

## ON A SPECIAL CASE OF HADWIGER'S CONJECTURE

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### Abstract

Hadwiger's Conjecture seems difficult to attack, even in the very special case of graphs  $G$  of independence number  $\alpha(G) = 2$ . We present some results in this special case.

**Keywords:** Hadwiger's Conjecture, complete minor, independence number, connected matching.

**2000 Mathematics Subject Classification:** 05C15, 05C83.

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\*Work supported by INTAS (project code 97-1001).

<sup>†</sup>Work supported by the Danish Natural Science Research Council.

## 1. Introduction

Hadwiger's Conjecture is the major unsolved problem in graph coloring theory. Even for graphs of independence number  $\alpha = 2$  a proof has proved elusive so far. This has led to speculation that the conjecture might be false, even in this special case.

The special case was first considered by Duchet and Meyniel [7], but it was W. Mader, who in a private communication a few years ago made clear to us how interesting the special case is.

Graphs  $G$  with independence number  $\alpha(G) \leq 2$  may at first seem rather restricted, but noticing that this is equivalent to  $\overline{G}$  being  $K_3$ -free and remembering the wide variety of  $K_3$ -free graphs, one may consider  $\alpha \leq 2$  as a mild restriction only.

The purpose of this paper is to investigate how far one can go by standard methods in an attempt to solve the special case  $\alpha = 2$ .

By the determination of the order of magnitude of the Ramsey number  $r(3, n)$  by Kim [11] there is a constant  $c > 0$  such that there exist graphs  $G$  on  $n$  vertices, with  $\alpha(G) = 2$  and clique number  $\omega(G) \leq c\sqrt{n \log n}$ . Again this indicates the non-triviality of Hadwiger's Conjecture for  $\alpha = 2$ : The chromatic number  $\chi(G)$  is at least  $n/2$  (since every color can be used at most twice), so although we want  $G$  to have at least a  $K_{\lceil n/2 \rceil}$  as a minor, it may have only a complete graph of order  $\sqrt{n \log n}$  as a subgraph.

For such a graph  $G$  to have  $K_{\lceil n/2 \rceil}$  as a minor one needs to make at least  $n/2 - c\sqrt{n \log n}$  contractions into single vertices of connected subgraphs on  $\geq 2$  vertices. Most of these connected subgraphs will be complete 2-graphs (because there are only  $n$  vertices altogether). That is,  $G$  will have a large matching with every two matching edges joined by at least one edge. We shall call such a matching *connected*. Thus the problem to find large connected matchings in graphs  $G$  with  $\alpha(G) = 2$  is closely related to Hadwiger's Conjecture for  $\alpha = 2$ . The problem of finding a large connected matching in a general graph is NP-hard, as we shall see in Section 7.

Duchet and Meyniel [7] proved that a graph  $G$  always has  $K_{\lceil n/(2\alpha(G)-1) \rceil}$  as a minor, i.e., a graph  $G$  with  $\alpha(G) \leq 2$  has  $K_{\lceil n/3 \rceil}$  as a minor. (This is easily proved by induction, contracting an induced path of length 2 when possible.) P. Seymour has asked if one can at least prove that there is a positive  $\epsilon$  such that any graph  $G$  with  $\alpha(G) \leq 2$  has  $K_{\lceil (1/3+\epsilon)n \rceil}$  as a minor.

In the present paper we shall present a large number of properties possessed by a smallest counterexample to Hadwiger's Conjecture for  $\alpha = 2$ . Moreover, we shall prove that the conjecture is true for several infinite families of  $\alpha = 2$  graphs and their inflations. An *inflation* of a graph is obtained by replacing its vertices by complete graphs. Note that inflations of  $\alpha = 2$  graphs likewise have  $\alpha = 2$ .

Our results support the following extended conjecture:

**EH.** (Extended Hadwiger's Conjecture for  $\alpha = 2$ ). Every graph  $G$  having  $\alpha(G) = 2$  has a connected matching  $M$  such that the contractions of the edges in  $M$  to  $|M|$  single vertices result in a graph containing a  $K_{\lfloor |V(G)|/2 \rfloor}$ .

## 2. Notation

Let  $G$  denote a finite simple undirected graph with vertex set  $V(G)$  and edge set  $E(G)$ . We shall often denote  $|V(G)|$  by  $n$ . A vertex set is *independent* if no two of its members are adjacent. The cardinality of any largest independent set in  $G$  is called the *independence number* of  $G$  and is denoted by  $\alpha(G)$  or just  $\alpha$  when graph  $G$  is understood. A graph  $G$  is said to be  $\alpha$ -*critical* if for every edge  $e \in E(G)$ ,  $\alpha(G - e) > \alpha(G)$ . An edge  $e = xy$  in  $G$  is said to be a *dominating edge* if every vertex of  $G$  different from  $x$  and from  $y$  is adjacent to at least one of  $x$  and  $y$ . A *matching* in  $G$  is a set of edges no two of which share a vertex. A matching  $M$  in  $G$  is said to be *connected* if every pair of edges of  $M$  are joined by at least one edge. A matching  $M$  in  $G$  is said to be *dominating* if every vertex in  $G - V(M)$  is adjacent to at least one endvertex of every edge of  $M$ . We shall write  $x \sim y$  ( $x \not\sim y$ ) when vertices  $x$  and  $y$  are (are not) adjacent. A graph  $H$  obtained from a graph  $G$  by deletions (of vertices and/or edges) and/or contractions (of edges) is a *minor* of  $G$ . We express this relation between the graphs  $G$  and  $H$  by  $G \succeq H$  (or by  $H \preceq G$ ). As usual, the chromatic number of  $G$  is denoted by  $\chi(G)$ , the vertex connectivity by  $\kappa(G)$  and the minimum degree of  $G$  by  $\delta(G)$ . For  $X \subseteq V(G)$ , we denote by  $G[X]$  the subgraph of  $G$  induced by  $X$ , further,  $G - X = G[V(G) - X]$ .

## 3. Hadwiger's Conjecture

H. Hadwiger presented his conjecture in a colloquium of the Eidgenössische Technische Hochschule in Zürich on December 15, 1942. The conjecture

resulted from Hadwiger's suggestion that graph coloring should be studied in terms of a combinatorial classification of graphs, rather than in terms of classification based upon embeddings as was the more common approach of the time. The new classification used the maximum  $k$  for which  $G$  has  $K_k$  as a minor and the conjecture simply stated that the chromatic number  $\chi(G)$  is at most this number  $k$ .

**H1.** Hadwiger's Conjecture [10].  $\forall G: G \succeq K_{\chi(G)}$ .

Toft [18] gave a comprehensive survey of H1. We shall consider the conjecture in the special case where the independence number  $\alpha(G)$  of  $G$  is  $\leq 2$ :

**H2.** Hadwiger's Conjecture for  $\alpha = 2$ .  $\forall G: \alpha(G) \leq 2 \Rightarrow G \succeq K_{\chi(G)}$ .

Since  $\alpha(G) \leq 2$  implies that  $\chi(G) \geq |V(G)|/2$ , it follows immediately that  $H2 \Rightarrow H3$ , where H3 is the following conjecture:

**H3.** Hadwiger's Conjecture for  $\alpha = 2$ .  $\forall G: \alpha(G) \leq 2 \Rightarrow G \succeq K_{\lceil |V(G)|/2 \rceil}$ .

As we shall see below,  $H3 \Rightarrow H2$  also; hence we have given H2 and H3 the same name.

Let us suppose in the following that the graph  $G$  is a smallest possible counterexample for H2 in terms of the number of vertices (i.e., we assume that H2 is false and that  $G$  is a counterexample with a smallest possible  $|V(G)|$ ). In the following we shall obtain properties (1), (2),  $\dots$ , (19) about such a graph  $G$ . Of course  $\alpha(G) = 2$ . Since each color class in a coloring has size at most 2, it follows that  $|V(G)| \leq 2\chi(G)$ .

If  $|V(G)| = 2\chi(G)$  then for an arbitrary  $x \in V(G)$  we have that  $2\chi(G) - 1 = |V(G - x)| \leq 2\chi(G - x)$  hence  $\chi(G) - \frac{1}{2} \leq \chi(G - x) \leq \chi(G)$ , i.e.,  $\chi(G - x) = \chi(G)$ . But  $G \succeq G - x$  and  $G - x \succeq K_{\chi(G-x)} = K_{\chi(G)}$  (by the minimality of  $G$ ). Hence  $G \succeq K_{\chi(G)}$ , contradicting the fact that  $G$  is a counterexample to H2. Therefore  $|V(G)| \leq 2\chi(G) - 1$ .

If  $\chi(G - x) = \chi(G)$  for a vertex  $x \in V(G)$  then by the minimality of  $G$  we get a contradiction as above. So

- (1)  $\forall x \in V(G): \chi(G - x) < \chi(G)$ , i.e.,  $G$  is *vertex-critical*.

Moreover

- (2) The complement  $\overline{G}$  of  $G$  is connected.

Assume, to the contrary, that the complement of  $G$  is disconnected. This means that  $G$  consists of two disjoint graphs  $G_1$  and  $G_2$  completely joined by edges. Moreover, by the minimality of  $G$ ,  $G_1 \succeq K_{\chi(G_1)}$  and  $G_2 \succeq K_{\chi(G_2)}$ , hence  $G \succeq K_{\chi(G_1)+\chi(G_2)} = K_{\chi(G)}$ , contradicting that  $G$  is a counterexample to H2. This proves (2).

Assume now that  $|V(G)| \leq 2\chi(G) - 2$ . By a deep theorem of Gallai [8],  $\overline{G}$  is disconnected, contradicting (2) above. Hence

$$(3) \quad |V(G)| = 2\chi(G) - 1.$$

From  $G \not\succeq K_{\chi(G)}$  and (3) above, it follows that  $G \not\succeq K_{(|V(G)|+1)/2} = K_{\lceil |V(G)|/2 \rceil}$ . Hence  $G$  is a counterexample also to H3. This proves that H3  $\Rightarrow$  H2. Hence

**Theorem 3.1.** *The conjectures H2 and H3 are equivalent. More precisely: any counterexample to H3 is a counterexample to H2, and any smallest (in terms of  $|V(G)|$ ) counterexample to H2 is a counterexample to H3.*

Let us now assume that  $G_1$  is a counterexample to H2 which has a smallest chromatic number. Then  $\chi(G_1) \leq \chi(G)$  (by the minimality of  $G_1$ ) and  $|V(G_1)| \geq |V(G)|$  (by the minimality of  $G$ ). Then from  $\alpha(G_1) = 2$ , we get  $2\chi(G) - 1 = |V(G)| \leq |V(G_1)| \leq 2\chi(G_1) \leq 2\chi(G)$ . The only option is that  $\chi(G_1) = \chi(G)$ . This proves:

**Theorem 3.2.** *If  $G$  is a smallest counterexample to H2 in terms of  $|V(G)|$ , then  $G$  is a smallest counterexample to H2 in terms of  $\chi(G)$ .*

Theorem 3.2 is interesting since it is not known if the statement holds with H2 replaced by H1. This is a problem due to A.A. Zykov (see Toft [18]).

