Similarity Solutions of the Burgers Equation with Linear Damping

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Abstract—The Burgers equation with linear damping has been subjected to Lie's group theoretic method of infinitesimal transformation to derive its solutions. To the best of our knowledge, we are the first to obtain a new similarity variable for an equation of the Burgers type and therefore provide new solutions. © 2004 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

The classical method for finding similarity reductions of a given partial differential equation is to use the Lie group method of infinitesimal transformations initially developed by Lie [1]. The monographs by Bluman and Cole [2], Bluman and Kumai [3], and Olver [4] provide an excellent description of Lie's classical group theoretic method of obtaining similarity solutions. Though the method is fully algorithmic, it often involves a large amount of tedious algebra and auxiliary calculations which are virtually unmanageable manually. Symbolic manipulation programs have been developed, particularly in MACSYMA [5,6] and REDUCE [7] in order to facilitate the determination of the associated similarity reductions.

Bluman and Cole [8] proposed a generalization of Lie's method and defined it as “nonclassical method of group invariant solutions”, which itself has been generalized by Olver and Rosenau [9,10]. All these methods determine Lie point transformations of a given partial differential equation.

According to Noether [11], Lie's method could be generalized by allowing the transformation to depend upon the derivatives of the dependent variable as well as the independent and dependent variables. The associated symmetries, called Lie-Bäcklund symmetries, can also be determined by an algorithmic method.

Bluman, Kumei and Reid [12] introduced an algorithmic method which yields new classes of symmetries of a given partial differential equation that are neither Lie point nor Lie-Bäcklund...
symmetries. A common characteristic of all these methods for finding symmetries and associated similarity reductions of a given partial differential equation is the use of group theory.

The Burgers equation with a linear damping which describes the plane motion of a continuous medium for which the constitutive relation for the stress contains a large linear term proportional to the strain, a small term which is quadratic in the strain, and a small dissipative term proportional to the strain-rate, has been studied for single hump conditions by singular perturbation approach by Lardner and Arya [13]. The N-wave solutions for this equation have been obtained by Sachdev and Joseph [14].

The scheme of the present paper is as follows. In Section 2, we obtain similarity solutions of the Burgers equation with linear damping using the group theoretic method. The conclusion of the present study is set forth in Section 3.

2. SIMILARITY SOLUTIONS

The Burgers equation with linear damping is

\[ u_t + uu_x + \sigma u = u_{xx}, \tag{1} \]

where \( \sigma > 0 \) is a constant. We seek to obtain Lie group of infinitesimal transformations which takes the \((x, t, u)\) space into itself and under which (1) is invariant, viz.,

\[ x^* = x + \varepsilon X(x, t, u) + o(\varepsilon^2), \]
\[ t^* = t + \varepsilon T(x, t, u) + o(\varepsilon^2), \]
\[ u^* = u + \varepsilon U(x, t, u) + o(\varepsilon^2). \tag{2} \]

Invariance of equation (1) under (2) gives

\[ \theta [U_x - U_u \sigma + 2X_x \sigma] + \theta_x [U - X_t - 2U_{xx} + X_{xx}] + \theta_t [T_{xx} - T_t + 2X_x] \]
\[ + \theta_x \theta_t [T_{uu} + 2X_u] + \theta \theta_x [X_x + 3X_u \sigma] + \theta \theta_t [T_u \sigma - T_x] + \theta \theta_x^2 2X_u + \theta x^2 [2X_{uu} - U_{uu}] \]
\[ + \theta_x^3 X_{uuu} + \theta_x^2 \theta_t T_{uu} + \theta_x X_{xx} + \theta \theta_x 2T_u + [U \sigma + U_t - U_{xx}] = 0. \tag{3} \]

Successively equating to zero the coefficients of \( \theta_x \theta_t, \theta_x, \theta^2 \theta_t, \theta_x^2, \theta \theta_x^2 \) in (3), we find that

\[ T_u = T_x = T_{uu} = X_{uu} = X_x = 0. \tag{4} \]

Equating the coefficient of \( \theta \theta_x \) in (3) to zero and using (4), we get

\[ X_x = 0. \tag{5} \]

