Multirate Sampled-Data Systems: All $\mathcal{H}_\infty$ Suboptimal Controllers and the Minimum Entropy Controller

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Abstract—For a general multirate sampled-data (SD) system, the authors characterize explicitly the set of all causal, stabilizing controllers that achieve a certain $\mathcal{H}_\infty$ norm bound; moreover, they give explicitly a particular controller that further minimizes an entropy function for the SD system. The characterization lays the groundwork for synthesizing multirate control systems with multiple/mixed control specifications.

Index Terms—Digital control, $\mathcal{H}_\infty$ optimization, matrix factorization, multirate systems, nest operators, sampled-data systems.

I. INTRODUCTION

MULTIRATE systems are abundant in industry [17]; there are several reasons for this.

• In multivariable digital control systems, often it is unrealistic, or sometimes impossible, to sample all physical signals uniformly at one single rate. In such situations, one is forced to use multirate sampling.

• In general one gets better performance if one can use faster A/D and D/A conversions, but this means a higher cost in implementation. For signals with different bandwidths, better tradeoffs between performance and implementation cost can be obtained using A/D and D/A converters at different rates.

• Multirate controllers are in general time-varying. Thus multirate control systems can outperform single-rate systems; for example, gain margin improvement [27], [16], simultaneous stabilization [27], and decentralized control [2], [44].

The study of multirate systems started in the late 1950’s [29], [25], [26]. Early studies were focused on analysis and were solely for purely discrete-time systems; see also [32]. A renaissance of research on multirate systems has occurred since late 1980 with an increased interest in multirate controller design, e.g., stabilizing controller design and parameterization of all stabilizing controllers [11], [30], [36], LQG/LQR control [8], [1], [31], $\mathcal{H}_2$ optimal control [42], [43], [34], $\mathcal{H}_\infty$ control [42], [43], [10], $\ell_1$ optimal control [15], and the work in [3], [21], and [38]. With the recognition that many industrial control systems consist of an analog plant and a digital controller interconnected via A/D and D/A converters, direct optimal control of multirate systems has been studied in this sampled-data setting [42], [10], [34]. The existing techniques for multirate $\mathcal{H}_\infty$ control allow for computation of one $\mathcal{H}_\infty$ controller via a numerical convex optimization [43] or more easily via an explicit design [10]. The purpose of this paper is to characterize in an explicit way the set of all $\mathcal{H}_\infty$ suboptimal controllers and to find a particular $\mathcal{H}_\infty$ suboptimal controller which minimizes an entropy function.

In this paper we shall treat a general multirate setup. For this, we define the periodic sampler $S_\tau$ and the (zero-order) hold $H_\tau$ (the subscript denotes the period) as follows: $S_\tau$ maps a continuous signal to a discrete signal and is defined via

$$\psi = S_\tau y \Leftrightarrow \psi(k) = y(k\tau).$$

$H_\tau$ maps discrete to continuous via

$$u = H_\tau v \Leftrightarrow u(t) = v(k), \quad k\tau \leq t < (k+1)\tau.$$

(The signals may be vector-valued.) Note that the sampler and hold are synchronized at $t = 0$.

The general multirate system is shown in Fig. 1. We have used continuous arrows for continuous signals and dotted arrows for discrete signals. Here, $G_e$ is the continuous-time generalized plant with two inputs, the exogenous input $w$ and the control input $u$, and two outputs, the signal $z$ to be controlled and the measured signal $y$. $S$ and $H$ are multirate sampling and hold operators and are defined as follows:

$$S = \begin{bmatrix} S_{m_1 h} & \cdots & S_{m_p h} \\ \vdots & \ddots & \vdots \\ S_{m_p h} & \cdots & S_{m_1 h} \end{bmatrix}, \quad H = \begin{bmatrix} H_{m_1 h} & \cdots & H_{m_p h} \end{bmatrix}.$$  

These correspond to sampling $p$ channels of $y$ periodically with periods $m_i h$, $i = 1, \ldots, p$, respectively, and holding $q$
channels of \( v \) with periods \( n_j \), \( j = 1, \ldots, q \), respectively. Here \( m_i \) and \( n_j \) are different integers and \( h \) is a real number referred to as the base period. If we partition the signals accordingly

\[
y = \begin{bmatrix}
y_1 \\
\vdots \\
y_p 
\end{bmatrix}, \quad \psi = \begin{bmatrix}
\psi_1 \\
\vdots \\
\psi_q 
\end{bmatrix}, \quad v = \begin{bmatrix}
v_1 \\
\vdots \\
v_q 
\end{bmatrix}, \quad u = \begin{bmatrix}
u_1 \\
\vdots \\
u_q 
\end{bmatrix}
\]

then

\[
\psi_i(k) = y_i(km_i h), \quad i = 1, \ldots, p
\]

\[
u_j(t) = v_j(k), \quad kn_j h \leq t < (k+1)n_j h,
\]

\[j = 1, \ldots, q.\]

\( K_{mr} \) is a discrete-time multirate controller, implemented via a microprocessor; it is synchronized with \( m_i h \) and \( n_j h \) in the sense that it inputs a value from the \( i \)th channel at times \( km_i h \) and outputs a value to the \( j \)th channel at \( kn_j h \).

In the general multirate setup of Fig. 1, we assume throughout that \( G_{a} \) and \( K_{mr} \) are causal and linear. Furthermore, \( G_{a} \) is assumed to be time-invariant and finite-dimensional, and \( K_{mr} \) is assumed to satisfy certain periodic property and to be finite-dimensional.

For periodicity of \( K_{mr} \), let \( l \) be the least common multiple of the sampling and hold indexers, \( \{m_1, \ldots, m_p, n_1, \ldots, n_q\} \). Thus \( \sigma = lh \) is the least common period for all sampling and hold channels. The multirate controller \( K_{mr} \) can be chosen so that \( \mathcal{H}(K_{mr} S) \) is \( \sigma \)-periodic in continuous time. For this, we need a few definitions.

Let \( \ell \) be the space of sequences, perhaps vector-valued, defined on the time set \( \{0, 1, 2, \ldots\} \). Let \( U \) be the unit time delay on \( \ell \) and \( U^* \) the unit time advance. Define the integers

\[
m_i = l/m_i, \quad i = 1, 2, \ldots, p
\]

\[
n_j = l/n_j, \quad j = 1, 2, \ldots, q.
\]

We say \( K_{mr} \) is \( \sigma \)-periodic in real time if

\[
\begin{bmatrix}
(U^*)^{m_1} & 0 & \cdots & 0 \\
0 & \ddots & \cdots & 0 \\
0 & \cdots & (U^*)^{m_p} & 0 \\
0 & \cdots & 0 & (U^*)^{n_q}
\end{bmatrix}
K_{mr}
\begin{bmatrix}
(U^*)^{m_1} & 0 & \cdots & 0 \\
0 & \ddots & \cdots & 0 \\
0 & \cdots & (U^*)^{m_p} & 0 \\
0 & \cdots & 0 & (U^*)^{n_q}
\end{bmatrix}^{-1} = K_{mr}.
\]

This means shifting \( \psi_i \) by \( m_i \) time units \( (m_i m_i h = \sigma) \) corresponds to shifting \( v_j \) by \( n_j \) units \( (n_j n_j h = \sigma) \). Thus \( \mathcal{H}(K_{mr} S) \) is \( \sigma \)-periodic in continuous time iff \( K_{mr} \) is \( \sigma \)-periodic in real time.

Since \( G_{a} \) is linear time-invariant (LTI), it follows that the sampled-data system in Fig. 1 is \( \sigma \)-periodic if \( K_{mr} \) is \( \sigma \)-periodic in real time. We shall refer to \( \sigma \) as the system period. We shall assume throughout the paper that \( K_{mr} \) is \( \sigma \)-periodic in real time. With all these assumptions, the controller \( K_{mr} \) can be implemented via difference equations [10]

\[
\eta(k+1) = A \eta(k) + \sum_{i=0}^{p} \sum_{s=0}^{m_i-1} (B_i)^{s} \psi_i(km_i + s)
\]

\[
u_j(kn_j + r) = (C_j)^{r} \eta(k) + \sum_{i=1}^{p} \sum_{s=0}^{m_i-1} (D_{ji})^{rs} \psi_i(km_i + s), \quad j = 1, 2, \ldots, q
\]

where causality requires \( (D_{ji})^{rs} = 0 \) if \( rn_j < sm_i \).

Our goal in this paper is two-fold: 1) characterize all feasible multirate controllers which internally stabilize the feedback system shown in Fig. 1 and make the \( L_{2} \)-induced norm less than a prespecified value, such controllers are called \( H_{\infty} \) suboptimal controllers and 2) among all \( H_{\infty} \) suboptimal controllers, find one which further minimizes an entropy function. Used with other optimization techniques, such a characterization, like its LTI counterpart [14], [22], is essential in designing control systems with simultaneous \( H_{\infty} \) and other performance requirements. The minimum entropy control, also like its LTI counterpart [33], [23], [24], gives a particular example of such multi-objective control problem in which an analytic solution exists.

Although the overall system shown in Fig. 1 is hybrid (involving both continuous-time and discrete-time signals) and time-varying, the recently developed lifting technique enables us to convert the problem into an equivalent LTI discrete-time problem. However, the resulting control problem will have an undesirable and unconventional constraint on the LTI controller due to the causality requirement. This constraint is the main difficulty in designing optimal multirate systems.

The recent introduction of the nest operators has proven to be effective in handling causality constraints in multirate design [10]. The results of this paper will be built on the nest operator technique.