A further property of  $G$  can be derived from  $\alpha(G - x) \leq \alpha(G) = 2$  and  $2\chi(G - x) = 2\chi(G) - 2 = |V(G)| - 1 = |V(G - x)|$ , namely that  $G - x$  has a  $(\chi(G) - 1)$ -coloring in which each color class has size exactly 2. That means

$$(4) \quad \forall x \in V(G): \overline{G} - x \text{ has a perfect matching, i.e., the complement } \overline{G} \text{ of } G \text{ is factor-critical.}$$

Choose  $xy \in E(G)$  and let  $H$  denote the graph obtained from  $G$  by contracting  $xy$  to a new vertex  $z$ . Then  $H$  has one vertex less than  $G$ . Hence  $G \succeq H \succeq K_{\chi(H)}$  by the minimality of  $G$ . Therefore  $\chi(H) < \chi(G)$ . A  $(\chi(G) - 1)$ -coloring of  $H$  gives immediately a  $(\chi(G) - 1)$ -coloring of  $G - xy$

by giving  $x$  and  $y$  the color of  $z$  and retaining all other colors. But in a  $(\chi(G) - 1)$ -coloring of the  $2\chi(G) - 1$  vertices of  $G - xy$  at least one color class must have size  $\geq 3$ , and hence:

(5)  $\forall xy \in E(G): \alpha(G - xy) = 3 > 2 = \alpha(G)$ , i.e.,  $G$  is  $\alpha$ -critical.

Finally, let  $H$  be a proper minor of  $G$ ; i.e.,  $G \succeq H$  and  $G \neq H$ . If  $H$  has fewer vertices than  $G$ , then  $H \succeq K_{\chi(H)}$ , so  $\chi(H) < \chi(G)$ . If  $|V(H)| = |V(G)|$  then  $H \subseteq G - xy$  for some edge  $xy$  of  $G$  since  $H$  is a proper minor, so  $\chi(H) \leq \chi(G - xy) < \chi(G)$ . In any case  $\chi(H) < \chi(G)$ . So

(6)  $\forall H, G \succeq H$  and  $G \neq H: \chi(H) < \chi(G)$ , i.e.,  $G$  is *contraction-critical*.

We still assume that  $G$  is a smallest counterexample to H2 (and hence to H3 by Theorem 1). By (3) and (6),  $G$  is non-complete contraction-critical. A large number of properties follow from this, as listed in the paper by Toft [18]:

(7)  $\chi(G) \geq 7$  (Robertson, Seymour and Thomas [16]).

(8)  $G$  is 7-vertex-connected (Dirac [6] and Mader [14]).

(9)  $\delta(G) \geq \chi(G)$  (Dirac [6]) and  $G$  is  $\chi(G)$ -edge-connected (Toft [17]).

The proof of result (7) above depends on the truth of the Four Color Theorem. However, for the special case  $\alpha = 2$ , this can of course be established in a much more elementary way. By (3) above, the cases when  $\chi \leq 6$  deal with graphs having at most 11 vertices and as we shall see, these are easy to handle.

From (3) and (9) and a well known theorem of Dirac [5], we get (10), which in turn, together with (3), implies (11).

(10)  $G$  is Hamiltonian.

(11)  $G$  is factor-critical.

Concerning matchings, the following property is easily proved.

(12)  $G$  does not contain a non-empty connected dominating matching.

**Proof.** Suppose  $M \neq \emptyset$  is a connected dominating matching in  $G$ . Then  $G' = G - V(M)$  is smaller than  $G$  and hence  $G' \succeq K_{\lceil n'/2 \rceil} = K_{\lceil n/2 \rceil - |M|}$ . Contracting the edges of  $M$  into  $|M|$  single vertices, we thus obtain a  $K_{\lceil n/2 \rceil}$

as a minor of  $G$ , since  $M$  is connected and dominating. But  $G$  is a counterexample to H2, so this is a contradiction. ■

Let  $H$  be an arbitrary graph. By a 2-path of  $H$  we mean an induced subpath of length 2 in  $H$ . Clearly,  $H$  does not contain any 2-path if and only if every component of  $H$  is a complete graph. If  $H$  has a 2-path  $P$  and  $\alpha(H) = 2$ , then contracting  $P$  to one vertex results in that vertex being joined to all other vertices. This simple observation leads to the following result.

**Theorem 3.3.** *Let  $G$  be any graph with  $\alpha(G) \leq 2$ . Let  $n = |V(G)|$  and  $\omega = \omega(G)$ . Then the following statements hold.*

- (a)  $G \succeq K_{\lceil(\omega+n)/3\rceil}$ .
- (b) If  $n \geq 2k - 1$  and  $\omega \geq k - 2$ , then  $G \succeq K_k$ .

**Proof.** If  $\alpha(G) = 1$ , part (a) is trivial. So we assume  $\alpha(G) = 2$ . We proceed to prove statement (a) by induction on  $n$ . Let  $K = K_\omega$  be a maximum complete subgraph of  $G$  and let  $H = G - V(K)$ . If  $H$  contains a 2-path  $P$ , then the induction hypothesis implies that  $G' = G - V(P)$  has  $K_{\lceil(\omega+(n-3))/3\rceil} = K_{\lceil(\omega+n)/3\rceil-1}$  as a minor. (Note that  $K \subseteq G'$  and hence  $\omega(G') = \omega(G) = \omega$ .) Since  $\alpha(G) = 2$ , this gives  $G \succeq K_{\lceil(\omega(G')+n)/3\rceil} = K_{\lceil(\omega(G)+n)/3\rceil}$ . If  $H$  does not contain any 2-path, then  $H$  is either a complete graph or the disjoint union of two complete graphs. In both cases we claim that  $\omega \geq \lceil n/2 \rceil$ . In the first case, this is evident. In the second case,  $H$  is the disjoint union of two complete graphs, say  $H_1$  and  $H_2$  and, because of  $\alpha(G) = 2$ , every vertex of the complete subgraph  $K$  is either joined to all vertices of  $H_1$  or to all vertices of  $H_2$ . This implies the claim. Consequently,  $G \succeq K_\omega \succeq K_{\lceil(\omega+n)/3\rceil}$ . Thus statement (a) is proved.

For the proof of (b) suppose, on the contrary, that  $G \not\succeq K_k$ . Then  $\alpha(G) = 2$ , and from (a) it follows that  $n = 2k - 1$  and  $\omega = k - 2$ . Since the non-neighbors of any vertex in  $G$  induce a complete graph, this implies that  $\delta(G) \geq k$ . Then, because the Ramsey number  $r(3, 3) = 6$ , we conclude that  $k \geq 6$ . Note that in case  $k = 5$  we have  $n = 9$  and  $\delta(G) \geq 5$  implying that one vertex of  $G$  has degree at least 6 and thus  $\omega \geq 4$ .

Now, consider an arbitrary maximum complete subgraph  $K = K_{k-2}$  of  $G$  and let  $H = G - V(K)$ . If  $H$  does not contain any 2-path, then  $H$  is either a complete graph or the disjoint union of two complete graphs and, as in the proof of (a), we conclude that  $\omega \geq \lceil n/2 \rceil = k$ , a contradiction. If  $H$  has two vertex disjoint 2-paths, then contraction of both these 2-paths results in a  $K_k$ , a contradiction, too. Consequently,  $H$  has one, but not two

vertex disjoint, 2-paths. Moreover, if  $P$  is any 2-path of  $H$ , then  $H - V(P) = G - V(K) - V(P)$  is either a complete graph or the disjoint union of two complete graphs.

Next, consider an arbitrary 2-path  $P = xzy$  of  $G - V(K)$ . We claim that  $G - V(P)$  contains two vertex disjoint maximum complete subgraphs, say  $K'$  and  $K''$ . This is evident if  $G - V(K) - V(P)$  is a complete graph. Otherwise,  $G - V(K) - V(P)$  is the disjoint union of two complete graphs, say  $H_1$  and  $H_2$ , and, because of  $\alpha(G) = 2$ , every vertex of  $K$  is joined to all vertices of either  $H_1$  or  $H_2$ . Since  $|V(K)| + |V(H_1)| + |V(H_2)| = 2k - 4$  and  $\omega = k - 2$ , this also implies the claim. Thus the claim is proved.

Since  $\omega = k - 2$ , vertex  $x$  has a non-neighbor  $x'$  in  $K'$  and  $y$  has a non-neighbor  $y'$  in  $K'$ . Moreover,  $\alpha(G) = 2$  implies that  $x' \neq y'$  and that  $x'y$  and  $xy'$  are edges of  $G$  (see Figure 3.1).

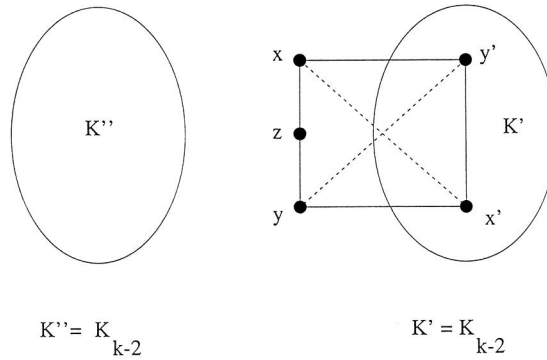


Figure 3.1

Since  $x'y'x$  and  $y'x'y$  are 2-paths and there are not two disjoint 2-paths in  $G - V(K'')$ , the vertices  $x, z$  and  $y$  are each either joined to all vertices of  $H = K' - x' - y'$  or to none. Since  $\alpha(G) = 2$ , we may assume that  $x$  is joined to all vertices of  $H$ . If  $y$  is also joined to all vertices of  $H$ , then any two vertices  $x''$  and  $y''$  of  $H$  produce two disjoint 2-paths  $x'x''x$  and  $y'y''y$  of  $G - V(K'')$ , a contradiction. Note that  $|V(H)| = |V(K')| - 2 = k - 4 \geq 2$ . If, on the other hand, vertex  $y$  is not joined to any vertex of  $H$ , then the degree of  $y$  in  $G$  is at most  $k - 3 + 2 = k - 1$ , contrary to  $\delta(G) \geq k$ . Note that  $y$  is not joined to all vertices of  $K'' = K_{k-2}$ , since  $\omega = k - 2$ .

