MMSE Filtering for Amplify & Forward SIMO Multiple Access Channel with Ill-conditioned Second Hop

Symeon Chatzinotas, Björn Ottersten
SnT - securityandtrust.lu, University of Luxembourg.
Email: {Symeon(Chatzinotas, Bjorn.Ottersten}@uni.lu

Abstract—Relaying has been extensively studied during the last decades and has found numerous applications in wireless communications. The simplest relaying method, namely amplify and forward, has shown potential in MIMO multiple access systems, when Gaussian fading channels are assumed for both hops. However, in some cases ill conditioned channels may appear on the second hop. For example, this impairment could affect cooperative BS systems with microwave link backhauling, which involves strong line of sight channels with insufficient scattering. In this paper, we consider a large system analysis of such as system model focusing on the joint MMSE filtering receiver. Analytical methods based on free probability are presented for calculating the MMSE error and average SINR, while the performance degradation of the system throughput due to second hop ill-conditioning is studied.

I. INTRODUCTION

The Dual Hop (DH) Amplify & Forward (AF) relay channel has attracted a great deal of attention mainly due to its low complexity and its manifold benefits, such as coverage extension and decreased outage probability. Although the DH AF channel has been extensively studied in the literature [1], [2], [3], the effect of the condition number of the second hop channel on the throughput performance is not well quantified yet. Assuming Gaussian channel model in both hops, authors in [1] approached the problem asymptotically using Silverstein’s fixed-point equation and found closed-forms expressions for the Stieltjes transform. Under similar assumptions, a finite analysis was recently performed by [2]. On the other hand, authors in [3] following a replica analysis tackled the problem of Kronecker correlated Gaussian models.

In addition, the MIMO MAC has been studied heavily during the last decades since it comprises a fundamental channel model for multiuser uplink cellular [4] and multibeam return link communications [5], [6]. The work in [7], [8] has combined AF relaying with a MAC and has performed a free-probablistic analysis for channel capacity[7] and MMSE filtering[8]. Furthermore, the work in [9] has combined AF relaying with collaborative Base Stations and has performed a replica analysis for channel capacity and MMSE throughput. In our scenario, we study a DH AF SIMO MAC modelling collaborative BSs with microwave link backhauling and we focus on the impact of ill-conditioned channel in the second hop. More specifically, due to line of sight and lack of scattering the resulting multiple-dimensional channel may appear ill-conditioned limiting the distributed MIMO gains. In this direction, we investigate the joint MMSE filtering performance of such a system and we compare it to a conventional system which employs orthogonal resource division access to eliminate multiuser interference.

The remainder of this paper is structured as follows: Section II introduces the system model and section III describes the free probability derivations and the main MMSE performance results. Section IV illustrates the accuracy of the analysis by comparing with Monte Carlo simulations and evaluates the effect of various system parameters on the performance. Section V concludes the paper.

Throughout the formulations of this paper, normal x, lower-case boldface x and upper-case boldface X font is used for scalars, vectors and matrices respectively. \( \mathbb{E}[\cdot] \) denotes the expectation, \( (\cdot)^H \) denotes the conjugate matrix transpose, and \( \odot \) denotes the Hadamard product. The Frobenius norm of a matrix or vector is denoted by \( \|\cdot\| \), the absolute value of a scalar is denoted by \( |\cdot| \) and the delta function is denoted by \( \delta(\cdot) \). \( \cdot^+ \) is equivalent to \( \max(0, \cdot) \) \{\cdot\} is the indicator function and \( \Rightarrow \) denotes almost sure (a.s.) convergence. The expression \( A \sim \mathcal{CN}(0, I) \) denotes a Gaussian matrix with independent identically distributed (i.i.d.) complex circularly symmetric (c.c.s.) elements.

II. SYSTEM MODEL

Figure 1 is a conceptual illustration of the input-output model. It can be seen that the BS-CP (Central Processor) microwave links (second hop) form an ill-conditioned SIMO MAC, whereas the user-BS-CP links can be modelled as...
SIMO AF MAC. Gaussian input is considered at the user-side, while neither users nor relays are aware of the Channel State Information (CSI). The described channel model can be expressed as follows:

\[
y_1 = H_1x_1 + z_1, \quad y_2 = H_2\sqrt{\nu}y_1 + z_2 \leftrightarrow y_2 = \sqrt{\nu}H_2H_1x_1 + \sqrt{\nu}H_2z_1 + z_2,
\]

where the \( M \times 1 \) vector \( x_1 \) denotes the user transmitted symbol vector with individual Signal to Noise Ratio (SNR) \( \mu \), \( y_1 \) denotes the \( K \times 1 \) received symbol vector by the BSs and the \( K \times 1 \) vector \( z_1 \) denotes AWGN at BS-side with \( E[z_1] = 0 \) and \( E[z_1^*] = I \). The received signal \( y_2 \) is amplified by \( \nu \) and forwarded and as a result \( y_2 \) denotes the \( K \times 1 \) received symbol vector by the CP and the \( K \times 1 \) vector \( z_2 \) denotes AWGN at CP-side with \( E[z_2] = 0 \) and \( E[z_2^*] = I \). It should be noted that for the remainder of this document \( \mu \) and \( \nu \) will be referred to as First Hop Power (FHP) and Second Hop Power (SHP) respectively.