We would like to remark here that the results in this paper extend directly to periodic discrete-time systems, i.e., direct application yields a characterization of all \( H_{\infty} \) suboptimal solutions which are periodic and causal; this result has not been obtained before.

The paper is organized as follows. The next section reviews some basic facts about continuous-time periodic systems, introduces the concept of entropy for such systems, and establishes the connection between the entropy and a linear, exponential, quadratic, Gaussian cost function. Section III addresses topics on nest operators and nest algebra, which are the main tools to handle causality in this paper. Section IV briefly discusses the procedure of converting our hybrid problem into an equivalent LTI problem with a causality constraint. Section V gives a characterization of all \( H_{\infty} \) suboptimal controllers and the minimum entropy controller. The Appendices contain two long and involved proofs.

Preliminary results in this paper have been presented at several conferences: the Asian Control Conference (Tokyo, Japan, 1994), the IEEE Conference on Decision and Control (Florida, USA, 1994), and the International Conference on Operator Theory and its Applications (Manitoba, Canada, 1994).

Finally, we introduce some notation. Given an operator \( K \) and two operator matrices

\[
P = \begin{bmatrix}
P_{11} & P_{12} \\
P_{21} & P_{22}
\end{bmatrix}, \quad Q = \begin{bmatrix}
Q_{11} & Q_{12} \\
Q_{21} & Q_{22}
\end{bmatrix}
\]

the linear fractional transformation associated with \( P \) and \( K \) is denoted

\[
F(P, K) = P_{11} + P_{12} K (I - P_{22} K)^{-1} P_{21}
\]
and the star product of \( P \) and \( Q \) is shown in (a) at the bottom of the page. Here, we assume that the domains and codomains of the operators are compatible and the inverses exist. With these definitions, we have

\[
\mathcal{F}(P \ast Q) = \mathcal{F}(P \ast K) = \mathcal{F}(F \ast Q, K).
\]

II. Entropy of Periodic Systems

A multirate system as depicted in Section I is a continuous-time \( \sigma \)-periodic system. In this section, we review some basic concepts of periodic systems and introduce the concept of entropy.

Let \( \mathcal{X} \) and \( \mathcal{Y} \) be Hilbert spaces and \( f = \{ f(k); k = 0, 1, 2, \ldots \} \) be a sequence of bounded operators from \( \mathcal{X} \) to \( \mathcal{Y} \). Then

\[
\hat{F}(\lambda) = \sum_{k=0}^{\infty} f(k) \lambda^k
\]

is an operator-valued function on some subset of \( \mathcal{C} \). We say that \( \hat{F} \) belongs to \( \mathcal{H}_\infty(\mathcal{X}, \mathcal{Y}) \) if \( \hat{F} \) is analytic in \( \mathcal{D} \), the open unit disk, and

\[
\sup_{\lambda \in \mathcal{D}} \| \hat{F}(\lambda) \| < \infty.
\]

In this case, the left-hand side above is defined to be the \( \mathcal{H}_\infty \) norm of \( \hat{F} \); denoted by \( \| \hat{F} \|_\infty \), the operator \( \hat{F}(e^{j\omega}) \) is bounded for almost every \( \lambda \in [-\pi, \pi] \), and

\[
\text{ess sup}_{\omega \in [-\pi, \pi]} \| \hat{F}(e^{j\omega}) \| = \| \hat{F} \|_\infty.
\]

Now let \( f = \{ f(k); k = 0, 1, 2, \ldots \} \) be a sequence of Hilbert–Schmidt operators from \( \mathcal{X} \) to \( \mathcal{Y} \). The set of Hilbert–Schmidt operators equipped with the Hilbert–Schmidt norm, \( \| \cdot \|_{\text{HS}} \), is a Hilbert space [19]. Then

\[
\hat{F}(\lambda) = \sum_{k=0}^{\infty} f(k) \lambda^k
\]

is a Hilbert-space vector-valued function on some subset of \( \mathcal{C} \). We say that \( \hat{F} \) belongs to \( \mathcal{H}_2(\mathcal{X}, \mathcal{Y}) \) if

\[
\left( \sum_{k=0}^{\infty} \| f(k) \|_{\text{HS}}^2 \right)^{1/2} < \infty.
\]

In this case, the left-hand side above is defined to be the \( \mathcal{H}_2 \) norm of \( \hat{F} \); denoted by \( \| \hat{F} \|_2 \), the operator \( \hat{F}(e^{j\omega}) \) is Hilbert-Schmidt for almost every \( \lambda \in [-\pi, \pi] \), and

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \| \hat{F}(e^{j\omega}) \|_2^2 d\omega = \| \hat{F} \|_2^2.
\]

Assume \( \hat{F} \in \mathcal{H}_\infty(\mathcal{X}, \mathcal{Y}) \cap \mathcal{H}_2(\mathcal{X}, \mathcal{Y}) \) and \( \| \hat{F} \|_\infty < 1 \). Extending the entropy definition for matrix valued analytic functions [23], [24], we define the entropy of \( \hat{F} \) as

\[
I(\hat{F}) = -\frac{1}{2\pi} \int_{-\pi}^{\pi} \ln \det [I - \hat{F}(e^{j\omega}) \hat{F}(e^{j\omega})] d\omega.
\]

This entropy is well defined. Since \( \hat{F}(e^{j\omega}) \) is a Hilbert–Schmidt operator at almost every \( \lambda \in [-\pi, \pi] \), its singular values form a square-summable sequence \( \{ \sigma_k(e^{j\omega}) \} \). Hence

\[
\ln \det [I - \hat{F}(e^{j\omega}) \hat{F}(e^{j\omega})] = \sum_{k=1}^{\infty} \ln(1 - \sigma_k^2(e^{j\omega}))
\]

which converges to some number in \( (0, 1) \) due to square summability of \( \{ \sigma_k(e^{j\omega}) \} \) and the fact that \( \| \hat{F} \|_\infty < 1 \). This also shows that \( I(\hat{F}) \) is nonnegative.

Lemma 1: Assume \( \hat{F} \in \mathcal{H}_\infty(\mathcal{X}, \mathcal{Y}) \cap \mathcal{H}_2(\mathcal{X}, \mathcal{Y}) \) and \( \| \hat{F} \|_\infty < 1 \). Then

1) \( \| \hat{F} \|_2^2 \leq I(\hat{F}) \);
2) for \( \hat{U} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \in \mathcal{H}_\infty(\mathcal{X} \oplus \mathcal{Y}, \mathcal{Y} \oplus \mathcal{X}) \) with \( \hat{U}^* \hat{U} = I, U_{11} \in \mathcal{H}_2(\mathcal{X}, \mathcal{Y}), \) and \( U_{22} \in \mathcal{H}_\infty(\mathcal{X}, \mathcal{X}) \),

\[
I(\hat{F}, \hat{U}, \hat{F}) = I(\hat{F}) + I(\hat{U}_{11}) + 2 \ln | \det [I - \hat{U}_{22}(0) \hat{F}(0)] |.
\]

The proof of Lemma 1 is similar to that for the finite-dimensional continuous-time case [33].

Now let us return to periodic systems. Let \( F_a \) be a continuous-time, \( \sigma \)-periodic, causal system described by the following integral operator:

\[
(F_a w)(t) = \int_{0}^{t} f_a(t, \tau) w(\tau) d\tau.
\]

We assume that \( f_a \), the matrix-valued impulse response of \( F_a \), is locally square integrable, i.e., every element is square integrable on any compact subset of \( \mathbb{R}^2 \). The periodicity of \( F_a \) implies \( f_a(t + T, \tau + T) = f_a(t, \tau) \), and the causality implies that \( f_a(t, \tau) = 0 \) if \( \tau > t \).

The local square integrability of \( f_a \) guarantees that \( F_a \) is a linear map from \( L_{2e} \) to \( L_{2e} \), the space of locally square-integrable functions of \( t \). Given an arbitrary positive integer \( l \), let

\[
\mathcal{K} = L^2_2 [0, \frac{T}{l}].
\]

Denote the space of \( \mathcal{K} \)-valued sequences by \( \ell(\mathcal{K}) \). Define the lifting operator \( L_{\sigma}; \mathcal{L}_{2e} \rightarrow \ell(\mathcal{K}) \) via

\[
\omega = L_{\sigma} w \leftrightarrow \{ w(0), w(1), \ldots \}
\]

\[
= \begin{bmatrix} u(t) \\ \vdots \\ w(t + (l-1)\frac{\sigma}{l}) \end{bmatrix}, \quad \begin{bmatrix} \vdots \\ w(t + \sigma) \\ \vdots \\ w(t + (l-1)\frac{\sigma}{l}) \end{bmatrix},
\]

\[
\begin{bmatrix} \vdots \\ \vdots \\ \vdots \end{bmatrix}, \quad t \in [0, \frac{T}{l}].
\]

\[
P \ast Q = \begin{bmatrix} P_{11} + P_{12}Q_{11}(I - P_{22}Q_{11})^{-1}P_{21} & P_{12}(I - Q_{11}P_{22})^{-1}Q_{12} \\ Q_{21}(I - P_{22}Q_{11})^{-1}P_{21} & Q_{21}(I - P_{22}Q_{11})^{-1}P_{22}Q_{12} + Q_{22} \end{bmatrix}
\]
This lifting \( L_{\rho,t} \) gives an algebraic isomorphism between \( \mathcal{L}_{2\infty} \) and \( \ell(K) \) [42]. We use the obvious norm in \( K \)

\[
\kappa = \begin{bmatrix} \kappa_1 \\ \vdots \\ \kappa_q \end{bmatrix} \in K \Leftrightarrow \|\kappa\| = \left( \sum_{i=1}^{l} \|\kappa_i\|^2 \right)^{1/2}
\]

where \( \|\kappa\| \) is the norm on \( \mathcal{L}_2[0, \infty) \). Denote by \( \ell_2(K) \) the subset of \( \ell(K) \) consisting of all sequences \( \omega \) with

\[
\left( \sum_{k=0}^{\infty} \|\omega(k)\|^2 \right)^{1/2} < \infty
\]

and define the norm on \( \ell_2(K) \) to be the left-hand side of the above inequality. It is clear that \( \omega \in \ell_2(K) \) if and only if \( \omega \in \mathcal{L}_2 \) and \( L_{\rho,t} \) is a Hilbert-space isometric isomorphism from \( \mathcal{L}_2 \) to \( \ell_2(K) \).