Therefore, we have obtained a contradiction in all cases, and hence statement (b) is proved. ■



For the minimum counterexample  $G$  for H2, we then conclude from Theorem 3.3 and property (3) that

$$(13) \quad \omega(G) \leq \chi(G) - 3.$$

Since the non-neighbors of any vertex in  $G$  induce a complete graph, we conclude from properties (13) and (3) that

$$(14) \quad \delta(G) \geq \chi(G) + 1.$$

**Theorem 3.4.** *Let  $G$  be any connected graph with  $\alpha(G) = 2$  and  $\kappa(G) \leq |V(G)|/2$ . Then  $G$  contains a non-empty connected dominating matching.*

**Proof.** Let  $S$  be a minimum vertex cut in  $G$  and suppose that the (necessarily exactly) two components of  $G - S$  are  $A$  and  $B$  respectively. Then both induced subgraphs  $G[A]$  and  $G[B]$  are complete. Moreover, each vertex in  $S$  is either joined to all vertices in  $A$  or to all vertices in  $B$ .

Let  $S_A = \{v \in S | v \sim \text{every vertex in } A\}$  and  $S_B = \{v \in S | v \sim \text{every vertex in } B\}$ . Let  $s_A = |S_A|$ ,  $s_B = |S_B|$ ,  $s = |S| = \kappa(G)$ ,  $a = |A|$  and  $b = |B|$ .

*Case 1.* Suppose  $s_A \leq b$  and  $s_B \leq a$ . Let  $M_A$  be a complete matching of  $S_B$  into  $A$  and let  $M_B$  be a complete matching of  $S_A$  into  $B$ . (Both must exist by Hall's Theorem or Menger's Theorem.) Then  $M = M_A \cup M_B$  contains a connected dominating matching in  $G$ .

*Case 2.* Now suppose  $s_A > b$  and assume that  $a \geq s - b$ . Then let  $M_B$  be a complete matching of  $B$  into  $S_A$ . (Again such must exist by Hall's Theorem or Menger's Theorem.) Now let  $M_B(B)$  denote the set of all vertices in  $S_A$  matched by  $M_B$ . Then  $S - M_B(B)$  has size  $s - b \leq a$ . So there exists a complete matching  $M'$  of  $S - M_B(B)$  into  $A$ . Finally, let  $M = M_B \cup M'$ . Then  $M$  is a connected dominating matching in  $G$ .

So there remains to consider only the case when  $s_A > b$  and  $a < s - b$ . But then  $s > a + b$  and  $s + (a + b) = |V(G)|$ . Thus  $s = \kappa(G) > |V(G)|/2$ , a contradiction. ■

The next property then follows from Theorem 3.4 and properties (3) and (12).

$$(15) \quad \kappa(G) \geq \chi(G).$$

Now let  $x$  and  $y$  be two arbitrary non-adjacent vertices of  $G$ . Let  $B$  be the set of common neighbors of  $x$  and  $y$ , let  $A$  be the remaining (private) neighbors of  $x$  and let  $C$  be the remaining (private) neighbors of  $y$ . Then, since  $\alpha(G) = 2$ ,  $G[A]$  and  $G[C]$  are both complete. Moreover,  $B \neq \emptyset$  since  $G$  does not contain a  $K_{\lceil |V(G)|/2 \rceil}$ .

- (16) Let  $b$  be any member of  $B$ . Then  $b$  has at least one non-neighbor in  $A$  and at least one non-neighbor in  $C$ . (Hence, in particular, both  $A$  and  $C$  are non-empty.)

**Proof.** By (5), the edge  $xb$  is critical, i.e.,  $\alpha(G - xb) = 3$ . The common non-neighbor  $c$  of  $x$  and  $b$  must lie in  $C$ . Similarly, edge  $yb$  is critical and the common non-neighbor  $a$  of  $y$  and  $b$  must lie in  $A$ . ■

- (17) Let  $a \in A$  and  $c \in C$ . Then  $a$  is adjacent to  $c$  if and only if there is a common non-neighbor  $b \in B$  of  $a$  and  $c$ .

**Proof.** If  $a$  and  $c$  have a common non-neighbor  $b$ , they are joined by an edge, since  $\alpha(G) = 2$ . Conversely, if  $a$  and  $c$  are joined by an edge  $ac$ , then  $\alpha(G - ac) = 3$ , implying a common non-neighbor  $b$  of  $a$  and  $c$ . The vertex  $b$  must belong to  $B$ , since  $\alpha(G) = 2$ . ■

- (18) If sets  $A, B$  and  $C$  are as above, then  $2 \leq |A| \leq k - 4$ ,  $2 \leq |C| \leq k - 4$  and  $5 \leq |B| \leq 2k - 7$ , where  $k = \chi(G)$ .

**Proof.** Since  $x$  is joined to all vertices of  $A$ , (13) implies that  $|A| \leq k - 4$ . Similarly,  $|C| \leq k - 4$ . But then  $|B| = 2k - 3 - |A| - |C| \geq 5$ .

We know  $A$  and  $C$  are each non-empty, so suppose  $|A| = 1$ . Then by (16) there are no edges between  $A$  and  $B$ . Hence  $G[B]$  is complete. Then either  $B$  or  $C$  induces a complete graph of size at least  $k - 2$ , contradicting (13). ■

- (19) Any two non-adjacent vertices  $x$  and  $y$  in  $G$  are joined by at least five 2-paths. Moreover, every 2-path in  $G$  is part of an induced  $C_5$ .

**Proof.** The first part follows immediately from the fact that  $|B| \geq 5$  and the second part follows from (16) and (17) above. ■

## 4. Inflations

Given a graph  $G$ , we say that graph  $H = \text{inf}(G)$  is an *inflation* of  $G$  if each vertex  $v$  of  $G$  is replaced by a complete graph  $K^v$  (or the empty set) and if vertices  $u$  and  $v$  of  $G$  are adjacent, then in  $H$  every vertex of  $K^u$  is joined to every vertex of  $K^v$ . We call the complete graph  $K^u$  which replaces vertex  $u$  an *atom* of the inflation. Clearly, inflation preserves the property of having  $\alpha \leq 2$ . Hence, H3 implies that any inflation  $H$  obtained from a graph  $G$  with  $\alpha(G) \leq 2$  would satisfy  $H \succeq K_{\lceil |V(H)|/2 \rceil}$ .

We have been unable to prove that every inflation  $H$  obtained from a graph  $G$  with  $\alpha(G) = 2$  satisfies  $H \succeq K_{\lceil |V(H)|/2 \rceil}$ , even when  $G$  itself satisfies H3. However, in Section 6 we shall prove

**Theorem 4.1.** *In any inflation on  $n$  vertices of a graph  $G$  with  $\alpha(G) \leq 2$  and  $|V(G)| \leq 11$ , there exists a dominating connected matching  $M$  such that by contracting the edges of  $M$ , one obtains a graph containing a  $K_{\lceil n/2 \rceil}$ .*

We shall also prove Hadwiger's Conjecture for inflations of the following infinite family. For  $k \geq 1$ , we define a family of graphs  $C_{3k-1}^{k-1}$  as follows. Let  $C_2^0 = \overline{K_2}$ . Now suppose  $k \geq 2$ . Arrange  $3k - 1$  vertices in a cycle. Now for each  $k$  successive vertices on this cycle, join every pair.

**Theorem 4.2.** *In each inflation  $G$  of  $C_{3k-1}^{k-1}$  having  $k \geq 1$  and  $n$  vertices, there exists a connected dominating matching  $M$  such that by contracting the edges of  $M$ , one obtains a graph containing a  $K_{\lceil n/2 \rceil}$ .*

**Proof.** The proof is by induction on  $k$ . For  $k = 1$ , graph  $G$  consists of two disjoint complete graphs and hence  $G \supseteq K_{\lceil n/2 \rceil}$  and  $M = \emptyset$  suffices. Next suppose  $k = 2$ . Choose a smallest atom in  $C_5^1$  and label it  $B_1$ . Now let the remaining four atoms be labelled  $B_2, B_3, B_4$  and  $B_5$  either clockwise or counterclockwise in such a way that  $|B_4| \leq |B_3|$ . Now let  $M_1$  be a matching of  $B_1$  into  $B_2$  which covers all of  $B_1$  and let  $M_2$  be a matching of  $B_4$  into  $B_3$  which covers all of  $B_4$ . Then  $M_0 = M_1 \cup M_2$  is a connected dominating matching. The unmatched vertices form two complete graphs, namely  $B_5$  and an induced complete subgraph of  $B_2 \cup B_3$ . Take the larger of these two subgraphs and call it  $B_0$ . Then contracting all the edges of matching  $M_0$  we obtain a minor (containing  $B_0$ ) which has at least  $n/2$  vertices and is complete. So we are done when  $k = 2$ .

So suppose the theorem is true for all  $k' < k$  and consider an inflation of  $C_{3k-1}^{k-1}$ . Choose the smallest atom and denote it by  $B_1$ . Number the rest of the atoms (clockwise or counterclockwise) by  $B_2, \dots, B_{3k-1}$  so that  $|V(B_{2k})| \leq |V(B_{k+1})|$ .

Now let  $M_1$  be a matching of all of  $B_1$  into  $B_k$  and let  $M_2$  be a matching of all of  $B_{2k}$  into  $B_{k+1}$ . Observe that  $M_0 = M_1 \cup M_2$  is then a connected dominating matching. Next let  $D_k$  denote the complete graph spanned by all unmatched vertices of  $B_k \cup B_{k+1}$ . (See Figure 4.1.)

