected by the relations

\[
c_{r+q} = k_{0q}k_{aq} + a_1k_{1q}k_{1a} + a_2k_{2q}k_{2a} + \ldots
\]

where

\[
k_{00} = 1; k_{r,s} = 0 \text{ if } r > s \text{ and where the } k_{r,s} \text{ for } s \geq r \text{ are given recurrently with the help of the following matrix equation}
\]

\[
\begin{pmatrix}
k_{00} & 0 & 0 & \cdots & k_{01} & k_{11} & 0 & 0 & 0 & \cdots \\
k_{01} & k_{11} & 0 & \cdots & k_{02} & k_{12} & k_{22} & 0 & 0 & \cdots \\
k_{02} & k_{12} & k_{22} & \cdots & k_{03} & k_{13} & k_{23} & k_{33} & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
= 
\begin{pmatrix}
k_{01} & k_{11} & 0 & 0 & 0 & \cdots \\
k_{02} & k_{12} & k_{22} & 0 & 0 & \cdots \\
k_{03} & k_{13} & k_{23} & k_{33} & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

The above matrix relation can be transformed as

\[
k_{mn} = \begin{cases} 
0 & \text{if } m > n \\
1 & \text{if } m = n \\
k_{m-1,n-1} + b_{m+1}k_{m,n-1} + a_{m+1}k_{m+1,n-1} & \text{if } 0 < m < n 
\end{cases}
\]

In the following section, the power series coefficient \( k_{0n} \), \( n = 0, 1, 2, \ldots \) is obtained in closed form. The power series coefficient is used to evaluate the busy period distribution and time-dependent system size probabilities of M/M/1 queue.
4 Evaluation of power series coefficients

In this section, we aimed at obtaining the power series coefficients \(k_{0n}\) in terms of Catalan numbers by solving the recurrence relation (5) using generating functions.

**Theorem 1:** If

\[
\frac{1}{b+z - b + z - b + z} - \frac{a}{z^{n+1}} = \sum_{n=0}^{\infty} (-1)^n k_{0n}
\]  

(6)

then for \(n = 0, 1, 2, \ldots\)

\[k_{0n} = \sum_{r=0}^{\infty} C_r \left( \frac{n}{2r} \right) b^{n-2r} a^r
\]  

(7)

where \(C_r = \frac{1}{r+1} \binom{2r}{r}\) is the Catalan number.

**Proof:** Let us assume that \(a_0 = a; b_0 = b, n = 1, 2, 3, \ldots\)

Then (5) becomes,

\[k_{0n} = bk_{0,n-1} + ak_{1,n-1}
\]

\[k_{mn} = bk_{m-1,n-1} + ak_{m,n-1} + ak_{m+1,n-1}, \quad m = 1, 2, \ldots, n-1.
\]

(8)

Defining

\[Q(z,n) = \sum_{n=0}^{\infty} \frac{z^n}{z^{n+1}} k_{mn} = k_{0n} + zk_{1n} + z^2 k_{2n} + \cdots + z^n k_{nn}
\]

On multiplying both sides of (8) by \(z^n\) and adding, we get

\[Q(z,n) = \left( z + b + \frac{a}{z} \right) Q(z,n-1) - \frac{a}{z} k_{0,n-1}
\]

Iteration of the equation yields

\[Q(z,n) = \left( z + b + \frac{a}{z} \right)^n - \frac{a}{z^n} \sum_{i=1}^{n} \left( z + b + \frac{a}{z} \right)^{n-i} k_{0,n-i}
\]

\[Q(z,n) = \left( \sqrt{a} \right)^n \left[ \frac{z}{\sqrt{a}} + \frac{b}{\sqrt{a}} + \frac{1}{\sqrt{a}} \right]^n - \frac{\sqrt{a}}{z} \sum_{i=1}^{n} \left( \sqrt{a} \right)^{n-i} \left[ \frac{z}{\sqrt{a}} + \frac{b}{\sqrt{a}} + \frac{1}{\sqrt{a}} \right]^{n-i} k_{0,n-i}
\]