Now we lift \( F_a \) to get \( F := L_{\rho,t} F_a L_{\rho,t}^{-1} \). The lifted system \( F: \ell(K) \to \ell(K) \) can be described by

\[
\zeta = F\omega \Leftrightarrow \zeta(k) = \sum_{i=0}^{k} f(k-i)\omega(i), \quad k \geq 0
\]

where \( f(k); k = 0, 1, \ldots \), map \( K \) to \( K \) via the equation shown in (b) at the bottom of the page. The local square integrability of \( f_a(t, \tau) \) ensures that \( f(k); k \geq 0 \), are Hilbert-Schmidt operators [46].

For \( \sigma \)-periodic \( F_a \), the lifted system \( F \) is LTI in discrete time; its transfer function is defined as

\[
\hat{F}(\lambda) = \sum_{k=0}^{\infty} f(k)\lambda^k.
\]

So if \( \hat{F} \in \mathcal{H}_\infty(K, K) \cap \mathcal{H}_2(K, K) \) and \( \|\hat{F}\|_\infty < 1 \), its entropy can be defined.

We will define the \( \mathcal{H}_\infty \) norm, \( \mathcal{H}_2 \) norm, and entropy of \( F_a \) to be those of \( \hat{F} \) respectively. Actually, the \( \mathcal{H}_\infty \) norm defined this way is indeed the \( \mathcal{L}_2 \)-induced norm of \( F_a \) [7], [8], [40]; the \( \mathcal{H}_2 \) norm has natural interpretations in terms of impulse responses and white noise responses [6], [28]; the entropy not only provides an upper bound for the \( \mathcal{H}_2 \) norm as stated in Lemma 1, but also has a stochastic interpretation in terms of a linear exponential quadratic Gaussian (LEQG) cost function, similar to the case of matrix-valued transfer functions [18].

To avoid an unnecessary technicality, we will concentrate on finite-dimensional periodic systems, i.e., those \( F_a \) with finite-dimensional realizations, or equivalently, those \( F_a \) whose lifted transfer functions \( \hat{F} \) have only a finite number of poles. (The multirate systems to be studied in Fig. 1 fall in this class if both \( G_a \) and \( K_{mr} \) are finite-dimensional.) Let \( w \) be a Gaussian white noise with zero mean and unit covariance on the time interval \([0, \infty)\) and \( z \) the corresponding response: \( z = F_a w \). Define an LEQG cost function for \( F_a \) as

\[
\Omega_T = \frac{2}{T} \ln \mathbb{E} \left\{ \exp \left[ \frac{1}{2} \int_0^T z'(t)z(t) dt \right] \right\}
\]

where \( \mathbb{E}(\cdot) \) means the expectation. The proof of the following theorem is given in Appendix A.

**Theorem 1:** Given a finite-dimensional \( \sigma \)-periodic system \( F_a \), assume its lifted transfer function \( \hat{F} \) satisfies \( \hat{F} \in \mathcal{H}_\infty(K, K) \cap \mathcal{H}_2(K, K) \) and \( \|\hat{F}\|_\infty < 1 \). Then \( \lim_{T \to \infty} \Omega_T = \mathcal{I}(\hat{F})/\sigma \).

Now we are ready to state our control problems associated with Fig. 1 precisely.

1) characterize all feasible multirate controllers \( K_{mr} \) such that the feedback system is internally stable and \( \|F(G_a, HK_{mr}S)\|_\infty < 1 \);

2) find a particular controller from those obtained in (1) such that the entropy \( \mathcal{I}[F(G_a, HK_{mr}S)] \) is minimized.

These problems will be solved explicitly in Sections V and VI. Next, we present the required mathematical tool based on nest operators.

### III. Nest Operators

In this section, we address some issues on nest operators and nest algebra [4], [12], which are useful in the sequel. Our main purpose is to probe further the Arveson’s distance problem, that is, we characterize explicitly all nest operators which are within a fixed distance from a given operator; we also give one such nest operator which minimizes an auxiliary entropy function. The same problems were also studied in the mathematical literature [45], but the solutions are different.

Our results, based on the unitary dilation, provide further insight as well as certain numerical advantages; they take forms which are easily applicable to our control problems at hand.

Let \( X \) be a vector space. A nest in \( X \), denoted \( \{X_\ell\} \), is a chain of subspaces in \( X \), including \( \{0\} \) and \( X \), with the nonincreasing ordering

\[
X = X_0 \supseteq X_1 \supseteq \cdots \supseteq X_{n-1} \supseteq X_n = \{0\}.
\]

(A nest may be defined to contain an infinite number of spaces, but this generalization is not necessary in the sequel.)

\[
[f(k)\omega](t) = \int_0^{\sigma/t} \begin{bmatrix} f_a(t + k\sigma, \tau) \\ \vdots \\ f_a(t + k\sigma, (l-1)\sigma_\ell, \tau) \\ \vdots \\ f_a(t + k\sigma + (l-1)\sigma_\ell, \tau + (l-1)\sigma_\ell) \end{bmatrix} \begin{bmatrix} \kappa_1(\tau) \\ \vdots \\ \kappa_\ell(\tau) \end{bmatrix} d\tau,
\]

\[
t \in \left[0, \frac{\sigma_\ell}{T}\right]
\]

(b)
Let $\mathcal{X}$ and $\mathcal{Y}$ be both Hilbert spaces. Denote by $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ the set of bounded linear operators $\mathcal{X} \to \mathcal{Y}$ and abbreviate it as $\mathcal{L}(\mathcal{X})$ if $\mathcal{X} = \mathcal{Y}$. Assume that $\mathcal{X}$ and $\mathcal{Y}$ are equipped, respectively, with nests $\{\mathcal{X}_i\}$ and $\{\mathcal{Y}_i\}$ which have the same number of subspaces, say, $n + 1$ as above. An operator $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is said to be a nest operator if

$$T \mathcal{X}_i \subseteq \mathcal{Y}_{i+1}, \quad i = 0, 1, \ldots, n. \quad (1)$$

It is said to be a strict nest operator if

$$T \mathcal{X}_i \subseteq \mathcal{Y}_{i+1}, \quad i = 0, 1, 2, \ldots, n - 1. \quad (2)$$

Let $\Pi_{\mathcal{X}_i}: \mathcal{X} \to \mathcal{X}_i$ and $\Pi_{\mathcal{Y}_i}: \mathcal{Y} \to \mathcal{Y}_i$ be orthogonal projections. Then the condition in (1) is equivalent to

$$(I - \Pi_{\mathcal{Y}_i}) T \Pi_{\mathcal{X}_i} = 0, \quad i = 0, 1, \ldots, n$$

and the condition in (2) is equivalent to

$$(I - \Pi_{\mathcal{Y}_{i+1}}) T \Pi_{\mathcal{X}_i} = 0, \quad i = 0, 1, 2, \ldots, n - 1.$$
Proof: Since
\[
\begin{bmatrix}
T + P_{11} & P_{12} \\
P_{21} & P_{22}
\end{bmatrix}
\]
is unitary and \(P_{12}, P_{21}\) are invertible, it follows from [37] that the map
\[
U \leftrightarrow \mathcal{F} \left( \begin{bmatrix} T + P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, U \right) = T + \mathcal{F}(P,U)
\]
is a bijection from the open unit ball of \(L(\{X\}, \{Y\})\) onto itself. What is left to show is that \(\mathcal{F}(P,U) \in N(\{X\}, \{Y\})\) if \(U \in N(\{X\}, \{Y\})\). The “if” part follows from Lemma 2 by noting \(P \in N(\{X\} \otimes X_i, \{Y_i \otimes X_i\})\). For the “only if” part, assume \(N := \mathcal{F}(P, U) \in N(\{X\}, \{Y\})\) for some \(U \in L(\{X\}, \{Y\})\); we need to show that \(U\) too belongs to \(N(\{X\}, \{Y\})\). From
\[
N = P_{11} + P_{12}U(I - P_{22}U)^{-1}P_{21}
\]
we obtain after some algebra
\[
P_{12}^{-1}(N - P_{11})P_{21}^{-1} = [I + P_{12}^{-1}(N - P_{11})P_{21}^{-1}P_{22}]U. \tag{6}
\]
Since
\[
I + P_{12}^{-1}(N - P_{11})P_{21}^{-1}P_{22} = I + P_{12}^{-1}P_{21}U(1 - U^{-1}P_{22})^{-1}P_{21}^{-1}P_{22}
\]
\[
= I + U(I - P_{22}U)^{-1}P_{22}
\]
\[
= (I - U^{-1}P_{22})^{-1}
\]
it follows that \(I + P_{12}^{-1}(N - P_{11})P_{21}^{-1}P_{22}\) is invertible. Hence from (6)
\[
U = [I + P_{12}^{-1}(N - P_{11})P_{21}^{-1}P_{22}]^{-1}(N - P_{11})P_{21}^{-1}.
\]
Therefore \(U\) belongs to \(N(\{X_i\}, \{Y_i\})\) by Lemma 2. ❄️

The characterization in Theorem 3 also renders an easy solution to the second matrix problem.