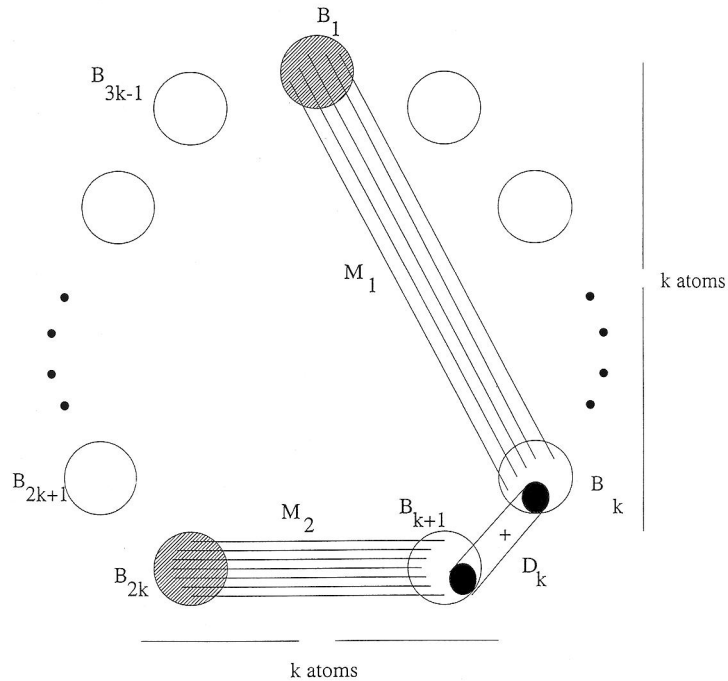


Figure 4.1

Then replacing  $B_k \cup B_{k+1}$  with  $D_k$  and deleting  $B_1$  and  $B_{2k}$ , we obtain an inflation of  $C_{3(k-1)-1}^{k-2}$ . But by induction hypothesis, this graph contains a connected dominating matching  $M'$  which contracts to a complete graph on half its number of vertices. So then  $M \cup M'$  is a connected dominating matching in the inflation of  $C_{3k-1}^{k-1}$  and hence upon contraction this matching yields a complete graph on half the number of vertices of the inflation and the theorem is proved. ■

## 5. Induced Subgraphs

**Theorem 5.1.** *Let  $G$  be a graph with  $n$  vertices and with  $\alpha(G) \leq 2$ . If  $G$  does not contain an induced  $C_5$ , then either  $G$  contains  $K_{\lceil n/2 \rceil}$  or else  $G$  contains a dominating edge.*

**Proof.** Assume that  $G$  does not contain a dominating edge. Then  $G$  is not complete, and there are two vertices  $x, y \in V(G)$  such that  $x \not\sim y$ . Denote by  $A$  the set of neighbors of  $x$  which are not neighbors of  $y$ , by  $C$  the set of neighbors of  $y$  which are not neighbors of  $x$  and by  $B$  the set of vertices adjacent to both  $x$  and  $y$ .

Since  $\alpha(G) \leq 2$ , both induced subgraphs  $G[A \cup \{x\}]$  and  $G[C \cup \{y\}]$  are complete. Therefore, if  $B = \emptyset$ , then  $G$  contains a  $K_{\lceil n/2 \rceil}$  and we are done. Otherwise, consider an arbitrary vertex  $b \in B$ . Since  $G$  does not contain a dominating edge, there is a common non-neighbor  $c$  of  $b$  and  $x$  as well as a common non-neighbor  $a$  of  $b$  and  $y$ . Clearly,  $a \in A$  and  $c \in C$  and, because  $\alpha(G) \leq 2$ ,  $ac$  is an edge of  $G$ . But then  $axbyc$  is an induced  $C_5$  in  $G$ , contrary to hypothesis. ■

**Corollary 5.2.** *Let  $G$  be a graph with  $n$  vertices and with  $\alpha(G) \leq 2$ . If  $G$  does not contain an induced  $C_5$ , then  $G \succeq K_{\lceil n/2 \rceil}$ .* ■

Figure 5.1 displays the six graphs  $G_4^1, G_4^2, G_4^3, G_4^4, G_4^5, G_4^6$  as well as graph  $H_7$ , which are used in the next theorem and the next corollary.

**Theorem 5.3.** *Let  $G$  be a graph with  $n$  vertices and with  $\alpha(G) \leq 2$ . If  $G$  does not contain an induced  $H_7$ , then  $G \succeq K_{\lceil n/2 \rceil}$ .*

**Proof.** We use induction on  $n$ . For  $n \leq 4$ , the result is clear since  $\alpha(G) \leq 2$ .

So suppose  $n \geq 5$  and suppose the result is true for graphs with fewer than  $n$  vertices and let  $G$  be a graph with  $n$  vertices. If  $G$  does not contain an induced  $C_5$ , we are done by Corollary 5.2. So suppose  $G$  contains an induced  $C_5 = abcdea$ . If  $M = \{ab, cd\}$  is a connected dominating matching in  $G$ , then since  $G - a - b - c - d \succeq K_{\lceil (n-4)/2 \rceil}$  by the induction hypothesis, we are done. Otherwise, since  $M$  is a connected matching, one of the edges, say  $ab$ , is not dominating and, therefore, there is a common non-neighbor  $z$  of  $a$  and  $b$  in  $G$ . But then  $z$  is adjacent to each of  $c, d$ , and  $e$ . If  $M' = \{ae, bc\}$  is a connected dominating matching, then again we are done. Otherwise, since  $M'$  is a connected matching, one of the two edges, say  $ae$ , is not dominating

and therefore there is a common non-neighbor  $z'$  of  $a$  and  $e$ . But then  $z'$  is adjacent to all of  $b, c, d$  and  $z$ , and  $G[\{a, b, c, d, e, z, z'\}] = H_7$  is an induced subgraph of  $G$ , contrary to the hypothesis. ■

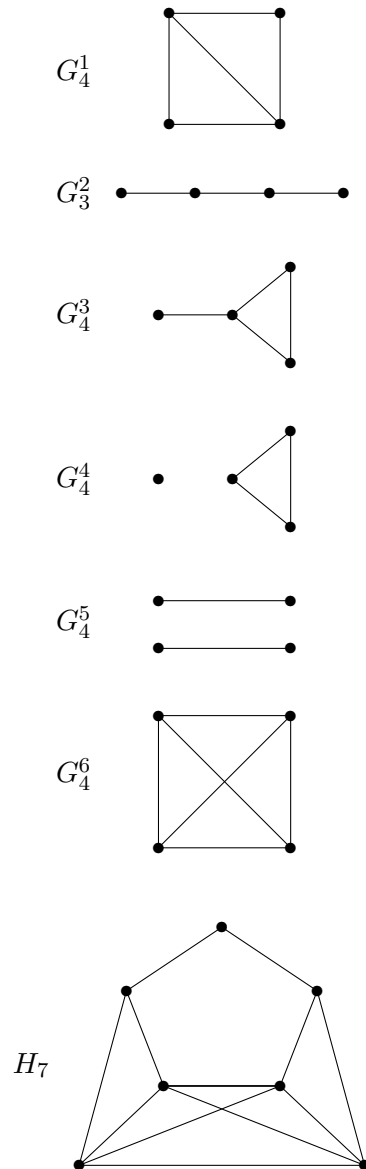


Figure 5.1

**Corollary 5.4.** *Let  $G$  be a graph with  $n$  vertices and with  $\alpha(G) \leq 2$ . Moreover, suppose graph  $H \in \{G_4^1, G_4^2, G_4^3, G_4^4, G_4^5, G_4^6\}$ . If  $G$  does not contain an induced  $H$ , then  $G \succeq K_{\lceil n/2 \rceil}$ . ■*

The list of forbidden induced subgraphs in Corollary 5.4 contains all 4-vertex graphs with  $\alpha \leq 2$ , except  $C_4$ . However  $C_4$  may be added to this list as we shall now show.

**Theorem 5.5.** *Let  $G$  be a graph with  $n$  vertices and with  $\alpha(G) \leq 2$ . If  $G$  does not contain an induced  $C_4$ , then  $G \succeq K_{\lceil n/2 \rceil}$ .*

**Proof.** We use induction on  $n$ . For  $n \leq 4$ , the statement is evident since  $\alpha(G) \leq 2$ .

So suppose  $n \geq 5$ , suppose the result is true for graphs with fewer than  $n$  vertices and let  $G$  be a graph with  $n$  vertices. If  $G$  does not contain an induced  $C_5$ , we are done by Corollary 5.2. So suppose  $G$  contains an induced  $C_5$ . Let  $H$  denote a largest induced inflated  $C_5$  in  $G$  with non-empty atoms. Let the five atoms of the inflated  $C_5$  be denoted by  $B_1, \dots, B_5$  in clockwise order.

First, we claim that every vertex outside  $H$  is adjacent to all vertices of  $H$ . For the proof, suppose on the contrary that there is a vertex  $z \in V(G) - V(H)$  such that  $z \not\sim b$  for some vertex  $b$  of  $H$ , say  $b \in B_1$ . Then  $\alpha(G) \leq 2$  implies that  $z$  is adjacent to all vertices of  $B_3 \cup B_4$ . If  $z$  has a neighbor  $b_2 \in B_2$  as well as a neighbor  $b_5 \in B_5$ , then  $(b, b_2, z, b_5)$  is an induced  $C_4$  in  $G$ , contrary to hypothesis. Therefore, by symmetry, we may assume that  $z$  has no neighbor in  $B_2$ . Then  $\alpha(G) \leq 2$  implies that  $z$  is adjacent to all vertices of  $B_5$  and, hence, to all vertices of  $B_3 \cup B_4 \cup B_5$ . But then  $z$  must be adjacent to some vertex  $b_1 \in B_1$ , since otherwise  $H$  would not be a largest induced inflated  $C_5$  in  $G$ . Thus, for every vertex  $b_2 \in B_2$  and every vertex  $b_3 \in B_3$ , we obtain an induced  $C_4 = (b_2, b_3, z, b_1)$ , contrary to hypothesis. This proves the claim.

Now we construct a matching  $M$  in  $H$  as follows. Let  $B_1$  be a smallest atom of  $H$  and let  $B_3$  and  $B_4$  be the two opposite atoms. By symmetry we may assume that  $|V(B_3)| \leq |V(B_4)|$ . Let  $M_1$  be a matching of all of  $B_3$  into  $B_4$  and let  $M_2$  be a matching of all of  $B_1$  into  $B_5$ . Then  $M = M_1 \cup M_2$  is a non-empty connected dominating matching in  $H$ . But then, since every  $z$  not in  $H$  is adjacent to all vertices in  $H$ , matching  $M$  dominates all such  $z$ 's. Now let  $H' = G - V(M)$ . Now  $\alpha(H') \leq 2$ , so by induction hypothesis,  $H' \succeq K_{\lceil n'/2 \rceil}$ , where  $n' = |V(H')|$ . Now contract all edges of matching  $M$

and we have a complete graph with  $\geq |V(H)|/2 + |M|/2 = |V(G)|/2$  vertices as desired. ■

## 6. $\alpha$ -Criticality

To prove Hadwiger's Conjecture for  $\alpha \leq 2$ , clearly it is sufficient to do so for those graphs with  $\alpha \leq 2$  which are  $\alpha$ -critical.

**Theorem 6.1.** *Suppose  $G$  is a connected graph with  $\alpha(G) \leq 2$  and suppose that  $G$  is  $\alpha$ -critical. Then if  $x$  and  $y$  are any two vertices in  $G$ ,  $d(x, y) \leq 2$ . Moreover, every pair of adjacent edges in  $G$  lie either in a  $K_3$  or a chordless  $C_5$ .*

**Proof.** Suppose  $x$  and  $y$  are any two non-adjacent vertices in  $G$ . Clearly, since  $\alpha(G) \leq 2$ , we have  $d(x, y) \leq 3$  and moreover,  $N(x) \cup N(y) = V(G)$ . So suppose  $d(x, y) = 3$  and let  $P$  be a path of length 3 joining  $x$  and  $y$ . Then  $N(x) \cap N(y) = \emptyset$  and each of  $N(x)$  and  $N(y)$  is a complete graph. Let  $e$  be the edge of  $P$  joining  $N(x)$  and  $N(y)$ . Then  $\alpha(G - e) = 2$ , a contradiction.