On substitution of \(y = \frac{z}{\sqrt{a}}\) in the above equation, we get

\[Q(\sqrt{a}y,n) = \left( \sqrt{a} \right)^n \left[ y + \frac{b}{\sqrt{a}} + \frac{1}{\sqrt{a}} \right]^n - \frac{\sqrt{a}}{y} \sum_{i=1}^{n} \left( \sqrt{a} \right)^{n-i} \left[ y + \frac{b}{\sqrt{a}} + \frac{1}{\sqrt{a}} \right]^{n-i} k_{0,n-i}
\]

(9)
Replace $y$ by $\frac{1}{y}$, we get
\[
Q\left(\frac{\sqrt{a}}{y}, n\right) = \left(\sqrt{a}\right)^y \left[y + \frac{b}{\sqrt{a}} + \frac{1}{y}\right]^n - \sqrt{a} y \sum_{i=1}^{n} \left(\sqrt{a}\right)^{i-1} \left[y + \frac{b}{\sqrt{a}} + \frac{1}{y}\right]^{i-1} k_{0,n-i} \tag{10}
\]

From (9) and (10), we get
\[
Q(\sqrt{a} y, n) = \left(\sqrt{a}\right)^y \left[1 - \frac{1}{y^2}\right] \left[y + \frac{b}{\sqrt{a}} + \frac{1}{y}\right]^n + \frac{1}{y^2} Q\left(\frac{\sqrt{a}}{y}, n\right) \tag{11}
\]

Replace $\sqrt{a} y$ by $z$ in (11), we obtain
\[
Q(z, n) = \left(\sqrt{a}\right)^y \left[1 - \frac{a}{z^2}\right] \left[z + \frac{b}{\sqrt{a}} + \frac{\sqrt{a}}{z}\right]^n + \frac{a}{z^2} Q\left(\frac{a}{z}, n\right)
\]

The above equation can be rewritten as
\[
\sum_{m=0}^{\infty} z^m k_{mn} = \left[1 - \frac{a}{z^2}\right] \left[z + b + \frac{a}{z}\right]^n + \frac{a}{z^2} \sum_{m=0}^{\infty} \left(\frac{a}{z}\right)^m k_{mn} \tag{12}
\]

Equating constant term in (12), we get
\[
k_{0n} = \sum_{j=0}^{\infty} \binom{n}{j} \frac{2r}{2r} b^{n-2} a^r - a \sum_{r=1}^{\infty} \binom{n}{2r} \frac{2r}{r-1} b^{n-2} a^{r-1}
\]

The simplification of above result yields the power series coefficients $k_{0n}$ given in (7).

Another special case is considered in the following theorem which in turns can be used to obtain the transient solution of M/M/1 queue.

\textbf{Theorem 2:} If
\[
\frac{1}{b_n + z - b + z - z} \ldots = \sum_{n=0}^{\infty} (-1)^n k'_{0n} \tag{13}
\]

then for $n = 0, 1, 2, \ldots$
\[
k'_{0n} = k_{0n} + (b_n - b) \sum_{i=1}^{n} k_{0,j-1} k'_{0,n-i} \tag{14}
\]

where $k_{0n}$ is given in (7).

\textbf{Proof:} If $b_n = b$ if $n \geq 2$ and $a_n = a$ if $n \geq 1$ then equation (12) is of the form
\[ Q(z, n) = \left( 1 - \frac{a}{z} \right)^n \left( z + b + \frac{a}{z} \right) + (b_1 - b) \left( 1 - \frac{a}{z} \right)^{n-1} \sum_{i=1}^{n} \left( z + b + \frac{a}{z} \right) k_{0\beta}^{i-1} + \frac{a}{z^n} Q\left( \frac{a}{z}, n \right) \]

Equating constant term on both sides, we get

\[ k_{0\beta} = \sum_{i=0}^{\infty} C_i \left( \frac{n}{2r} \right) b^{n-2r} a^r + (b_1 - b) \sum_{i=1}^{\infty} \sum_{r=0}^{i-1} C_i \left( \frac{i-1}{2r} \right) b^{i-1-2r} a^r k_{0\beta}^{i-1} \]

Substituting the value of \( k_{0\beta} \) in the above equation, we get (14).