**Theorem 4:** Let \(T \in L(\{X\}, \{Y\})\) and assume condition 3) in Theorem 2 is satisfied. Then the unique \(N\) which satisfies \(\|T + N\| < 1\) and minimizes \(\mathcal{I}(T + N)\) is given by \(N = P_{11}\).

**Proof:** According to Theorem 3, all \(N\) satisfying \(\|T + N\| < 1\) are characterized by (5). Consequently, all resulting \(T + N\) are given by
\[
\mathcal{F} \left( \begin{bmatrix} T + P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, U \right), \quad U \in N(\{X_i\}, \{Y_i\}) \quad \text{and} \quad \|U\| < 1 \right) \cup \mathbb{R}.
\]

By Lemma 1, we obtain
\[
\mathcal{I}(T + N) = \mathcal{I}(U) + \mathcal{I}(T + P_{11}) + 2 \ln |\det(I - P_{22}U)|.
\]
Notice that the second term is independent of \(U\) and \(P_{22}U \in N(\{Y_i\}, \{Y_i\})\), which implies that the third term is zero. Therefore the minimizing \(U\) is zero and hence \(N = P_{11}\). ❄️

One implication of Theorem 4 is that although \(P\) in condition 3) of Theorem 2 is not unique, \(P_{11}\) is uniquely determined.

IV. EQUIVALENT LTI SYSTEMS

Our main problems deal with hybrid time-varying systems. Following [10] and [42], we can reduce the control problem to an equivalent one involving only finite-dimensional LTI systems. In this section we briefly review the reduction process. The detailed justification is referred to [10], [42], and [5]. Our emphasis here is on the relationship between the entropy of the original system and the equivalent LTI system.

We start with a state model of \(G_a\)
\[
\hat{G}_a(s) = \begin{bmatrix} A_a & B_{a1} & B_{a2} \\ C_{a1} & 0 & D_{a12} \\ C_{a2} & 0 & 0 \end{bmatrix}.
\]

For an integer \(m > 0\), define the discrete lifting operator \(L_m\) via
\[
L_m\{\psi(0), \psi(1), \ldots\} = \left\{ \begin{bmatrix} \psi(0) \\ \vdots \\ \psi(m - 1) \end{bmatrix}, \begin{bmatrix} \psi(m) \\ \vdots \\ \psi(2m - 1) \end{bmatrix} \right\}.
\]

Denote
\[
L_m = \begin{bmatrix} L_{m_{11}} & \cdots & L_{m_{1q}} \\ \vdots & \ddots & \vdots \\ L_{m_{q1}} & \cdots & L_{m_{qq}} \end{bmatrix}, \quad L_N = \begin{bmatrix} L_{N_{11}} & \cdots & L_{N_{1q}} \\ \vdots & \ddots & \vdots \\ L_{N_{q1}} & \cdots & L_{N_{qq}} \end{bmatrix}
\]
and recall the continuous lifting operator \(L_{\sigma_L}\) in Section II: Here we take \(\sigma = I_{L_4}\). We lift \(G_a\) and \(K_{mr}\) by defining
\[
\hat{G} = \left[ L_{\sigma_L} L_{\sigma_L} S \right] G_a \left[ \begin{bmatrix} L_{L_{11}}^{-1} \\ \cdots \\ L_{L_{qq}}^{-1} \end{bmatrix} \right] H_L^{-1}
\]
and
\[
K = L_N K_{mr} L_{N_L}^{-1}.
\]

It is easy to check that \(\hat{G}\) and \(K\) are LTI systems, so they have transfer functions \(\hat{G}(\lambda)\) and \(K(\lambda)\). By definitions
\[
\|\mathcal{F}(G_a, H_{K_{mr}S})\|_\infty = \|\mathcal{F}(\hat{G}, \hat{K})\|_\infty
\]
\[
\mathcal{I}[\mathcal{F}(G_a, H_{K_{mr}S})] = \mathcal{I}[\mathcal{F}(\hat{G}, \hat{K})],
\]
A state-space realization of \(\hat{G}\) can be computed
\[
\hat{G}(\lambda) = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix}.
\]

Due to the causality of \(G_a\) and \(K_{mr}\), the lifted systems \(\hat{G}\) and \(K\) have some special structures which can be easily characterized using nest operators.

Write
\[
\hat{w} = L_{\sigma_L} w, \quad \hat{z} = L_{\sigma_L} z, \quad e = L_N v, \quad \phi = L_{\sigma_L} \psi.
\]
Then
\[
\phi(0) = [\psi_1(0)' \cdots \psi_q(0)' \cdots \psi_p(0)' \cdots \psi_p(m - 1)']'.
\]
Note that \(\psi_i(k)\) is sampled at \(t = km_i\). Similarly
\[
e(0) = [v_1(0)' \cdots v_1(0)' \cdots v_q(0)' \cdots v_q(m - 1)']'.
\]
Fig. 2. The lifted system.

and \( u_j(k) \) occurs at \( t = k\gamma_1 h \). For \( r = 0, 1, \ldots, l \), define

\[
\hat{W}_r = \{ \hat{w}(0); \hat{w}_1(0) = \hat{w}_2(0) = \cdots = \hat{w}_r(0) = 0 \}
\]
\[
\hat{Z}_r = \{ \hat{z}(0); \hat{z}_1(0) = \hat{z}_2(0) = \cdots = \hat{z}_r(0) = 0 \}
\]
\[
\mathcal{U}_r = \{ \nu(0); \nu_j(k) = 0 \text{ if } k \gamma_1 < r \}
\]
\[
\mathcal{Y}_r = \{ \psi(0); \psi_j(k) = 0 \text{ if } k \gamma_2 < r \}.
\]

Then the \( D \)-blocks in the lifted plant satisfy

\[
\begin{align*}
\hat{D}_{11} & \in \mathcal{N}(\{\hat{W}_r\}, \{\hat{Z}_r\}) \\
\hat{D}_{12} & \in \mathcal{N}(\{\mathcal{U}_r\}, \{\hat{Z}_r\}) \\
\hat{D}_{21} & \in \mathcal{N}_s(\{\hat{W}_r\}, \{\mathcal{Y}_r\}) \\
\hat{D}_{22} & \in \mathcal{N}_s(\{\mathcal{U}_r\}, \{\mathcal{Y}_r\})
\end{align*}
\]

and for \( K_{nn} \) to be causal

\[
\hat{K}(0) \in \mathcal{N}(\{\mathcal{Y}_r\}, \{\mathcal{U}_r\}).
\]

Hence we have arrived at an equivalent LTI problem, shown in Fig. 2, with plant \( \hat{G} \) and controller \( \hat{K} \). Note that (7)–(10) give special structures of \( \hat{G} \) that can be exploited, whereas (11) is a design constraint on \( \hat{K} \) that has to be respected in order for \( \hat{K} \) to correspond to a causal \( K_{nn} \).

The signals \( \hat{w} \) and \( \hat{z} \) in Fig. 2 take values in infinite-dimensional spaces. In other words, \( \hat{D}_1, \hat{C}_1, \hat{D}_{11}, \hat{D}_{12}, \hat{D}_{21} \) are operators with either domain or codomain being infinite-dimensional spaces. To overcome this difficulty, we observe that all these operators except \( \hat{D}_{11} \) have finite rank.

Due to the particular choice of decomposition of \( \mathcal{W} \) and \( \mathcal{Z} \), the operator \( \hat{D}_{11} \) takes a lower-triangular Toeplitz form

\[
\hat{D}_{11} = \begin{bmatrix}
(D_{11})_0 & 0 \\
(\cdots)_{\hat{D}_{11}} & \cdots \\
(\cdots)_{\hat{D}_{11}} & (D_{11})_0
\end{bmatrix}.
\]

The only block with infinite rank is \( (\hat{D}_{11})_0 \). Our next step is to get rid of this by a linear fractional transformation. Since \( \hat{D}_{21} \in \mathcal{N}(\{\hat{W}_r\}, \{\mathcal{Y}_r\}) \), the diagonal blocks of

\[
\mathcal{F}(\hat{G}, \hat{K})(0) = \mathcal{F}(\hat{G}(0), \hat{K}(0))
\]

\[
= \hat{D}_{11} + \hat{D}_{22} \hat{K}(0)[I + \hat{D}_{22} \hat{K}(0)]^{-1} \hat{D}_{21}
\]

are invariant for any \( \hat{K} \) satisfying (11). Therefore \( \| (\hat{D}_{11})_0 \|_1 < 1 \) is a necessary condition for the solvability of our \( H_{\infty} \) control problem. From now on we assume this condition is satisfied.

Define a diagonal operator matrix

\[
U_{11} = \begin{bmatrix}
-(\hat{D}_{11})_0 & 0 \\
\vdots & \ddots \\
0 & \cdots & -(\hat{D}_{11})_0
\end{bmatrix}
\]

and a Julian operator matrix

\[
U = \begin{bmatrix}
U_{11} & U_{12} \\
U_{21} & U_{22}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
(I - U_{11} U_{11}^{\dagger})^{1/2} \\
(I - U_{11} U_{11}^{\dagger})^{-1/2}
\end{bmatrix}
\]

Let

\[
\bar{G} = U \ast \hat{G}.
\]

Then it is well known [37] that \( \| \mathcal{F}(\hat{G}, \hat{K}) \|_\infty < 1 \) iff \( \| \mathcal{F}(\bar{G}, \hat{K}) \|_\infty < 1 \). The relationship between the entropies is given in the following lemma.