Let  $e$  and  $f$  be adjacent edges in  $G$ . In any  $\alpha$ -critical graph, every pair of adjacent edges share a chordless odd cycle. (See [1, 13].) If this cycle were a  $C_7$  or larger, we would have  $\alpha(G) \geq 3$ , a contradiction. ■

For an  $\alpha$ -critical graph  $G$  and two non-adjacent vertices  $x$  and  $y$  of  $G$ , we cannot produce any restriction on the structure of the subgraph induced by the common neighborhood of  $x$  and  $y$ . The next theorem explains why. More particularly, we show that *any* graph with  $\alpha \leq 2$  can be embedded in a graph having  $\alpha = 2$  which is  $\alpha$ -critical.

**Theorem 6.2.** *Let  $H$  be any graph with  $\alpha(H) \leq 2$ . Then there exists an  $\alpha$ -critical graph  $G$  with  $\alpha(G) = 2$  which contains  $H$  as an induced subgraph. Furthermore,  $H$  can be embedded in  $G$  so that there exist two non-adjacent vertices  $x$  and  $y$  in  $G$  such that  $N_G(x) \cap N_G(y) = H$ .*

**Proof.** Let  $V(H) = \{h_1, \dots, h_r\}$  and define two new sets of vertices  $A' = \{a_1, \dots, a_r\}$  and  $C' = \{c_1, \dots, c_r\}$ . Join vertices  $a_i$  and  $h_j$  if and only if  $i \neq j$  and join vertices  $c_i$  and  $h_j$  if and only if  $i \neq j$ .

Now for each edge  $e_{ij} = h_i h_j$  in  $H$ , insert a new vertex  $v_{ij}$  into either  $A'$  or  $C'$  thus obtaining vertex sets  $A \supseteq A'$  and  $C \supseteq C'$ . Then join  $v_{ij}$  to all  $h_k$ ,  $k \neq i$  and  $k \neq j$ . (Note that since each  $v_{ij}$  may be put into either  $A$  or  $C$ , the graph  $G$  under construction is by no means unique.)



Let  $x$  and  $y$  be two new non-adjacent vertices. Join both  $x$  and  $y$  to all vertices of  $H$ , join  $x$  to all vertices of  $A$  and join  $y$  to all vertices of  $C$ .

Now join all vertices of  $A$  to each other and all vertices of  $C$  to each other. Finally, suppose  $a \in A$  and  $c \in C$ . Join  $a$  to  $c$  if and only if there exists an  $h_i$  such that  $a \not\sim h_i \not\sim c$ .

It is then routine to check that  $\alpha(G) = 2$  and that all edges of  $G$  are critical. ■

The following theorem is stated in [2, 3] in complementary form. The equivalence of (i) and (ii) is due to Brandt and the equivalence of (ii) and (iii) is due to Pach [15].

**Theorem 6.3.** *The following three statements are equivalent for any graph  $G$ :*

- (i) *Graph  $G$  is an inflation of the graph  $C_{3k-1}^{k-1}$  (where empty atoms are not allowed), for  $k \geq 1$ .*
- (ii)  *$\alpha(G) = 2$ ,  $G$  is  $\alpha$ -critical and  $G$  does not contain the triangular prism as an induced subgraph.*
- (iii)  *$\alpha(G) = 2$ ,  $G$  is  $\alpha$ -critical and any complete subgraph of  $G$  is in the non-neighborhood of some vertex of  $G$ .* ■

We have shown in Theorem 4.2 that Hadwiger's Conjecture is true for all graphs which satisfy Theorem 6.3. So now let us assume that  $\alpha(G) = 2$ , that  $G$  does contain the triangular prism as an induced subgraph and that  $G$  is  $\alpha$ -critical. Let  $P_6$  denote the triangular prism subgraph of  $G$  and let  $e_1, e_2$  and  $e_3$  be the three edges of  $P_6$  which do not lie in the two triangles of  $P_6$ . Since edge  $e_1$  is critical, there must be a seventh vertex  $v_1$  such that  $v_1$  is not adjacent to either endvertex of  $e_1$ . Similarly there must be vertices  $v_2$  and  $v_3$  relative to  $e_2$  and  $e_3$  respectively. But then since  $\alpha(G) = 2$ , for each  $i = 1, 2, 3$ ,  $v_i$  must be adjacent to the other four vertices of  $P_6$  different from the two endvertices of edge  $e_i$ . In particular, all three  $v_i$  must be distinct and hence  $|V(G)| \geq 9$ . In particular, this implies that Hadwiger's Conjecture is true for all graphs  $G$  having  $\alpha(G) \leq 2$  and  $|V(G)| \leq 8$ , as well as their inflations, since any such graph contains a spanning  $\alpha$ -critical subgraph with  $\alpha \leq 2$  satisfying the conditions of Theorem 6.3. Moreover, we have shown that the  $K_{\lceil n/2 \rceil}$  minor can be obtained by contracting the edges of a connected dominating matching.

But now the set  $\{v_1, v_2, v_3\}$  cannot be independent, so there exists at least one edge among the three  $v_i$ 's. Thus we may suppose we have as an induced subgraph of  $G$  one of the three graphs designated as  $\Gamma_1, \Gamma_2$  and  $\Gamma_3$

shown in Figure 6.1 where the vertices  $v_1, v_2$  and  $v_3$  have been relabeled  $v_5, v_7$  and  $v_9$  respectively.

We label by Cases 1, 2 and 3 the situations when  $G$  contains graphs  $\Gamma_1, \Gamma_2$  and  $\Gamma_3$ , respectively, as induced subgraphs. Note that these three

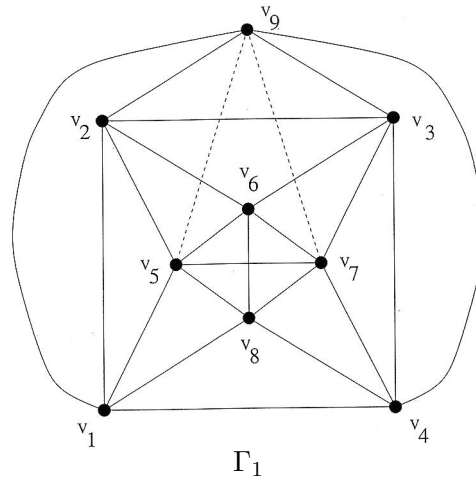


Figure 6.1(a)

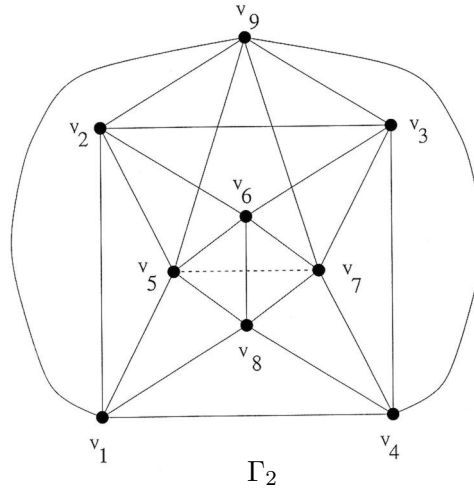


Figure 6.1(b)

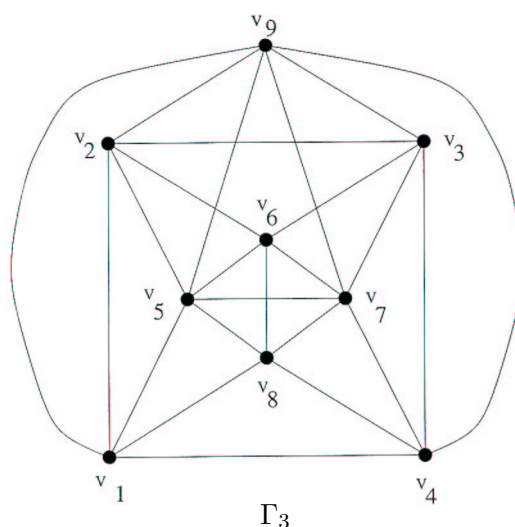


Figure 6.1(c)

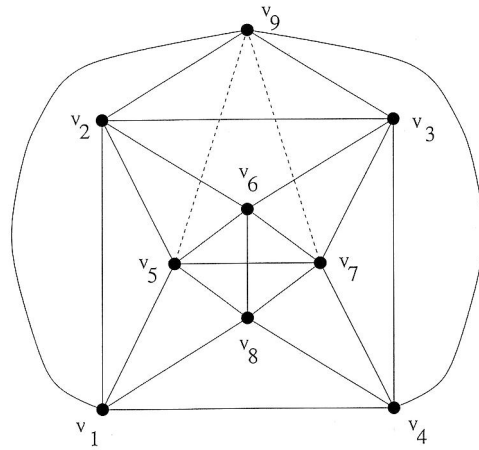
cases are *not* mutually exclusive! We shall use these three cases to prove Theorem 4.1; in particular, we demonstrate that Hadwiger's Conjecture is true for all graphs  $G$  with  $\alpha(G) \leq 2$  and  $|V(G)| \leq 11$  and all inflations of such graphs.

**Theorem 6.4.** *For each  $i = 1, 2, 3$ , a smallest  $\alpha$ -critical graph with  $\alpha = 2$  and containing  $\Gamma_i$  as an induced subgraph is unique. If  $G_i$  denotes this graph, then  $|V(G_1)| = 9$ ,  $|V(G_2)| = 11$  and  $|V(G_3)| = 10$ . The graphs  $G_1, G_2$  and  $G_3$  are shown in Figure 6.2. Moreover,  $G_i$  and any inflation of  $G_i$  satisfies Hadwiger's Conjecture for  $i = 1, 2, 3$ .*

**Proof.** In Case 1, the graph  $\Gamma_1$  is already  $\alpha$ -critical; hence  $G_1 = \Gamma_1$ .

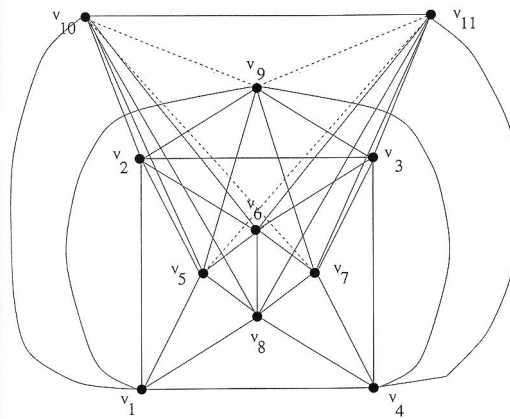
In Case 3, edges  $v_5v_7, v_5v_9$  and  $v_7v_9$  are not critical in  $\Gamma_3$ . Hence  $G_3$  contains at least one new vertex joined neither to  $v_5$  nor  $v_7$ , one new vertex adjacent to neither  $v_5$  nor  $v_9$  and one new vertex joined neither to  $v_7$  nor  $v_9$ . Suppose that these three new vertices are equal; call it  $v_{10}$ . Then  $v_{10}$  is joined to none of  $v_5, v_7$  and  $v_9$ . Since  $\alpha(\Gamma_3) = 2$ ,  $v_{10}$  is joined to all other vertices of  $\Gamma_3$ . It's easy now to check that the resulting graph is  $\alpha$ -critical. Hence the smallest possible  $G_3$  is unique and is, in fact, equal to the complement of the Petersen graph,  $\overline{P_{10}}$ . This completes Case 3.