5 State-dependent Markovian queue

Let \( \{X(t), t \geq 0\} \) be a state-dependent Markovian queue with arrival and service rates \( \lambda_{n-1} \) and \( \mu_n \), \( n = 1, 2, 3, \ldots \), respectively defined on a probability space \( (\Omega, \mathcal{F}, P) \). Then \( P(X(t) = n | X(0) = 0) = P_n(t) \) satisfy the forward Kolmogorov equations

\[
P_n^{(0)}(t) = -\lambda_0 P_n(t) + \mu_1 P_1(t),
\]

\[
P_n^{(n)}(t) = \lambda_{n-1} P_{n-1}(t) - (\lambda_n + \mu_n) P_n(t) + \mu_{n+1} P_{n+1}(t), \quad n = 1, 2, \ldots \tag{15}\]

Taking Laplace transforms

\[ f_n(s) = \int_0^\infty e^{-st} P_n(t) \, dt, \quad n = 0, 1, 2, \ldots \]

for \( \text{Re}(s) > 0 \) of the system of equations given by (15) and which yield the following expressions:

\[ f_0(s) = \frac{1}{s + \lambda_0 - \mu_0} \frac{f_1(s)}{f_0(s)} \]

and

\[
\frac{f_n(s)}{f_{n-1}(s)} = \frac{\lambda_{n-1}}{s + \lambda_n + \mu_n - \mu_{n+1}} \frac{f_{n+1}(s)}{f_n(s)}
\]

\[ n = 1, 2, 3, \ldots \tag{16}\]

Iteration of the equation yields a J-fraction (or Jacobi fraction),

\[ f_0(s) = \frac{1}{s + \lambda_0 - s + \lambda_1 + \mu_1 - s + \lambda_2 + \mu_2 - s + \lambda_3 + \mu_3 - \ldots} \]

In the following section, a new approach is presented to determine the steady state probabilities from the CFs.
6 Steady state probabilities

In this section, a new approach is used to determine the steady state probabilities of both finite and infinite capacity state-dependent queues. A CF can be expressed as a rational function of two tridiagonal determinants and the tridiagonal determinants can be expressed as polynomials. One can also express a rational function as power series and its inversion yields a time-dependent system size probabilities of finite state-dependent system. Parthasarathy and Sudhesh (2005) have given the details to find the power series expression from the rational function of two tridiagonal determinants in the context of overflow process.

The applications of limit theorems in Laplace transform yields the stationary probabilities of finite capacity state-dependent queues. Further, the stationary probabilities of infinite systems are also deduced.

Let the arrival and service rates be assumed as follows:

\[ \lambda_n = a_{2n}, \mu_n = a_{2n+1}, n = 0, 1, 2, \ldots \]

Using the CFs given in section (5), for a state-dependent queue with state space \( S_f = \{0, 1, 2, \ldots, N\} \),

\[
f_0(s) = \frac{1}{s + a_0 - s + a_1 + a_2 - s + a_3 + a_4 - \cdots - s + a_{2N-2}a_{2N-1}}
\]

and for \( n = 1, 2, \ldots, N \),

\[
f_n(s) \frac{f_{n-1}(s)}{f_{n-1}(s)} = \frac{a_{2n-2}}{s + a_{2n-1}} \frac{a_{2n}a_{2n+1}}{s + a_{2n+1}} \cdots \frac{a_{2N-2}a_{2N-1}}{s + a_{2N-1}}
\]