Equating the coefficients of \( \theta_t, \theta_x, \theta, \) and \( \theta^0 \) in (3) to zero and using (4) and (5), we have

\[ U_{uu} = 0, \tag{6} \]
\[ T_t = 0, \tag{7} \]
\[ U - X_t - 2U_{xx} = 0, \tag{8} \]
\[ U_x - U_u \sigma = 0, \tag{9} \]
\[ \sigma U + U_t - U_{xx} = 0. \tag{10} \]

Equations (4), (5), and (7) lead to

\[ X = X(t), \quad T = c, \tag{11} \]

where \( c \) is a constant.
Equation (6) requires that
\[ U(x, t, u) = f(x, t)u + g(x, t). \] (12)

In view of (12), equations (8)–(10) take the form
\[ fu + g - X' + 2fx = 0, \] (13)
\[ f_xu - f_g + g_x = 0, \] (14)
\[ \sigma f u + \sigma g + f_t u + g_t - f_{xx} u - g_{xx} = 0. \] (15)

Equation (13) is meaningful only if
\[ f = 0, \] (16)
and reduces to
\[ g = X'(t), \] (17)
which in turn reveals that \( g_x = 0. \)

Substituting (16) and (17) into (14) and solving for \( \sigma, \) we find that the latter reduces to an identity.

On insertion of (16) and (17), equation (15) becomes
\[ \sigma g + g' = 0, \] (18)
whose general solution is
\[ g = a \exp[-\sigma t], \] (19)
where \( a \) is an arbitrary constant.

Substituting (19) into (17) and integrating with respect to \( t, \) we get
\[ X(t) = -\frac{a}{\sigma} \exp[-\sigma t] + b, \] (20)
where \( b \) is an arbitrary constant.

On substituting (19), equation (12) gives
\[ U = a \exp[-\sigma t]. \] (21)

The invariant surface condition is
\[ \frac{dx}{X} = \frac{dt}{T} = \frac{du}{U}. \] (22)

On inserting (11), (20), and (21) into (22), we have
\[ \frac{dx}{b - (a/\sigma) \exp[-\sigma t]} = \frac{dt}{c} = \frac{du}{a \exp[-\sigma t]}. \] (23)

Integration of the first two ratios of (23) gives rise to the similarity variable
\[ z(x, t) = cx - bt - \frac{a}{\sigma^2} \exp[-\sigma t]. \] (24)

In a similar manner, the second and third ratios give the similarity form of \( u \) as
\[ u(x, t) = -\frac{a}{\sigma \sigma} \exp[-\sigma t] + f(z). \] (25)

Thus, the similarity transform of (1) is
\[ u(x, t) = -\frac{a}{\sigma \sigma} \exp[-\sigma t] + f(z), \]
\[ z(x, t) = cx - bt - \frac{a}{\sigma^2} \exp[-\sigma t]. \] (26)
Putting (26) in (1), we get the following ordinary differential equation for the similarity function $f(z)$:

$$f'' - \frac{1}{c} f f' - \frac{\sigma}{c^2} f + \frac{b}{c^2} f' = 0. \quad (27)$$

Equations of the form (27) have been subjected to extensive analysis by Sachdev and his collaborators [15-17]. However, we also study (27) under two circumstances, namely, $b = 0$ and $b \neq 0$. When $b = 0$, we provide an intermediate integral as well as a simple algebraically linear solution. As an application of the results reported in [18], we derive a solution of the cylindrical Burgers equation from a solution of (27), with $b = 0$. On the other hand, when $b \neq 0$, an exact representation of the solution of (27) in terms of an integral is obtained.

2.1. The Case $b = 0$

With $b = 0$, (27) assumes the form

$$f'' - \frac{1}{c} f f' = 0. \quad (28)$$

A first integral of (28) is (see [19])

$$f' - \frac{\sigma}{c} \log \left| b_0 f' + \frac{\sigma b_0}{c} \right| = \frac{1}{2c^2} f^2 + q, \quad (29)$$

where $b_0$ and $q$ are arbitrary constants. And an exact solution of (28) is found to be

$$f(z) = -\frac{\sigma}{c} z + l, \quad (30)$$

where $l$ is a free constant.