In Case 2, the edges  $v_5v_9$  and  $v_7v_9$  are not critical in  $\Gamma_2$ . Hence  $G_2$  contains a new vertex  $v_{10}$  joined neither to  $v_7$  nor to  $v_9$  and a new vertex  $v_{11}$  joined to neither  $v_5$  nor to  $v_9$ . The vertices  $v_{10}$  and  $v_{11}$  are not equal, for if they were, the vertices  $v_5, v_7$  and  $v_{10}$  would be independent and this is impossible.



$$G_1 = \Gamma_1$$

Figure 6.2(a)



$$G_2$$

Figure 6.2(b)

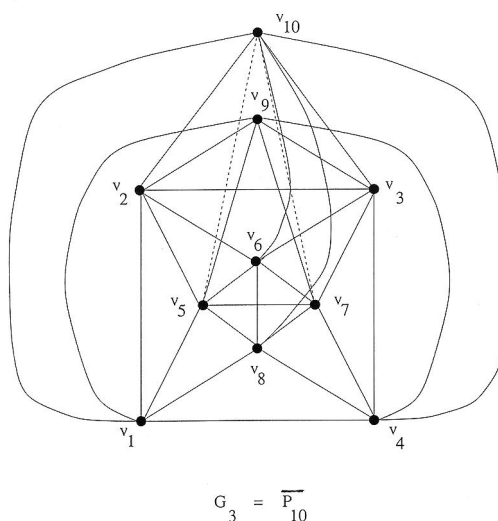


Figure 6.2(c)

Suppose now that  $G_2$  has only these eleven vertices. Then, since  $\alpha = 2$ , vertex  $v_{10}$  is adjacent to  $v_1, v_2, v_5, v_6, v_8$  and  $v_{11}$  and  $v_{11}$  is adjacent to  $v_3, v_4, v_6, v_7, v_8$  and  $v_{10}$ . Moreover,  $v_{10} \not\sim v_4$ , for if  $v_{10} \sim v_4$ , edge  $v_4v_{10}$  is not critical. Similarly,  $v_{11} \not\sim v_1$ . So  $G_2$  is an  $\alpha$ -critical graph on eleven vertices and is hence unique.

Now we turn to inflations of the graphs  $G_1, G_2$  and  $G_3$ . Let us begin by considering an inflation  $H$  of  $G_1$ . Let the atoms of  $H$  be labeled  $A_i, i = 1, \dots, 9$  so as to correspond to vertices  $v_1, \dots, v_9$  as shown in Figure 6.2(a). We henceforth adopt the notation  $A_i \leq A_j$  to mean  $|A_i| \leq |A_j|$ .

*Case 1(a).* First suppose that  $A_1 \leq A_2$  and  $A_7 \leq A_3$ . Let  $M_1$  be a complete matching of  $A_1$  into  $A_2$  and  $M_2$ , a complete matching of  $A_7$  into  $A_3$ . Then  $M_1 \cup M_2$  is a connected dominating matching. Let  $A_1^c$  and  $A_7^c$  denote the atoms resulting from the contraction of each edge of the matching  $M_1 \cup M_2$ . Then the atoms not involved in this contraction may be designated by  $A'_2, A'_3, A_4, A_5, A_6, A_8$ , and  $A_9$ , where ( $A'_2$  and  $A'_3$  denote the “left-over” vertices of the original atoms  $A_2$  and  $A_3$  which were not involved in the contraction.) But these seven atoms are the vertex set of an inflation of a seven-vertex graph having  $\alpha \leq 2$  and we have already shown that such a graph can be contracted to a complete graph  $K_{\lceil n'/2 \rceil}$ , where  $n'$  denotes the number of vertices in the graph. Thus  $H$  satisfies Hadwiger's Conjecture.

*Case 1(b).* Suppose now that  $A_1 \leq A_5$  and  $A_3 \leq A_7$ . In this case, let  $M_1$  be a complete matching of  $A_1$  into  $A_5$  and  $M_2$ , a complete matching of  $A_3$  into  $A_7$ . Then  $M_1 \cup M_2$  is a connected dominating matching and again the “left-overs” form an inflation of a graph on seven vertices and having  $\alpha \leq 2$ . So again we are done.

We now claim that Cases 1(a) and 1(b) cover all possibilities. Suppose not. Thus suppose we are not in one of these cases, nor are we in any case symmetric to one of these cases. By symmetry, we may assume, without loss of generality, that  $A_1 \leq A_2$ . Then since we are not in Case 1(a), it follows that  $A_7 > A_3$ . Moreover, since we are not in Case 1(b),  $A_1 > A_5$ . Now if  $A_3 \leq A_4$ , we get a case which is symmetric to Case 1(a). So we assume  $A_4 < A_3$ . If, then,  $A_2 \geq A_5$ , we get a case symmetric to Case 1(a) again. Thus we may assume  $A_2 < A_5$ . But then the inequalities involving the sizes of atoms  $A_1, A_2$  and  $A_5$  violate transitivity.

Let us next consider Case 3. Let  $H$  be an inflation of  $G_3 = \overline{P}_{10}$ . Due to the symmetry of the Petersen graph, we may assume that atom  $A_{10}$  is a smallest atom. Again by symmetry, without loss of generality, we may suppose that atom  $A_9$  is smallest among  $A_5, A_7$  and  $A_9$ . Let  $M_1$  be a complete matching of  $A_{10}$  into  $A_6$  and  $M_2$ , a matching of  $A_9$  into  $A_5$ . Then again  $M_1 \cup M_2$  is a connected dominating matching and the remaining “left-overs” induce an inflation of an eight-vertex graph and hence we are done.

Finally, consider an inflation  $H$  of  $G_2$ .

*Case 2.1.* Suppose  $A_2 \leq A_3$  and  $A_8 \leq A_4$ . As usual, let  $M_1$  and  $M_2$  be complete matchings of  $A_2$  into  $A_3$  and  $A_8$  into  $A_4$  respectively. The graph made up of the “left-overs” has nine atoms and we know it contracts to a graph on half its total number of vertices by Case 1.

*Case 2.2.* Suppose  $A_4 \leq A_1$  and  $A_6 \leq A_2$ . This is symmetric with Case 2.1.

*Case 2.3.* Suppose  $A_1 \leq A_4$  and  $A_6 \leq A_3$ . This is also symmetric with Case 2.1.

*Case 2.4.* Suppose  $A_2 \leq A_6$  and  $A_4 \leq A_8$ . Then we let  $M_1$  and  $M_2$  be complete matchings of  $A_2$  into  $A_6$  and  $A_4$  into  $A_8$  respectively and proceed as before.

Now suppose none of the above four subcases occurs. Then without loss of generality we may assume that  $A_2 \leq A_3$ . So since we are not in Case 2.1, we may assume  $A_4 < A_8$ . Since we are not in Case 2.4, we may assume  $A_6 < A_2$ . Then by transitivity,  $A_6 < A_3$ . Then since we are not in Case 2.3, we may assume that  $A_4 < A_1$ . Then since we are not in Case 2.2, we may

assume that  $A_2 < A_6$ . But this is a contradiction. So Case 2 is complete and with it the proof of Theorem 6.4. ■

The preceding Theorem shows that all graphs  $G$  with  $\alpha(G) \leq 2$  and having  $|V(G)| \leq 9$  satisfy Hadwiger's Conjecture. In fact a  $K_{\lceil n/2 \rceil}$  minor can be obtained by contracting the edges of a connected dominating matching.

Suppose now that  $G$  is  $\alpha$ -critical,  $\alpha(G) = 2$ , and  $|V(G)| = 10$ . Then if  $G = C_{3k-1}^{k-1}, G_1, G_2$  or  $G_3$ , or any inflation thereof, we have shown that  $G$  satisfies Hadwiger's Conjecture. So suppose  $G$  is not one of these. Then  $G$  must contain  $\Gamma_1 = G_1$  as an induced subgraph. This 9-vertex subgraph  $G_1$  is  $\alpha$ -critical.

We proceed to investigate how vertex  $v_{10}$  is adjacent to the vertices  $v_1, \dots, v_4$ . Since  $\{v_1, v_3\}$  is independent,  $v_{10}$  is joined to vertex  $v_1$  and/or vertex  $v_3$ . Similarly, since  $\{v_2, v_4\}$  is independent,  $v_{10}$  is joined to vertex  $v_2$  and/or vertex  $v_4$ . So vertex  $v_{10}$  is joined to two, three or all four of  $\{v_1, \dots, v_4\}$ .

*Case 1.* Suppose  $v_{10}$  is adjacent to all four of  $\{v_1, \dots, v_4\}$ . Edge  $v_1v_{10}$  must be critical, so at least one of  $v_6$  and  $v_7$  is not adjacent to  $v_{10}$ . But then  $v_{10} \sim v_9$ . If  $v_{10} \sim v_5, v_6, v_7$  or  $v_8$ , then the corresponding edge would not be critical, so  $v_{10}$  is adjacent to none of  $v_5, \dots, v_8$ . So  $G$  is an inflation of  $G_1$  where  $\{v_9, v_{10}\}$  lie in the same atom and so we are done.

So let us assume that vertex  $v_{10}$  is not adjacent to at least one of  $v_1, \dots, v_4$ .

*Case 2.* Suppose now that  $v_{10} \sim v_2, v_3$  and  $v_4$ , but  $v_{10} \not\sim v_1$  (without loss of generality). Then  $v_{10} \sim v_6$  and  $v_{10} \sim v_7$ . Then  $v_{10} \not\sim v_8$  (since edge  $v_2v_{10}$  is critical),  $v_{10} \not\sim v_5$  (since edge  $v_4v_{10}$  is critical), and  $v_{10} \sim v_9$  (since  $\alpha = 2$ ). But then  $G$  must be an inflation of  $G_1$  where vertices  $v_3$  and  $v_{10}$  belong to the same atom and we are done.