The CF representation of \( f_0(s) \) and \( f_0(s) / f_{n-1}(s) \) can be written as a rational function of two tridiagonal determinants as follow:

\[
f_0(s) = \frac{A_0^{(1)}(s)}{B_{N+1}(s)},
\]

\[
f_n(s) = \left[ \prod_{i=1}^{n} a_{2i-2} \right] \frac{A_{N-n}^{(2n+1)}(s)}{A_n^{(1)}(s)} f_0(s), \quad n = 1, 2, \ldots, N,
\]

where

\[
B_{N+1}(s) = \begin{bmatrix}
  s + a_0 & a_0 \\
  a_1 & s + a_1 + a_2 & a_2 \\
  & a_3 & s + a_3 + a_4 \\
  & & \ddots \\
  & & & a_{2N-1} & s + a_{2N-1} \end{bmatrix}
\]

and for \( n = 0, 1, 2, \ldots, N \), \( A_{N-n}^{(2n+1)}(s) \) is obtained from \( B_{N+1}(s) \) by deleting the first \( n + 1 \) rows and first \( n + 1 \) columns with \( A_0^{(1)}(s) = 1 \).

\( B_n(s) \) is an orthogonal polynomial satisfying the three-term recurrence relation
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$B_n(s) = (s + a_{2n-3} + a_{2n-2})B_{n-1}(s) - a_{2n-4}a_{2n-3}B_{n-2}(s), \quad n = 2, 3, 4, \ldots, k,$

with $B_0(s) = 1$ and $B_1(s) = s + a_0$.

We can write,

$$B_n(s) = \sum_{m=0}^{n} \phi_m(2n+1)s^{n-m}, \quad (22)$$

where $\phi_0(n) = 1$, $\phi_0(n) = 0$ if $m \geq \left\lceil \frac{n+1}{2} \right\rceil$, for $n = 1, 2, 3, \ldots$ and $m \geq 1$,

$$\phi_m(n+1) = \sum_{k=0}^{n-2m} a_k \sum_{l=k+2}^{n-2m+2} \sum_{l_1=k+2}^{n-2m+4} \cdots \sum_{l_m=k_{m+1}+2}^{n-2} a_{l_m}, \quad (23)$$

The proof of the above result has been given in Parthasarathy and Sudhesh (2006b) in the context of orthogonal polynomials.

From (22),

$$B_{N+1}(s) = s \sum_{m=0}^{N} \phi_m(2N+1)s^{N-m}, \quad (24)$$

$$A^{(2n+1)}_{N-n}(s) = \sum_{m=0}^{N-n} \psi_m^{(2n+1)}(2N+1)s^{N-m}, \quad n = 0, 1, \ldots, N, \quad (25)$$

where

$$\phi_m(2N+1) = \sum_{k=0}^{2N-2m+1} a_k \sum_{l=0}^{2N-2m+3} \sum_{l_1=0}^{2N-2m+5} \cdots \sum_{l_m=0}^{2N-1} a_{l_m}, \quad m = 1, 2, \ldots, N,$$

and for $n = 0, 1, \ldots, N$,

$$\psi_m^{(2n+1)}(2N+1) = \sum_{k=0}^{2N-2m+1} a_k \sum_{l=0}^{2N-2m+3} \sum_{l_1=0}^{2N-2m+5} \cdots \sum_{l_m=0}^{2N-1} a_{l_m},$$

with $\psi_0^{(2n+1)}(2N+1) = 1 = \phi_0(2N+1)$.

It is well known that,

$$\lim_{t \to \infty} P_n(t) = \pi_n = \lim_{s \to 0} \mathcal{L}s f_n(s), \quad \text{if } n = 0, 1, 2, \ldots,$$

where $f_n(s)$ is the Laplace transform of $P_n(t)$.