**Lemma 6:**

\[
\mathcal{I}[\mathcal{F}(\bar{G}, \hat{K})] = \mathcal{I}[\mathcal{F}(\hat{G}, \hat{K})] + l \ln \det [I - (D_{11})_0 (D_{11})_0^\dagger].
\]

**Proof:** By Lemma 1

\[
\mathcal{I}[\mathcal{F}(\bar{G}, \hat{K})] = \mathcal{I}[\mathcal{F}(\hat{G}, \hat{K})] + \mathcal{I}(U_{11})
\]

\[
= \mathcal{I}[\mathcal{F}(\hat{G}, \hat{K})] + l \ln \det [I - U_{22} \mathcal{F}(\hat{G}(0), \hat{K}(0))].
\]

Since \( U_{11} \) is a constant operator function

\[
\mathcal{I}(U_{11}) = -\ln \det [I - U_{11}^* U_{11}]
\]

\[
= -\ln \det [I - (\hat{D}_{11})_0 (\hat{D}_{11})_0^\dagger]
\]

\[
= -\ln \det [I - (\hat{D}_{11})_0 (\hat{D}_{11})_0^\dagger].
\]

Note that \( U_{22} \in \mathcal{N}(\{\hat{Z}_r\}, \{\hat{W}_r\}) \) and

\[
\mathcal{F}(\hat{G}(0), \hat{K}(0)) = \hat{D}_{11} + \hat{D}_{12} \hat{K}(0)[I + \hat{D}_{22} \hat{K}(0)]^{-1} \hat{D}_{21}
\]

whose first term is in \( \mathcal{N}(\{\hat{W}_r\}, \{\hat{Z}_r\}) \) and second term in \( \mathcal{N}_s(\{\hat{W}_r\}, \{\hat{Z}_r\}) \). Hence

\[
\ln | \det [I - U_{22} \mathcal{F}(\hat{G}(0), \hat{K}(0))] | = \ln | \det [I - (\hat{D}_{11})_0 (\hat{D}_{11})_0^\dagger] |
\]

\[
= \ln | \det [I - (\hat{D}_{11})_0 (\hat{D}_{11})_0^\dagger] |
\]

The result then follows.

A state-space model of \( \bar{G} \) can again be computed

\[
\hat{G}(\lambda) = \begin{bmatrix}
\hat{G}_{11}(\lambda) & \hat{G}_{12}(\lambda) \\
\hat{G}_{21}(\lambda) & \hat{G}_{22}(\lambda)
\end{bmatrix} = \begin{bmatrix}
A & B_1 & B_2 \\
C_1 & D_{11} & D_{12} \\
C_2 & D_{21} & D_{22}
\end{bmatrix}.
\]

Since \( U_{11} \) is diagonal, i.e., \( U_{11} \in \mathcal{N}(\{\hat{W}_r\}, \{\hat{Z}_r\}) \) and \( U_{11}^* \in \mathcal{N}(\{\hat{W}_r\}, \{\hat{Z}_r\}) \), it follows:

\[
\hat{D}_{11} = U_{11} + U_{12} \hat{D}_{11}(I - U_{22} \hat{D}_{11})^{-1} U_{21}
\]

\( \in \mathcal{N}(\{\hat{W}_r\}, \{\hat{Z}_r\}) \)

\[
\hat{D}_{12} = U_{12}(I - \hat{D}_{11} U_{22})^{-1} \hat{D}_{12} \in \mathcal{N}(\{\mathcal{U}_r\}, \{\hat{Z}_r\})
\]

\[
\hat{D}_{21} = \hat{D}_{21}(I - U_{22} \hat{D}_{11})^{-1} U_{21} \in \mathcal{N}_s(\{\hat{W}_r\}, \{\mathcal{Y}_r\})
\]

\[
\hat{D}_{22} = \hat{D}_{22}(I - U_{22} \hat{D}_{11})^{-1} U_{22} \hat{D}_{22} \hat{D}_{22} + \hat{D}_{22}
\]

\( \in \mathcal{N}_s(\{\mathcal{U}_r\}, \{\mathcal{Y}_r\}) \).

Note that the diagonal blocks of \( \hat{D}_{11} \) have been cancelled by the linear fractional transformation, resulting in a strictly
Then the advantage of \( G \) over \( \hat{G} \) is that all operators \( \bar{D}_{11}, \bar{C}_{11}, \bar{D}_{12} \), and \( \bar{D}_{21} \) are of finite rank. Therefore, if we define

\[
\mathcal{Z} = \text{Im} [\bar{C}_1 \bar{D}_{11} - \bar{D}_{12}], \quad \mathcal{W} = \left( \text{Ker} \begin{bmatrix} \bar{D}_1 \\ \bar{D}_{11} \\ \bar{D}_{21} \end{bmatrix} \right) ^{\perp}
\]

and

\[
G = \begin{bmatrix} \Pi_{2} \bar{G}_{11} |_{\mathcal{W}} & \Pi_{2} \bar{G}_{12} \\ \bar{C}_{21} |_{\mathcal{W}} & \bar{C}_{22} \end{bmatrix}
\]

then \( G \) has finite-dimensional input and output spaces and

\[
\| F(\hat{G}, \hat{K}) \|_{\infty} = \| F(G, \hat{K}) \|_{\infty}
\]

\[
\mathcal{T}[F(\hat{G}, \hat{K})] = \mathcal{T}[F(G, \hat{K})]
\]

The nests \( \{ \hat{V}_r \} \) and \( \{ \hat{Z}_r \} \) induce nests in \( \mathcal{W} \) and \( \mathcal{Z} \) in a natural way

\[
\mathcal{W}_r = \mathcal{W} \cap \hat{V}_r, \quad \mathcal{Z}_r = \mathcal{Z} \cap \hat{Z}_r.
\]

Assume that a state-space model of \( G \) is

\[
\hat{G}(\lambda) = \begin{bmatrix} \hat{G}_{11}(\lambda) & \hat{G}_{12}(\lambda) \\ \hat{G}_{21}(\lambda) & \hat{G}_{22}(\lambda) \end{bmatrix} = \begin{bmatrix} A \\ C_1 \\ C_2 \end{bmatrix} \begin{bmatrix} B_1 \\ B_{12} \\ B_{21} \end{bmatrix}.
\]

The following structure of \( G \) is inherited from that of \( \hat{G} \):

\[
D_{11} \in \mathcal{N}(\mathcal{W}_r), \{ \mathcal{Z}_r \}) \quad (12)
\]

\[
D_{12} \in \mathcal{N}(\mathcal{U}_r), \{ \mathcal{Z}_r \}) \quad (13)
\]

\[
D_{21} \in \mathcal{N}(\mathcal{V}_r), \{ \mathcal{Y}_r \}) \quad (14)
\]

\[
D_{22} \in \mathcal{N}(\mathcal{U}_r), \{ \mathcal{Y}_r \}) \quad (15)
\]

In summary, our original hybrid time-varying control problem with plant \( G_a \) and controller \( K_{mp} \) can be converted into a finite-dimensional LTI control problem with plant \( G \) and controller \( K \), as shown in Fig. 3, in the sense that the system in Fig. 3 is internally stable iff the system in Fig. 1 is internally stable

\[
\| F(G_a, \mathcal{K} K_{mp} S) \|_{\infty} < 1
\]

and

\[
\mathcal{T}[F(\hat{G}, \hat{K})] = \mathcal{T}[F(G, \hat{K} K_{mp} S)]
\]

\[
+ \ln \det [I - (D_{11})^* (D_{11})_0].
\]

A state-space model of \( G \) can be computed from that of \( G_a \) using the techniques developed in [5]. Any \( \hat{K} \) satisfying (11) resulted from the design can be converted into a feasible multirate controller \( K_{mp} \). We would like to emphasize, however, that the finite-dimensional LTI problem has a nonconventional constraint on the controller \( K \) given by (11). This constraint is the causality constraint. Also, the LTI plant \( G \) obtained from \( G_a \) will automatically satisfy (12)–(15).

In order for the \( H_\infty \) problem for the finite-dimensional LTI generalized plant \( G \) to be solvable, we need the following.

1) \( (A, B_2, C_2) \) is stabilizable and detectable. Some of the existing techniques to solve the \( H_\infty \) problem for \( G \) require \( G \) to satisfy the following additional conditions.

2) \[
\ker \begin{bmatrix} A - \lambda I & B_2 \\ C_1 & D_{12} \end{bmatrix} = \{0\} \text{ for all } |\lambda| = 1.
\]

3) \[
\left( \text{range} \begin{bmatrix} A - \lambda I & B_1 \\ C_2 & D_{21} \end{bmatrix} \right) ^{\perp} = \{0\} \text{ for all } |\lambda| = 1.
\]

First it is shown in [35] that if:

1a) \( (A_a, B_{a2}, C_{a2}) \) is stabilizable and detectable and \( \sigma \) is nonpathological with respect to \( A_a \);

2a) \( (C_{a1}, A_a) \) has no unobservable modes on the imaginary axis, \( (A_a, B_{a2}, C_{a1}, D_{a12}) \) is right-invertible and has no zero at 0;

3a) \( (A_a, B_{a1}) \) has no uncontrollable modes on the imaginary axis and \( (A_a, B_{a1}, C_{a2}, 0) \) is left-invertible;

then:

1) \( (\hat{A}, \hat{B}_2, \hat{C}_2) \) is stabilizable and detectable.