*Case 3.* So suppose  $v_{10}$  is adjacent to *exactly* two of the vertices  $v_1, v_2, v_3, v_4$ . Then these two must in turn be adjacent. So suppose  $v_{10} \sim v_3, v_4$  and  $v_{10} \not\sim v_1, v_2$ . Then  $v_{10} \sim v_8, v_6$  and  $v_7$ . If  $v_{10} \sim v_5$ , then  $v_{10} \not\sim v_9$ . But then  $G$  is an inflation of  $G_1$  (where vertices  $v_{10}$  and  $v_7$  belong to the same atom) and once again we are done.

So suppose  $v_{10} \not\sim v_5$ . Thus  $v_{10} \sim v_9$ . So  $G$  is isomorphic to the graph  $G_4$  pictured in Figure 6.3(a) and redrawn in Figure 6.3(b) to better exhibit its symmetries.

This is a new  $\alpha$ -critical graph which we haven't encountered before. Let us now consider any inflation  $H$  of graph  $G_4$ .

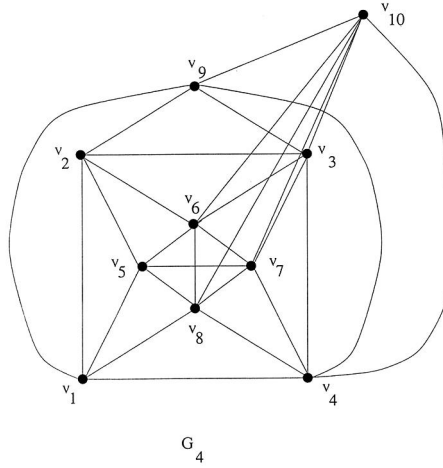


Figure 6.3(a)

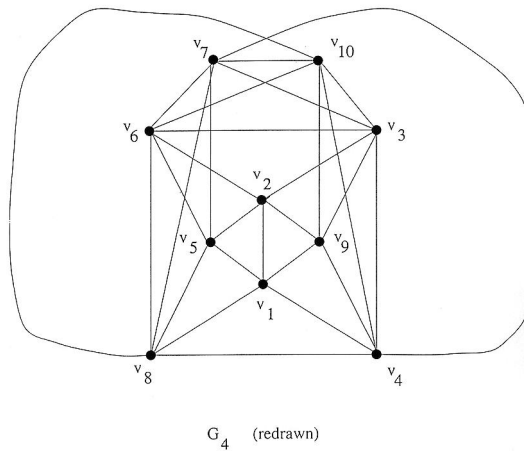


Figure 6.3(b)

*Case 3.1.* Suppose  $A_6 \leq A_2$  and  $A_4 \leq A_1$ . Let  $M_1$  and  $M_2$  be complete matchings of  $A_6$  into  $A_2$  and  $A_4$  into  $A_1$ , respectively. Clearly  $M_1 \cup M_2$  is a connected dominating matching. The vertices not spanned by  $M_1 \cup M_2$  induce an 8-atom graph  $H'$  of “left-over vertices” which is an inflation of an 8-vertex graph with  $\alpha \leq 2$ . Then  $H'$  satisfies HC and thus has  $K_{\lceil n'/2 \rceil}$  as



a minor. Contracting the edges of  $M_1 \cup M_2$  into single vertices results in a graph containing  $K_{\lceil n/2 \rceil}$ . Hence  $H$  satisfies HC.

*Case 3.2.* Suppose  $A_6 \leq A_3$  and  $A_1 \leq A_4$ . Let  $M_1$  be a complete matching of  $A_6$  into  $A_3$  and  $M_2$ , a complete matching of  $A_1$  into  $A_4$  and continue as before.

*Case 3.3.* Suppose  $A_8 \leq A_4$  and  $A_2 \leq A_3$ . Let  $M_1$  be a complete matching of  $A_8$  into  $A_4$  and  $M_2$ , a complete matching of  $A_2$  into  $A_3$  and continue as before.

*Case 3.4.* Suppose  $A_4 \leq A_8$  and  $A_2 \leq A_6$ . This case is symmetric to Case 3.2.

Now also suppose that  $H$  does not satisfy HC. Without loss of generality, we may assume that  $A_6 \leq A_3$ . By Case 3.2 we may assume that  $A_4 < A_1$ . Then by Case 3.1 it follows that  $A_2 < A_6$ . But then by Case 3.4,  $A_8 < A_4$  and by Case 3.3,  $A_3 < A_2$ . But then  $A_3 < A_2 < A_6 \leq A_3$ , a contradiction.

So we may conclude that any inflation of a graph with  $\alpha \leq 2$  and having no more than ten vertices (and any inflation of such a graph) satisfies Hadwiger's Conjecture and moreover we can contract such a graph to a graph containing a clique on at least half the number of its vertices by contracting the edges of a connected dominating matching.

We have carried out a similar investigation when  $|V(G)| = 11$ . As  $\alpha$ -critical graphs with  $\alpha = 2$  and having eleven vertices we obtain two new graphs,  $G_5$  and  $G_6$ , (shown below in Figures 6.4 and 6.5 respectively) not covered before. More specifically,  $G_5$  arises in Case 1 and  $G_6$  arises in Case 3.

We assert that any inflation  $H$  of graph  $G_6$  satisfies Hadwiger's Conjecture. This follows by an argument similar to those above. More particularly, let Case 1 denote the situation when  $A_2 \leq A_3$  and  $A_8 \leq A_4$ ; Case 2, the situation when  $A_4 \leq A_1$  and  $A_6 \leq A_2$ ; Case 3, the situation when  $A_1 \leq A_4$  and  $A_6 \leq A_3$ ; and Case 4, the situation when  $A_2 \leq A_6$  and  $A_4 \leq A_8$ . Assume then that  $H$  does not satisfy Hadwiger's Conjecture. Without loss of generality, we may assume that  $A_2 \leq A_3$ . But then by Case 1,  $A_4 < A_8$ ; by Case 2,  $A_1 < A_4$ ; by Case 3,  $A_3 < A_6$  and by Case 4,  $A_6 < A_2$ . But then we have  $A_6 < A_2 \leq A_3 < A_6$ , a contradiction.

We now turn our attention to graph  $G_5$ . The reader is directed to the second drawing of  $G_5$  shown in Figure 6.4 to more clearly see the symmetries which we shall appeal to below. We prove that any inflation  $H$  of  $G_5$  satisfies HC.

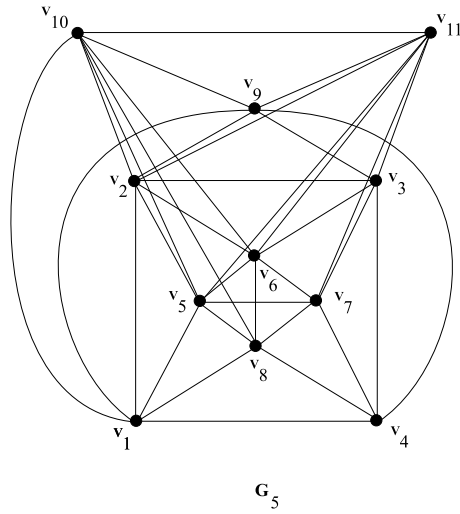


Figure 6.4(a)

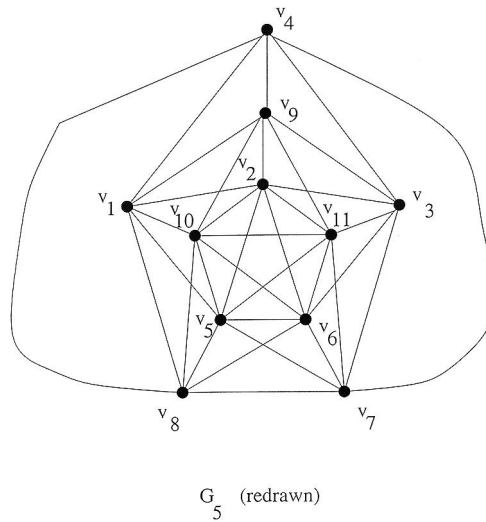


Figure 6.4(b)

Let us denote by Case A, the situation when  $A_1 \leq A_5$  and  $A_3 \leq A_6$ . The associated complete matchings of  $A_1$  into  $A_5$  and  $A_3$  into  $A_6$  give the desired result. The reader sees that by (rotational) symmetry in Figure 6.4 (b), four other cases follow.

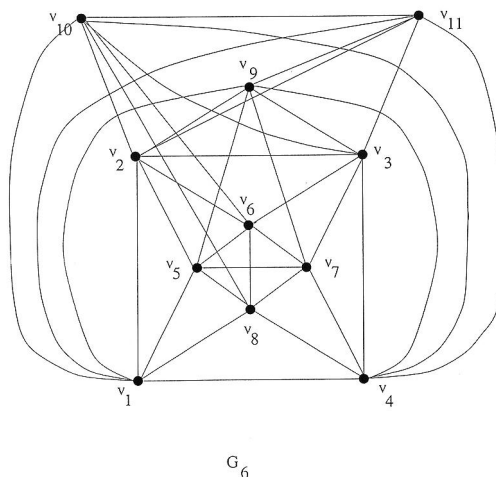


Figure 6.5

Let Case B denote the situation in which  $A_4 \leq A_9$ ,  $A_5 \leq A_2$  and  $A_6 \leq A_{10}$ . The three associated complete matchings then give the desired result. This time there are four additional cases which follow by rotating  $G_5$  clockwise succesively through angles of  $2\pi/5$  radians and another five cases which follow by reflecting the graph about a vertical axis of symmetry such as the axis through atom  $A_2$  and the midpoint of the family of edges joining  $A_5$  and  $A_6$ .

Case C denotes the situation in which  $A_9 \leq A_4$ ,  $A_5 \leq A_7$  and  $A_6 \leq A_8$ . Again the associated complete matchings serve to give the desired result. Also here there are an additional four cases which are settled by rotating the graph by multiples of  $2\pi/5$  radians clockwise.