With $b = 0$ and $f(z)$ given by (30), a solution of the Burgers equation with linear damping (1) is written below by inserting (26) and (30), with $b = 0$, in (25):

$$u(x, t) = l - \sigma x, \quad (31)$$

which is actually a solution of

$$u u_x + \sigma u = 0. \quad (32)$$

According to Sachdev and Mayil Vaganan [18], the solutions of the damped Burgers equation (1) and the cylindrical Burgers equation

$$w_{\tau} + w w_\xi + \frac{w}{2\tau} = w_{\xi\xi} \quad (33)$$

are related by

$$u(x, t) = \sqrt{2\sigma} e^{\sigma t} w(\xi, \tau) - \sigma x, \quad (34)$$

$$\xi(x, t) = \sqrt{2\sigma} x \exp \sigma t, \quad (35)$$

$$\tau(t) = \exp 2\sigma t. \quad (36)$$

We recall that the solution (30) of (28) gives the following solution of (1):

$$u(x, t) = l - \sigma x. \quad (37)$$

The corresponding solution of (33) is obtained by substitution of (37) into (34) as

$$w(\xi, \tau) = \frac{l}{\sqrt{2\sigma \tau}}. \quad (38)$$
2.2. The Case \( b \neq 0 \)

We first write (27) as an 'inhomogeneous' ordinary differential equation,

\[
f'' + \frac{b}{c^2} f' - \frac{\sigma}{c^2} f = \frac{1}{c} f f'.
\]

(39)

The 'homogeneous' equation

\[
f'' + \frac{b}{c^2} f' - \frac{\sigma}{c^2} f = 0
\]

has two linearly independent solutions, namely,

\[
f_1(z) = e^{\left(-b + \frac{\sqrt{b^2 + 4\sigma^2}}{2c^2}\right) \frac{z}{c}}
\]

and

\[
f_2(z) = e^{\left(-b - \frac{\sqrt{b^2 + 4\sigma^2}}{2c^2}\right) \frac{z}{c}},
\]

with the Wronskian

\[
W(f_1, f_2) = \frac{-\sqrt{b^2 + 4\sigma^2}}{c^2} e^{\frac{bs}{c^2}}.
\]

(43)

By the method of variation of parameters, the general solution of (39) can be written as

\[
f(z) = A e^{\left(-b + \frac{\sqrt{b^2 + 4\sigma^2}}{2c^2}\right) \frac{z}{c}} + B e^{\left(-b - \frac{\sqrt{b^2 + 4\sigma^2}}{2c^2}\right) \frac{z}{c}}
\]

+ \int_{z}^{\infty} k(s, z) R(s) ds,

(44)

where

\[
k(s, z) = \frac{\sigma}{\sqrt{\pi}} e^{-\frac{(s-z)^2}{2}} \left[f_1(z) f_2(s) - f_1(s) f_2(z)\right],
\]

(45)

\[
R(s) = \frac{1}{c} f(s) f'(s).
\]

(46)

3. CONCLUSION

Classical Lie group method of infinitesimal transformation has been successfully applied to the Burgers equation with linear damping (1) to derive a new similarity transformation given by

\[
u(x, t) = -\frac{a}{2\sigma} e^{-\sigma t} + f(z),
\]

(47)

\[
z(x, t) = c x - b t - \frac{a}{\sigma^2} e^{-\sigma t}.
\]

(48)

A similarity variable of the form (48) has not been reported previously, as often the similarity variable of any equation of the Burgers type is of the form \(x/\sqrt{t}\).

A result of Sachdev and Mayil Vaganan [18] has been exploited here to derive a solution of the cylindrical Burgers equation (33) from the solution (37) of the Burgers equation with linear damping. Significantly, for \( b \neq 0 \), another solution of the Burgers equation with linear damping (1) has been determined as

\[
u(x, t) = -\frac{a}{2\sigma} e^{-\sigma t} + A e^{\left(-b + \frac{\sqrt{b^2 + 4\sigma^2}}{2c^2}\right) \frac{z}{c}}
\]

+ \int_{z}^{\infty} k(s, z) R(s) ds,

(49)
where

\[ k(s, z) = \frac{\sqrt{\pi}}{2} \exp \left[ -\frac{s^2 - z^2}{2} \right] \left[ f_1(z)f_2(s) - f_1(s)f_2(z) \right], \tag{50} \]

\[ R(s) = \frac{1}{C} f(s)f'(s). \tag{51} \]

Here the similarity variable \( z \) is given by (26).

We close this paper with the remark that despite the difficulties posed by the nonlinear term \( uu_x \) in (1), we are able to obtain new solutions of the Burgers equation with linear damping term.

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