2) \[
\ker \begin{bmatrix} \hat{A} - \lambda \hat{I} & \hat{B}_2 \\ \hat{C}_1 & \hat{D}_{12} \end{bmatrix} = \{0\} \text{ for all } |\lambda| = 1.
\]

3) \[
\left( \text{range} \begin{bmatrix} \hat{A} - \lambda \hat{I} & \hat{B}_1 \\ \hat{C}_2 & \hat{D}_{21} \end{bmatrix} \right) ^{\perp} = \{0\} \text{ for all } |\lambda| = 1.
\]

Now assume that 1a)–3a) and hence \( \hat{1})–\hat{3}) \), are satisfied. Then it follows from the same argument as in [20, Section IV-F] that conditions 1)–3) are satisfied if there exists an internally stabilizing multirate controller \( K_{mp} \) such that

\[
\| F(G_a, \mathcal{H} K_{mp} S) \|_{\infty} < 1.
\]

V. ALL \( H_\infty \) SUBOPTIMAL CONTROLLERS

AND THE MINIMUM ENTROPY CONTROLLER

In this section, we first characterize all \( \hat{K} \) satisfying the causality constraint (11) such that the system shown in Fig. 3 is internally stable and \( \| F(\hat{G}, \hat{K}) \|_{\infty} < 1 \). This problem differs from the standard \( H_\infty \) problem only in the causality constraint on \( \hat{K} \) and is hence called a constrained \( H_\infty \) problem. Our strategy in solving this problem is first to characterize all \( \hat{K} \) such that the system in Fig. 3 is internally stable and \( \| F(\hat{G}, \hat{K}) \|_{\infty} < 1 \) without considering the causality constraint (this is a standard \( H_\infty \) problem) and then choose, if possible, from this characterization all those satisfying the causality constraint.

Several solutions to the standard \( H_\infty \) problem exist in the literature. Here we adopt the solution in [22]. Note that it is assumed in [22] that \( D_{a2}^* D_{21} > 0 \) and \( D_{21} D_{21}^* > 0 \); these assumptions are not satisfied for the equivalent LTI system \( G \). However, they are not essential and the solution in [22] can be modified accordingly by following, e.g., the idea in [39]. Assume the solvability conditions are satisfied, then
all stabilizing controllers $K$ satisfying $||\mathcal{F}(\hat{G}, \hat{K})||_{\infty} < 1$ are characterized by

$$\{ \hat{K} = \mathcal{F} \left( \begin{bmatrix} 0 & I \\ I & -D_{22} \end{bmatrix} \right) \ast \hat{M}, \hat{\Phi} : \hat{\Phi} \in \mathcal{R}H_{\infty}, ||\hat{\Phi}||_{\infty} < 1, \\ I + D_{22} \mathcal{F}[\hat{M}(0), \hat{\Phi}(0)] \text{ is invertible} \}$$

(16)

where $\hat{M} = \begin{bmatrix} \hat{M}_{11} \\ \hat{M}_{12} \end{bmatrix}$ is not uniquely given in [22] and by using Lemma 5 we can always choose $\hat{M}$ so that

$$\hat{M}_{12}(0) \in \mathcal{N}(\{U_r\}), \hat{M}_{21}(0) - 1 \in \mathcal{N}(\{Y_r\}), \hat{M}_{22}(0) = 0$$

and furthermore, $\hat{M}_{12}(0)$ and $\hat{M}_{21}(0)$ are invertible.

**Theorem 5:** The constrained $H_\infty$ problem is solvable iff the corresponding unconstrained problem is solvable and

$$\max_{\hat{K}} ||(I - \Pi_{\hat{u}_r})\hat{M}_{12}(0)^{-1}\hat{M}_{11}(0)\hat{M}_{21}(0)^{-1}||_{\infty} < 1. \quad (17)$$

**Proof:** Obviously, the corresponding unconstrained problem has to be solvable in order for the constrained problem to be solvable. Assume that the unconstrained problem is solvable. Since $D_{22} \in \mathcal{N}_s(\{U_r\}, \{Y_r\})$, it follows that $\hat{K}(0) \in \mathcal{N}(\{Y_r\}, \{U_r\})$ iff

$$\mathcal{F}[\hat{M}(0), \hat{\Phi}(0)] = \hat{M}_{11}(0) + \hat{M}_{12}(0)\hat{\Phi}(0)\hat{M}_{21}(0)$$

$$\in \mathcal{N}(\{Y_r\}, \{U_r\}).$$

Pre- and postmultiply this by $\hat{M}_{12}(0)^{-1}$ and $\hat{M}_{21}(0)^{-1}$, respectively, to get

$$\hat{M}_{12}(0)^{-1}\mathcal{F}[\hat{M}(0), \hat{\Phi}(0)]\hat{M}_{21}(0)^{-1} = \hat{M}_{12}(0)^{-1}\hat{M}_{11}(0)\hat{M}_{21}(0)^{-1} + \hat{\Phi}(0).$$

It follows from Theorem 1 that in order to have $\mathcal{F}[\hat{M}(0), \hat{\Phi}(0)] \in \mathcal{N}(\{Y_r\}, \{U_r\})$ and $||\hat{\Phi}(0)||_{\infty} < 1$, we must have (17). Conversely, if (17) is true, then there exists a constant matrix $\hat{\Phi}$ with $||\hat{\Phi}||_{\infty} < 1$ such that

$$\hat{M}_{12}(0)^{-1}\hat{M}_{11}(0)\hat{M}_{21}(0)^{-1} + \hat{\Phi} \in \mathcal{N}(\{Y_r\}, \{U_r\}).$$

Hence

$$\hat{K} = \mathcal{F} \left( \begin{bmatrix} 0 & I \\ I & -D_{22} \end{bmatrix} \right) \ast \hat{M}, \hat{\Phi}$$

achieves $\hat{K}(0) \in \mathcal{N}(\{Y_r\}, \{U_r\})$. \hfill \Box

If the conditions in Theorem 5 are satisfied, then there exists

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \in \mathcal{N}(\{Y_r \oplus U_r\}, \{U_r \oplus Y_r\})$$

with $P_{22} \in \mathcal{N}_s(\{U_r\}, \{Y_r\})$ and $P_{12}$ and $P_{21}$ invertible such that

$$U = \begin{bmatrix} -\hat{M}_{12}^{-1}(0)\hat{M}_{11}(0)\hat{M}_{21}^{-1}(0) + P_{11} \\ P_{21}^{-1} \end{bmatrix}$$

is unitary. Define

$$\hat{N} = \begin{bmatrix} 0 & I \\ I & -D_{22} \end{bmatrix} \ast \hat{M} \ast U,$$

It is easy to check that $\hat{N}(0) \in \mathcal{N}(\{Y_r \oplus U_r\}, \{U_r \oplus Y_r\})$, $\hat{N}_{12}(0)$, and $\hat{N}_{21}(0)$ are invertible, and $\hat{N}_{22}(0) \in \mathcal{N}_s(\{U_r\}, \{Y_r\})$. By setting $\hat{\Phi} = \mathcal{F}(U, \hat{\Psi})$, the set (16) can be rewritten as

$$\{ \hat{K} = \mathcal{F}(\hat{N}, \hat{\Psi}) : \hat{\Psi} \in \mathcal{R}H_{\infty}, ||\hat{\Psi}||_{\infty} < 1, \\ I - \hat{N}_{22}(0)\hat{\Psi}(0) \text{ is invertible} \}.$$

Now we can state the main result of this paper.

**Theorem 6:** Assume the solvability of the constrained $H_\infty$ problem. Then the set of all controllers solving the problem is given by

$$\{ \hat{K} = \mathcal{F}(\hat{N}, \hat{\Psi}) : \hat{\Psi} \in \mathcal{R}H_{\infty}, ||\hat{\Psi}||_{\infty} < 1, \\ \hat{\Psi}(0) \in \mathcal{N}(\{Y_r\}, \{U_r\}) \},$$

(18)

**Proof:** First notice that $I - \hat{N}_{22}(0)\hat{\Psi}(0)$ is always invertible if $\hat{\Psi}(0) \in \mathcal{N}(\{Y_r\}, \{U_r\})$. Since $\hat{N}(0) \in \mathcal{N}(\{Y_r \oplus U_r\}, \{U_r \oplus Y_r\})$ and $\hat{N}_{12}(0)$ and $\hat{N}_{21}(0)$ are invertible, it follows that $\hat{K}(0) \in \mathcal{N}(\{Y_r\}, \{U_r\})$ iff $\hat{\Psi}(0) \in \mathcal{N}(\{Y_r\}, \{U_r\})$, then the result follows immediately. \hfill \Box

Theorem 6 gives a characterization of all $H_\infty$ controllers in terms of a linear fractional transformation of an attractive $H_\infty$ function satisfying the causality constraint. Clearly, this characterization is not unique in general, although the set of such controllers is unique. It is then of interest to explicitly characterize the nonuniqueness. Let $\hat{N}$ be another matrix satisfying $\hat{N}(0) \in \mathcal{N}(\{Y_r \oplus U_r\}, \{U_r \oplus Y_r\})$ and $\hat{N}_{22}(0) \in \mathcal{N}_s(\{U_r\}, \{Y_r\})$ such that

$$\{ \mathcal{F}(\hat{N}, \hat{\Psi}) : \hat{\Psi} \in \mathcal{R}H_{\infty}, ||\hat{\Psi}||_{\infty} < 1, \hat{\Psi}(0) \in \mathcal{N}(\{Y_r\}, \{U_r\}) \}$$

$$= \{ \mathcal{F}(\hat{\tilde{N}}, \hat{\Phi}) : \hat{\Phi} \in \mathcal{R}H_{\infty}, ||\hat{\Phi}||_{\infty} < 1, \hat{\Phi}(0) \in \mathcal{N}(\{Y_r\}, \{U_r\}) \}.$$

Then it can be shown using the standard theory on linear fractional transformation (LFT) (see [20, Ch. 4] for example) that

$$\hat{N} = \hat{N} \ast \begin{bmatrix} 0 & W_{12} \\ W_{21} & 0 \end{bmatrix}$$

where $W_{12}$ is a unitary matrix belonging to $\mathcal{N}(U_r) \cap \mathcal{N}(Y_r)$ and $W_{21}$ is a unitary matrix belonging to $\mathcal{N}(Y_r) \cap \mathcal{N}(U_r)$, where $\mathcal{N}(U_r)$ is the set of adjoints of members in $\mathcal{N}(U_r)$ and similarly for $\mathcal{N}(Y_r)$, i.e., $W_{12}$ and $W_{21}$ are block diagonal unitary matrices. In particular, this implies $\hat{N}_{11} = \hat{\tilde{N}}_{11}$. Since $\hat{N}_{11}$ is a particular $H_\infty$ suboptimal controller by setting $\hat{\Phi} = 0$ in (18), we call it the central $H_\infty$ controller. Notice that the central controller with the causality constraint is different from the central controller without the causality constraint.