Finally, let Case D denote the situation in which  $A_4 \leq A_9 \leq A_{10}$ ,  $A_9 \leq A_{11}$ ,  $A_2 \leq A_1$ ,  $A_2 \leq A_3$ ,  $A_5 \leq A_7$  and  $A_6 \leq A_8$ . First, let  $M_1$  and  $M_2$  be complete matchings of  $A_5$  into  $A_7$  and  $A_6$  into  $A_8$ , respectively. Then let  $A'_7$  and  $A'_8$  be the vertices of  $A_7$  and  $A_8$ , respectively, that are not covered by the matching  $M_1 \cup M_2$ . By symmetry, we may assume that  $A'_8 \leq A'_7$ . This implies that there is a complete matching  $M_3$  of  $A'_8$  into  $A'_7$ . Furthermore, there are complete matchings  $M_4$  and  $M_5$  of  $A_4$  into  $A_9$  and  $A_2$  into  $A_3$ , respectively. Eventually, let  $A'_9$  be the vertices of  $A_9$  not covered by  $M_4$ . Then there is a complete matching  $M_6$  of  $A'_9$  into  $A_{11}$ . Clearly,  $M = \bigcup_{i=1}^6 M_i$  is a matching that covers all vertices from  $A_2 \cup A_4 \cup A_5 \cup A_6 \cup A_8 \cup A_9$  and the

reader can easily check that the matching  $M$  is connected and dominating. Then it gives the desired result. Also here there are four additional cases.

To finish the proof, we claim that these four cases are sufficient. Suppose that this is not true, i.e., none of the situations described in Case A,B,C or D occur in the inflation  $H$  of  $G_5$ . To arrive at a contradiction, we first choose the smallest atom among the five atoms  $A_2, A_5, A_6, A_{10}$  and  $A_{11}$  that belong to the  $K_5$  of  $G_5$ . By symmetry, we may assume that this atom is  $A_2$ . Then  $A_2 \leq A_i$  for  $i \in \{5, 6, 10, 11\}$ . Furthermore, by symmetry, we may assume that  $A_{11} \leq A_{10}$ . Then we conclude that  $A_8 < A_4$ , since otherwise we have Case B, because of  $A_{11} \leq A_{10}$  and  $A_2 \leq A_5$ . Now, we distinguish two cases.

*Case 1.* Suppose  $A_3 \leq A_2$ . Then  $A_{10} < A_8$ , since otherwise we have Case A. Because of  $A_3 \leq A_2 \leq A_{10} < A_8 \leq A_4$ , it follows that  $A_3 < A_4$ . Also  $A_3 \leq A_6$  follows by transitivity and, therefore,  $A_5 < A_1$ , since otherwise we have Case A. But then, because of  $A_3 < A_4$ ,  $A_5 < A_1$  and  $A_{10} < A_8$ , we have Case C, a contradiction.

*Case 2.* Suppose  $A_2 < A_3$ . Then  $A_9 < A_{11}$ , since otherwise, because of  $A_8 < A_4$ , we have Case C. Then, using Case A, we conclude that  $A_6 < A_8$  and, using transitivity, we conclude that  $A_9 < A_{10}$ . This, by Case A, implies  $A_5 < A_7$ . Then, using Case C, we infer that  $A_4 < A_9$ . Since  $A_8 < A_4 < A_9 < A_{10}$ , it follows that  $A_8 < A_{10}$  and, therefore,  $A_6 < A_{10}$ , by transitivity. Furthermore, we infer that  $A_1 < A_2$ , since otherwise we have Case D with  $A_5 < A_7$ ,  $A_6 < A_8$ ,  $A_4 < A_9 < A_{10}$ ,  $A_9 < A_{11}$  and  $A_2 \leq A_3$ . Because of  $A_1 < A_2$ , it follows from Case A, that  $A_{11} < A_7$ . Since  $A_4 < A_9 < A_{11} < A_7$ , we have  $A_4 < A_7$ . Furthermore, since  $A_1 < A_2 \leq A_5$ , we have  $A_1 < A_5$ . Then, using Case A, we conclude that  $A_6 < A_3$ . Since  $A_{11} < A_7$ , it then follows by Case C that  $A_4 < A_1$ . Now, we have  $A_8 < A_4 < A_1$  and  $A_1 < A_2 \leq A_6 < A_8$ , a contradiction.

Thus, in both cases we arrived at a contradiction. This proves our claim and completes the proof of Theorem 4.1. ■

## 7. Connected Matchings

Let us denote by the name CONNECTED MATCHING the following problem (already posed in our Introduction):

Given a graph  $G$  and a positive integer  $k$ , is there a connected matching  $M$  in  $G$  such that  $|M| \geq k$ ?

A well-known NP-complete problem called CLIQUE is stated as follows [9]:

Given a graph  $G$  and a positive integer  $k$ , is there a clique  $K$  in  $G$  with  $|K| \geq k$ ?

**Theorem 7.1.** CONNECTED MATCHING is NP-complete.

*Proof.* Clearly the problem is in NP and we shall reduce CLIQUE to CONNECTED MATCHING.

Let  $G$  be any graph and construct a new graph  $H$  as follows. Given two disjoint copies  $G_1$  and  $G_2$  of  $G$ , join all vertices of  $G_1$  to all vertices of  $G_2$ . Now attach to each vertex  $v$  of  $V(G_1) \cup V(G_2)$  a new edge joining  $v$  to a new vertex  $v'$  and let  $H$  be the resulting graph on  $4|V(G)|$  vertices.

Suppose first that  $H$  has a connected matching  $M$ . Then  $M$  is composed of edge set  $M_{11}$  consisting of edges of the form  $vv'$  where endvertex  $v$  lies in  $G_1$ , edge set  $M_1$  having both endvertices in  $G_1$ , edge set  $M_{12}$  consisting of edges having one endvertex in  $G_1$  and the other in  $G_2$ , edge set  $M_2$  consisting of edges having both endvertices in  $G_2$  and  $M_{22}$  made up of edges of the form  $vv'$  where  $v$  lies in  $G_2$ . Then for  $i = 1, 2$ , the endvertices of  $M_{ii}$  in  $G_i$  are a complete subgraph  $Q_i$  of  $G_i$ . Thus

$$\begin{aligned} |M| &= |M_{11}| + |M_{22}| + |M_1| + |M_{12}| + |M_2| \\ &\leq |V(Q_1)| + |V(Q_2)| + (2|V(G)| - |V(Q_1)| - |V(Q_2)|)/2 \\ &\leq |V(G)| + |V(Q)|, \end{aligned}$$

where  $Q$  is a maximum complete subgraph of  $G$ .

Conversely, it is easy to see that  $H$  has a connected matching of size  $|V(G)| + |V(Q)|$ , where  $Q$  is a largest complete subgraph in  $G$ . Thus  $G$  has a complete subgraph of size at least  $k$  if and only if  $H$  has a connected matching of size at least  $|V(G)| + k$ . Therefore CLIQUE has been reduced to CONNECTED MATCHING and thus the latter is NP-complete. ■

## 8. Concluding Remarks

When commencing this investigation, our feeling was that Hadwiger's Conjecture might fail for some  $\alpha = 2$  graphs and that a counterexample might possibly be obtained as an inflation of some small graph  $G$  having  $\alpha = 2$  (in the same way that a counterexample to the related conjecture of Hajós

turned out to be simply an inflation of the 5-cycle, as noted by Catlin [4], see also [12]).

The main outcome of our investigations is that this seems not to be so; at least  $G$  will have to have at least twelve vertices.

It is unfortunate that we have not been able to carry our investigation through to a final conclusion for Hadwiger's Conjecture with respect to  $\alpha = 2$  graphs. It has been likewise disappointing not even to be able to improve the trivial constant  $1/3$ . (See the Introduction.) A possible improvement would be to the value  $2/5$  perhaps by being able to repeatedly contract three edges of a 5-cycle into two vertices.

So Hadwiger's Conjecture seems to remain one of the great challenges of discrete mathematics, even for graphs with  $\alpha = 2$ .

## References

- [1] C. Berge, *Alternating chain methods: a survey*, in: Graph Theory and Computing, ed., R. Read (Academic Press, New York, 1972) 1–13.
- [2] S. Brandt, *On the structure of dense triangle-free graphs*, Combin. Prob. Comput. **8** (1999) 237–245.
- [3] S. Brandt and T. Pisanski, *Another infinite sequence of dense triangle-free graphs*, Elect. J. Combin. **5** (1998) #R43 1–5.
- [4] P.A. Catlin, *Hajós' graph-coloring conjecture: variations and counterexamples*, J. Combin. Theory (B) **26** (1979) 268–274.
- [5] G.A. Dirac, *Some theorems on abstract graphs*, Proc. London Math. Soc. **2** (1952) 69–81.
- [6] G.A. Dirac, *Trennende Knotenpunktmenngen und Reduzibilität abstrakter Graphen mit Anwendung auf das Vierfarbenproblem*, J. Reine Angew. Math. **204** (1960) 116–131.
- [7] P. Duchet and H. Meyniel, *On Hadwiger's number and the stability number*, Ann. Discrete Math. **13** (1982), 71–74.
- [8] T. Gallai, *Kritische Graphen II*, Publ. Math. Inst. Hung. Acad. **8** (1963) 373–395.
- [9] M.R. Garey and D.S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness* (W.H. Freeman and Company, San Francisco, 1979).
- [10] H. Hadwiger, *Über eine Klassifikation der Streckenkomplexe*, Vierteljahrsschrift der Naturf. Gesellschaft in Zürich **88** (1943) 133–142.

- [11] J.H. Kim, *The Ramsey number  $R(3, t)$  has order of magnitude  $t^2/\log t$* , Random Struct. Algorithms **7** (1995) 173–207.
- [12] U. Krusenstjerna-Hafström and B. Toft, *Some remarks on Hadwiger's Conjecture and its relation to a conjecture of Lovász*, in: The Theory and Applications of Graphs: Proceedings of the Fourth International Graph Theory Conference, Kalamazoo, 1980, eds., G. Chartrand, Y. Alavi, D.L. Goldsmith, L. Leśniak-Foster and D.R. Lick (John Wiley and Sons, 1981) 449–459.
- [13] L. Lovász and M.D. Plummer, Matching Theory, Akadémiai Kiadó, Budapest and Ann. Discrete Math. **29** (North-Holland, Amsterdam, 1986) 448.
- [14] W. Mader, *Über trennende Eckenmengen in homomorphiekritischen Graphen*, Math. Ann. **175** (1968) 243–252.
- [15] J. Pach, *Graphs whose every independent set has a common neighbor*, Discrete Math. **37** (1981) 217–228.
- [16] N. Robertson, P.D. Seymour and R. Thomas, *Hadwiger's conjecture for  $K_6$ -free graphs*, Combinatorica **13** (1993) 279–362.
- [17] B. Toft, *On separating sets of edges in contraction-critical graphs*, Math. Ann. **196** (1972) 129–147.
- [18] B. Toft, *A survey of Hadwiger's Conjecture*, Congr. Numer. **115** (1996) 249–283.

Received 29 September 2001

Revised 13 May 2002