From (20), (24) and (25),

$$\pi_0 = \lim_{s \to 0} \mathcal{L}s f_0(s) = \frac{\psi_0^{(2N+1)}(2N+1)}{\phi_0(2N+1)} = \frac{a_0 a_1 \cdots a_{2N-1}}{1 + \sum_{x=0}^{2N-1} \sum_{l=0}^{2N-1} a_{l+x}},$$

and from (21) and (25),
As \( N \to \infty \), we get the steady state probabilities of an infinite capacity state-dependent queue. Therefore, for infinite capacity models,

\[
\pi_0 = \lim_{N \to \infty} \frac{1}{1 + \sum_{r=1}^{\infty} \prod_{i=1}^{r} \frac{a_{2i-2}}{a_{2i-1}}} = \left[ 1 + \sum_{r=1}^{\infty} \prod_{i=1}^{r} \frac{a_{2i-2}}{a_{2i-1}} \right]^{-1},
\]

if \( \sum_{r=1}^{\infty} \frac{a_d a_2 \cdots a_{2r-2}}{a_d a_2 \cdots a_{2r-1}} < \infty \).

**7 Busy period distribution of M/M/1 queue**

Customers arrive at an M/M/1 queueing system at a Poisson process with parameter \( \lambda \) and service times are exponentially distributed with parameter \( \mu \). A busy period begins with the arrival of a customer to an idle channel and ends when the channel next becomes idle. The cumulative distribution function of the busy period is determined by considering the following differential-difference equations with an absorbing barrier imposed at zero system size.

Let \( P_n(t) \) be the probability that there are \( n \) customers in the system. The difference differential equations are:

\[
P_n'(t) = \mu P_n(t) \\
\lambda P_n(t) = -((\lambda + \mu) P_1(t) + \mu P_2(t)) \\
P_n(t) = \lambda P_{n-1}(t) - (\lambda + \mu) P_n(t) + \mu P_{n+1}(t), \quad n > 1
\]

with \( P_1(0) = 1 \).

On taking Laplace transform of (26) and after some simple algebra, we get

\[
f_1(s) = \frac{1}{s + \lambda + \mu - s + \lambda + \mu - s + \lambda + \mu - \cdots} \tag{27}
\]

If \( a = \lambda \mu, b = \lambda + \mu \) then the above J-fraction agrees with (6) and this can be expanded as a power series which is given by

\[
f_1(s) = \sum_{n=0}^{\infty} \frac{(-1)^n k_{0n}}{s^{n+1}}
\]

where the power series coefficient \( k_{0n} \) is given in (7).

On inversion, we get

\[
R(t) = \sum_{n=0}^{\infty} \frac{(-1)^n k_{0n} t^n}{n!}
\]
Hence, the busy period density function is given by

\[
b(t) = \mu P_1(t) = \mu \sum_{n=0}^{\infty} (-1)^n k_{0n} \frac{t^n}{n!}
\]  

(28)

where

\[
k_{0n} = \sum_{r=0}^{\infty} C_r \left( \frac{n}{2r} \right) (\lambda + \mu)^{n-2r} (\lambda \mu)^r
\]

One can also verify that the expression given in (28) is converging to the actual density function of the busy period and it equals

\[
\int_0^t e^{-(\lambda + \mu) t} \frac{I_1 \left( 2 \sqrt{\lambda \mu t} \right)}{t}, \ t > 0\]

where \(I_k(\cdot)\) is the modified Bessel function of the first kind of order \(n\).

8 Transient solution of single server queue

For an M/M/1 queue, the arrival rate is \(\lambda\), service rate is \(\mu\) and the system of differential-difference equation is written as follows:

\[
\begin{align*}
P_0(t) &= -\lambda P_0(t) + \mu R(t) \\
P_n(t) &= \lambda P_{n-1}(t) - (\lambda + \mu) P_n(t) + \mu P_{n+1}(t), n \geq 1
\end{align*}
\]  

(29)

with the assumption that initially there are no customers in the system.

On taking Laplace transform of (29) and manipulation yield the following CFs:

\[
\begin{align*}
f_0(s) &= \frac{1}{s + \lambda - s + \lambda + \mu} - \frac{\lambda \mu}{(s + \lambda + \mu)^2} - \ldots \\
f_n(s) &= \left( \frac{\lambda}{s + \lambda + \mu} - \frac{\lambda \mu}{(s + \lambda + \mu)^2} - \ldots \right) = \lambda G(s) \text{ (say)}.
\end{align*}
\]  

(30)

Therefore,

\[
f_n(s) = \lambda G(s) f_{n-1}(s),
\]

where \(G(s)\) is a CF which is given in (27).