In the rest of this section, we show that the central controller obtained by setting $\hat{\Phi} = 0$ in (18) is the controller which minimizes $I[\mathcal{F}(\hat{G}, \hat{K})]$. Now let us go back to the characterization given in [22]. It is known (see [33] for the continuous-time case) that if all $H_\infty$ suboptimal controllers are characterized by (16), then all $H_\infty$
suboptimal closed-loop transfer functions are characterized by
\[\mathcal{F}(\hat{G}, \hat{K}) = \{ \mathcal{F}\left(\hat{R}, \begin{bmatrix} \hat{\Phi} & 0 \\ 0 & 0 \end{bmatrix}\right) : \hat{\Phi} \in \mathcal{R}H_{\infty}, \|\hat{\Phi}\|_{\infty} < 1, \]
\[I + D_{22}\mathcal{F}[\hat{M}(0), \hat{\Phi}(0)] \text{ is invertible} \}\]
where
\[\hat{R} = \begin{bmatrix} \hat{R}_{11} & \hat{R}_{12} & \hat{R}_{13} \\ \hat{R}_{21} & \hat{R}_{22} & \hat{R}_{23} \\ \hat{R}_{31} & \hat{R}_{32} & \hat{R}_{33} \end{bmatrix} \in \mathcal{R}H_{\infty}\]
is para-unitary satisfying \(\hat{R}_{31}^{-1} \in \mathcal{R}H_{\infty}\). Clearly we have
\[\begin{bmatrix} \hat{R}_{11} & \hat{R}_{12} \\ \hat{R}_{21} & \hat{R}_{22} \end{bmatrix} = \begin{bmatrix} 0 & I \\ I & -D_{22} \end{bmatrix} \hat{M}\]
and \(\hat{R}_{22}(0) = 0\). Because of this, the \(\mathcal{H}_{\infty}\) controller without the causality constraint which minimizes the entropy
\[\mathcal{I}[\mathcal{F}(\hat{G}, \hat{K})]
\]
is conveniently given by \(\hat{K} = \mathcal{F}(\hat{M}, 0) = \hat{M}_{11}\). Notice that \(\hat{\Phi} = \mathcal{F}(\hat{U}, \hat{V})\) gives
\[\begin{bmatrix} \hat{\Phi} & 0 \\ 0 & 0 \end{bmatrix} = \mathcal{F}(V, \begin{bmatrix} \hat{\Psi} & 0 \\ 0 & 0 \end{bmatrix})\]
where
\[V = \begin{bmatrix} -\hat{M}_{21}^{-1}(0)\hat{M}_{11}(0)\hat{M}_{21}^{-1}(0) + P_{11} & 0 & P_{22} & 0 \\ 0 & P_{21} & 0 & P_{22} \end{bmatrix}\]
\[\hat{\Psi}(0) \in \mathcal{N}(\{\mathcal{V}_{\tau}\}, \{\mathcal{U}_{\tau}\})\}
where
\[\hat{S} = \begin{bmatrix} \hat{S}_{21} & \hat{S}_{22} & \hat{S}_{23} \\ \hat{S}_{21} & \hat{S}_{22} & \hat{S}_{23} \end{bmatrix} = \hat{R} \times V \in \mathcal{R}H_{\infty}\].
Since \(\hat{R}\) is para-unitary and \(V\) is unitary, it follows that \(\hat{S}\) is para-unitary. It can be checked that \(\hat{S}_{21}^{-1} \in \mathcal{R}H_{\infty}\) and \(S_{22}(0) \in \mathcal{N}_{\infty}(\{\mathcal{V}_{\tau}\})\). By Lemma 1
\[\mathcal{I}[\mathcal{F}(\hat{G}, \hat{K})] = \mathcal{I}(\begin{bmatrix} \hat{\Psi} & 0 \\ 0 & 0 \end{bmatrix}) + 2\ln \det \left( I + \begin{bmatrix} \hat{S}_{22}(0) & \hat{S}_{23}(0) \\ \hat{S}_{22}(0) & \hat{S}_{33}(0) \end{bmatrix} \begin{bmatrix} \hat{\Psi}(0) & 0 \\ 0 & 0 \end{bmatrix} \right)\]
\[= \mathcal{I}(\hat{\Psi}) + \mathcal{I}(\hat{S}_{11}) + 2\ln |\det [I - \hat{S}_{22}(0)\hat{\Psi}(0)]|\]
\[= \mathcal{I}(\hat{\Psi}) + \mathcal{I}(\hat{S}_{11}).\]
The last equality is due to the fact \(\hat{S}_{22}(0)\hat{\Psi}(0) \in \mathcal{N}_{\infty}(\{\mathcal{V}_{\tau}\})\).
Therefore, the minimum of \(\mathcal{I}[\mathcal{F}(\hat{G}, \hat{K})]\) is achieved at \(\hat{\Psi} = 0\). The following theorem is thus obtained.

**Theorem 7:** The minimum entropy controller is given by
\(\hat{K} = \hat{N}_{11}\).
That is, the minimum entropy controller (with the causality constraint) is given by the central \(\mathcal{H}_{\infty}\) controller (with the causality constraint).

**APPENDIX A**

**Proof of Theorem 1**

The proof of Theorem 1 follows from the idea in [18] but has two complications: 1) operator-valued transfer functions are treated, which requires dealing with random variables in Hilbert spaces [41] and 2) signals are defined on time \([0, \infty)\) instead of \((-\infty, \infty)\), which requires treating nonstationary stochastic processes. Since \(F_a\) is linear, it follows that \(z\) is a Gaussian process. Define \(z_T\) as the stochastic process on \([0, T]\) such that \(z_T(t) = z(t)\) for \(t \in [0, T]\). Then \(z_T\) can be considered as a Gaussian random variable in the Hilbert space \(L_2[0, T]\). The covariance operator \(V_T : L_2[0, T] \rightarrow L_2[0, T]\) is then given by \((t \in [0, T])\)
\[(V_T x)(t) = \mathbb{E} z_T(t) \int_0^T z_T(\tau) x(\tau) d\tau\]
\[= \int_0^T \mathbb{E} z_T(t) z_T(\tau) x(\tau) d\tau \int_0^T u(\tau) d\tau\]
\[= \int_0^T \mathbb{E} \int_0^T f_a(t, \tau) \mathcal{E}[w(\tau) u(\tau)] x(\tau) d\tau d\tau\]
\[= \int_0^T \int_0^T f_a(t, \tau) \mathcal{E}[w(\tau) u(\tau)] x(\tau) d\tau d\tau\]
\[= \int_0^T \int_0^T f_a(t, \tau) f_a(\tau, \tau) x(\tau) d\tau d\tau\]
\[= (F_a F_a^*) x(t)\]
This shows that \(V_T = \Pi_{L_2[0, T]} F_a F_a^*\). Since \(\Pi_{L_2[0, T]} F_a F_a^*\) is a contractive Hilbert–Schmidt operator and \(F_a\) is causal, it follows that \(V_T\) is a self-adjoint contractive nuclear operator. Let the Schmidt expansion of \(V_T\) be
\[V_T = \sum_{i=1}^\infty \sigma_i \mathcal{V}_i\]
Then \(z_T\) can be expressed as
\[z_T = \sum_{i=1}^\infty \alpha_i \mathcal{V}_i\]
and \( \alpha_i, i = 1, 2, \cdots \), are independent scalar Gaussian random variables with covariance \( \sigma_i \). Hence

\[
E \left\{ \exp \left[ \frac{1}{2} \int_0^T z'(t)z(t) \, dt \right] \right\} = E \left\{ \exp \left[ \frac{1}{2} \sum_{i=1}^\infty \alpha_i^2 \right] \right\}
\]

Now lift \( w \) to get \( \omega \) and lift \( z \) to get \( \zeta \). Then \( z = F_\omega w \) is equivalent to \( \zeta = F_\omega \omega \) and \( F \) has a matrix representation

\[
F = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\vdots & \vdots & \vdots
\end{bmatrix}
\]

Let \( F_K \) be the leading \( K \times K \) submatrix of \( F \). Then

\[
E \left\{ \exp \left[ \frac{1}{2} \int_0^{K\sigma} z'(t)z(t) \, dt \right] \right\} = \det (I - F_K F_K^* )^{-1/2}.
\]

Since \( \tilde{F} \) has only a finite number of poles, the infinite Hankel matrix

\[
H = \begin{bmatrix}
f(1) & f(1) & f(3) & \cdots \\
f(1) & f(3) & f(4) & \cdots \\
f(3) & f(4) & f(5) & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

has finite rank. Let \( H_K \) be the first \( K \) block rows of \( H \) and define

\[
W_K = F_K F_K^* + H_K H_K^*.
\]

Notice that \( W_K \) is a self-adjoint Toeplitz matrix, as shown in (c) at the bottom of the page, and \( w(i) \) is the \( i \)th Fourier coefficient of \( \tilde{F}\tilde{F}^* \), where \( \tilde{F}(\lambda) = \tilde{F}(\lambda^{-1})^* \). Denote by \( \sigma_i(W_K) \) and \( \sigma_i(F_K F_K^* ) \), \( i = 1, 2, \cdots \), the singular values of \( W_K \) and \( F_K F_K^* \) respectively assuming ordered nondecreasingly. Then

\[
\sum_{i=1}^\infty |\sigma_i(W_K) - \sigma_i(F_K F_K^* )| \leq \text{tr} H_K H_K^* \leq \text{tr} H H^* < \infty.
\]

Since \( \sigma_i(W_K) \) and \( \sigma_i(F_K F_K^* ) \) are all contained in \( [-\|\tilde{F}\|_2,\|\tilde{F}\|_2] \), it follows that

\[
\left| \ln \det (I - F_K F_K^*) - \ln \det (I - W_K) \right| \leq \sum_{i=1}^\infty |\sigma_i(W_K) - \sigma_i(F_K F_K^* )| \leq \frac{1}{1 - \|\tilde{F}\|_2^2} \sum_{i=1}^\infty |\sigma_i(W_K) - \sigma_i(F_K F_K^* )|
\]

for some \( \xi_i \in [-\|\tilde{F}\|_2,\|\tilde{F}\|_2] \). This shows that

\[
\left| \ln \det (I - F_K F_K^*) - \ln \det (I - W_K) \right| \leq \frac{1}{1 - \|\tilde{F}\|_2^2} \sum_{i=1}^\infty |\sigma_i(W_K) - \sigma_i(F_K F_K^* )| \leq \frac{1}{1 - \|\tilde{F}\|_2^2} \text{tr} H H^*.
\]

Hence by using the operator-valued strong Szego–Widom limit theorem [9, Th. 6.4]

\[
\lim_{K \to \infty} \Omega_{K\sigma} = -\lim_{K \to \infty} \frac{1}{K\sigma} \ln \det (I - F_K F_K^* ) = -\lim_{K \to \infty} \frac{1}{K\sigma} \ln \det (I - W_K) = -\frac{1}{2\pi\sigma} \int_{-\pi}^\pi \ln \det (I - \tilde{F} e^{i\omega}) \tilde{F}^*(e^{i\omega}) \, d\omega = \frac{1}{\sigma} \mathcal{I}(\tilde{F}).
\]

Notice that for \( K\sigma < T < (K + 1)\sigma \)

\[
\frac{K}{K + 1} \Omega_{K\sigma} \leq \Omega_T \leq \frac{K + 1}{K} \Omega_{(K + 1)\sigma}.
\]

Therefore

\[
\lim_{T \to \infty} \Omega_T = \mathcal{I}(\tilde{F})/\sigma.
\]

**APPENDIX B**

**PROOF OF THEOREM 2**

The equivalence of 1) and 2) follows from the Arveson’s distance formula [12]. That 3) implies 1) is obvious. It remains to show that 2) implies 3). For this, we need a technical lemma.

**Lemma 7:** Assume the matrices \( E, F, \) and \( H, \) of appropriate dimensions, satisfy the conditions

\[
\begin{bmatrix}
E & F \\
G & H
\end{bmatrix} = I, \quad \begin{bmatrix} F \\ H \end{bmatrix} = 0, \quad \begin{bmatrix} E^* \\ F^* \\ H^* \end{bmatrix} = 0.
\]

Then there exists a matrix \( G \) satisfying

\[
\begin{bmatrix}
E & F \\
G & H
\end{bmatrix} \leq 1, \quad \begin{bmatrix} E^* \\ F^* \\ H^* \end{bmatrix} = 0, \quad \begin{bmatrix} G^* \\ H^* \end{bmatrix} < 1.
\]

An explicit formula for such a matrix is \( G = -HF^*(EE^*)^{-1}E \). 

\[
W_K = \begin{bmatrix}
w(0) & w(-1) & w(-2) & \cdots & w(-K + 1) \\
w(1) & w(0) & w(-1) & \cdots & w(-K + 2) \\
w(2) & w(1) & w(0) & \cdots & w(-K + 3) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
w(K - 1) & w(K - 2) & w(K - 3) & \cdots & w(0)
\end{bmatrix}
\]
Proof: It follows from [13] that there exists a matrix $G$ such that
\[ \left\| \begin{bmatrix} E & F \\ G & H \end{bmatrix} \right\|^2 \leq 1. \]
Among all such $G$ characterized in [13] in terms of a free contractive matrix, the “central” one obtained by setting the free contractive matrix to zero is
\[ G = -HF^*(I - FF^*)^{-1}E = -HF^*(EE^*)^{-1}E. \]
Using this $G$, we have
\[ \begin{bmatrix} G & H \end{bmatrix} \begin{bmatrix} E^* \\ F^* \end{bmatrix} = -HF^*(EE^*)^{-1}EE^* + HF^* = 0 \]
and
\[ \begin{bmatrix} G & H \end{bmatrix} \begin{bmatrix} G^* \\ H^* \end{bmatrix} = H(I - F^*F)^{-1}H^* = H(I - F^*F)^{-1}H^* < I. \]
The last inequality follows from $\left\| \begin{bmatrix} H \\ F \end{bmatrix} \right\|^2 < 1$. \hfill \square
To avoid awkward notation in the proof of Theorem 2, we redefine
\[ \begin{bmatrix} A & B \\ C & D \end{bmatrix} := \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}. \]
Under the decompositions of $\mathcal{X}$ and $\mathcal{Y}$ in (3) and (4), we get the matrix representation shown in (d) at the bottom of the page. Statement 2) becomes
\[ \max_i \left\| \begin{bmatrix} T_{1(i+1)} & \cdots & T_{in} \\ \vdots & \ddots & \vdots \\ T_{n(i+1)} & \cdots & T_{mn} \end{bmatrix} \right\| < 1. \]
We need to decide $A_{ij}, B_{ij}, C_{ij}$ for $i \geq j$, and $D_{ij}$ for $i > j$. This will be done in the following order: In the $i$th step, determine those blocks in the $(n+i)$th row and the $i$th row.

Step 1: Set $C_{11} = I$, $T_{11} + A_{11} = 0$, and choose $B_{11}$ so that
\[ [T_{12} \cdots T_{1n} B_{11}] \]
is a co-isometry. Statement 2) implies that any $B_{11}$ chosen in this way is nonsingular.

Step $i, i = 2, \cdots, n - 1$: Set $C_{ii} = 0$ and choose the rest of the $(n+i)$th row so that it is a co-isometry and is orthogonal to all of the previously determined rows. This requires
\[ [C_{i2} \cdots C_{ii} D_{i1} \cdots D_{i(i-1)}] \]
to be an isometry onto the kernel of the matrix shown in (e) at the bottom of the page. Then set $T_{i1} + A_{i1} = 0$ and choose
\[ [T_{i2} + A_{i2} \cdots T_{ii} + A_{ii} B_{i1} \cdots B_{i(i-1)}] \]
in such a way so that the matrix shown in (f), at the bottom of the page, is a contraction and it is orthogonal to all previously determined block rows. This is possible following Lemma 7.
condition 3), and the fact that the matrix shown in (g), at the top of the page, is a co-isometry. Finally determine $B_{ii}$ so that

$$
\begin{bmatrix}
T_{12} & \cdots & T_{1i} & T_{1(i+1)} & \cdots & T_{1n} & B_{11} & \cdots & 0 \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\
T_{i-1}^{(i-1)2} + A_{i-1}^{(i-1)2} & \cdots & T_{i-1}^{(i-1)i} & T_{i-1}^{(i-1)(i+1)} & \cdots & T_{i-1}^{(i-1)n} & B_{i-1}^{(i-1)1} & \cdots & B_{i-1}^{(i-1)(i-1)} \\
C_{i2} & \cdots & 0 & \cdots & \cdots & 0 & D_{i1} & \cdots & D_{i(i-1)} \\
\end{bmatrix}
$$

is a co-isometry. By Lemma 7, any $B_{ii}$ chosen in such a way is nonsingular.

**Step n:** Set $C_{n1} = 0$ and choose the rest of the $2n$th row so that it is orthogonal to all the previously determined rows. This requires

$$
[C_{n2} \cdots C_{mn} D_{n1} \cdots D_{n(n-1)}]^*$$

to be an isometry onto the kernel of the matrix shown in (h) at the top of the page.

Finally set

$$
\begin{bmatrix}
T_{n1} + A_{n1} & \cdots & T_{nm} + A_{nm} & B_{n1} & \cdots & B_{n(n-1)} \\
\vdots & & \vdots & \vdots & & \vdots \\
\end{bmatrix} = 0
$$

and $B_{nm} = I$.

The above construction guarantees that the matrix

$$
\begin{bmatrix}
T & A \\
C & D
\end{bmatrix}
$$

is unitary, $B$ is invertible, and $D \in N_{\mathcal{S}}(\{\mathcal{X}_2\}, \{\mathcal{X}_2\})$. The invertibility of $C$ follows from that of $B$ and the fact that the matrix in (19) is unitary.

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