Iteration of the above equation yields,

\[
f_n(s) = \left[ \lambda G(s) \right]^n f_0(s).
\]  

(31)

Assuming \(a = \lambda \mu, b_1 = \lambda\) and \(b = \lambda + \mu\) in Theorem 2, then the power series expression of (30) is given by

\[
f_0(s) = \sum_{n=0}^{\infty} \frac{(-1)^n k_{0n}'}{s^{n+1}}
\]

(32)

Inversion yields,
\[ P_0(t) = \sum_{n=0}^{\infty} (-1)^n k'_0 t^n / n! \]  

(32)

where

\[ k'_0 = k_{0e} - \mu \sum_{i=1}^{n} k_{0e-i} k'_{0n-i} \]

On the inversion of (31) gives

\[ P_0(t) = (\lambda)^t g^*(t) * P_0(t) \]  

(33)

where

\[ g(t) = \sum_{n=0}^{\infty} (-1)^n k_{0n} t^n / n! \]

and \( g^*(t) \) is \( n \)-fold convolution with \( g(t) \).

Hence, (32) and (33) completely determine all the state probabilities of M/M/1 queue.

9 Numerical illustration

In the previous sections, the expressions for the busy period distribution and probability distribution of number of customers in the system for an M/M/1 queue are obtained analytically. However, it is important to visualise the solutions in practical situations. The numerical value of the busy period distribution and empty size probability \( P_0(t) \) of M/M/1 queue are computed at different time points for the power series expressions of (28) and (32), and the same is discussed through tables.

Table 1 provides numerical comparison between the power series expression is given in (28) and \( b(t) = \sqrt{2 \pi e^{-(\lambda+\mu)} \mu t} \left[ 2 t \sqrt{\lambda \mu} \right] \). The power series is truncated at different levels namely \( n = 50, 75, 100 \). We observe that when \( n \) increases, the numerical values obtained from the power series expressions are coincide correct to ten places of decimals.

Table 2 compares the analytical expression for \( P_0(t) \) given in Saaty (1961) and the power series expression of \( P_0(t) \) which is given in (32). The power series expression is truncated at different levels namely \( n = 30, 50, 100 \). As the number of terms considered in the power series increases, the numerical values of the power series expression coincides with the result obtained by Saaty (1961).
Table 1  Numerical computation between the existing result in the literature and our power series expression of busy period distribution of M/M/1 queue

<table>
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<tr>
<th>Time (t)</th>
<th>( b(t) = \frac{\sqrt{\mu} e^{\lambda t}}{2} \int \left[ 2 \sqrt{\mu} \right] )</th>
<th>( n = 30 )</th>
<th>( n = 50 )</th>
<th>( n = 100 )</th>
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Table 2  Numerical computation between Saaty (1961) result and power series expression of \( P_0(t) \) of M/M/1 queue

<table>
<thead>
<tr>
<th>Time (t)</th>
<th>Saaty expression of ( P_0(t) )</th>
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10 Conclusions

In this paper, a new approach to find the transient system size probabilities and busy period distribution of M/M/1 queue with infinite capacity and stationary probability of general state-dependent queue are presented. Most of the results available in the literature obtained by many distinguished researchers are in terms of infinite series (e.g., Gross et al., 2008). The explicit analytical expressions are obtained in closed form in terms of power series. Also the numerical results obtained by employing this new approach are discussed through two tables. As can be seen, the above results compare quite favourably with those obtained by numerical computation. A unique approach is used to determine the stationary probabilities of general state-dependent birth-death queuing models from the CFs. One can also extend this new approach to other queueing model such as M/M/c model of both finite and infinite capacity.

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References

Stationary and transient solution of Markovian queues