The \( K \times M \) channel matrix \( H_1 \) and the \( K \times K \) channel matrix \( H_2 \) represent the concatenated channel vectors for the user-BS and BS-CP links respectively. The first hop Rayleigh fading channel is assumed to be modelled as \( H_1 \sim CN(0, I) \). The BS-CP channel \( H_2 \) under line of sight suffers from correlation due to lack of scattering and thus it can be modelled as an ill-conditioned deterministic channel with variable condition number \( \zeta^2 = \lambda_{\text{max}}(H_2H_2^H)/\lambda_{\text{min}}(H_2H_2^H) \). The exact matrix models for \( H_2 \) are described in detail in sections III-2.

### A. Performance Metrics

The performance metric considered in this work is the average Minimum Mean Square Error (MMSE) achieved by joint MMSE filtering at the CP. It should be noted that the channel capacity achieved by successive interference cancellation at the CP has been already studied in [10]. Both of these receiver structures require multiuser processing at the CP. On the other hand, section II-B considers a conventional system where Frequency or Time Division Multiple Access is used in combination with single-cell decoding at the CP.

The performance of the MMSE receiver for \( K = M \) is dependent on the achieved MSE averaged over users and channel realizations and is given by:

\[
\text{mmse}_{\text{avg}} = E \left[ \frac{1}{M} \sum_{m=1}^{M} \text{mmse}_m \right]
\]

\[
= E \left[ \frac{1}{M} \sum_{m=1}^{M} \left( \mu_\text{m}\nu_\text{m}H_1^H \left( I + \nu_\text{m}H_2H_2^H \right)^{-1} H_2H_1 \right)^{-1} \right]_{m,m}
\]

\[
= E \left[ \frac{1}{M} \nu_\text{m} \left( \mu_\text{m}\nu_\text{m}H_1^H \left( I + \nu_\text{m}H_2H_2^H \right)^{-1} H_2H_1 \right)^{-1} \right]
\]

\[
= E \left[ \frac{1}{M} \nu_\text{m} \left( \mu_\text{m}\nu_\text{m} \left( I + \nu_\text{m}H_2H_2^H \right)^{-1} \right) \left( I + \nu_\text{m}H_2H_2^H \right)^{-1} \left( I + \nu_\text{m}H_2H_2^H \right)^{-1} \right] .
\]

The average SINR is given by:

\[
\text{SINR}_{\text{avg}} = E \left[ \frac{1}{M} \sum_{m=1}^{M} \text{mmse}_m \right] - 1
\]

and the achieved throughput per receive antenna using LMMSE by (4) at the top of the next page. Compared to existing literature, we follow a free probabilistic analysis as in [11], [4], [12], [13], [14] to derive the channel capacity, but we extend it for the described DH AF SIMO MAC including the noise amplification terms and ill-conditioned second hop modelling. More importantly, we consider the MMSE filtering receiver and we obtain a lower bound on the average MMSE performance.

To simplify the notations during the mathematical analysis, the following auxiliary variables are defined:

\[
M = I + \mu H_1 H_1^H
\]

\[
M = I + \nu H_2 H_2^H
\]

\[
N = H_1 H_1^H
\]

\[
\tilde{N} = H_2^H H_2
\]

\[
K = H_2 H_2^H (I + \mu H_1 H_1^H) = \tilde{N} M
\]

\[
\tilde{K} = H_2 (I + \mu H_1 H_1^H) H_2^H
\]

\[
\beta = \frac{M}{K}
\]

where \( \beta \geq 1 \) is the ratio of horizontal to vertical dimensions of matrix \( H_1 \) respectively.

### B. Conventional System

In a conventional fractional frequency reuse system, the available resources (frequency or time) would have to be split in \( K \) pieces in order to avoid multiuser interference from neighboring BSs. This entails that only \( K \) out of \( M \) users could be served simultaneously, namely one user per BS. On the plus side, each user or BS could concentrate its power on a smaller portion of the resource using \( K \mu \) and \( K \nu \) respectively. Assuming a single user per BS (\( K = 1 \)), the conventional channel model for a single user-BS-CP link can be written as:

\[
y_1 = h_1x_1 + z_1
\]

\[
y_2 = h_2 \sqrt{\nu}y_1 + z_2 \leftrightarrow y_2 = \sqrt{\nu}h_2y_1 + z_2
\]

(5)

with \( x_1 \) Gaussian input with \( E[x_1^2] = K \mu \) and \( z_1, z_2 \) AWGN with \( E[z_1^2] = E[z_2^2] = 1 \). In this case, the per-antenna capacity at the CP would be:

\[
C_{co} = E \left[ \log (1 + \text{SINR}) \right] = E \left[ \log \left( 1 + \frac{K^2 \nu h_2^2 h_1^2}{1 + K \nu h_2^2} \right) \right]
\]

(6)

where \( h_1 \) and \( h_2 \) are the channel coefficients of the first and second hop respectively. The first and second hop are modelled as Rayleigh fading and AWGN channels respectively and thus we can assume that \( h_1 \sim CN(0, 1) \) and \( h_2 = 1 \). The performance of the conventional and proposed transmission schemes are compared in section IV.

### III. PERFORMANCE ANALYSIS

In order to calculate the system performance analytically, we resort to asymptotic analysis which entails that the dimensions of the channel matrices grow to infinity assuming proper normalizations. It has already been shown in many occasions that asymptotic analysis yields results which also provide accurate
results for finite dimensions [15], [16], [17]. In other words, the expressions of interest converge quickly to a deterministic value as the number of channel matrix dimensions increases.

The average MMSE when $\beta = 1$ can be expressed as:

$$
\text{mmse}_\text{avg} = \lim_{K,M \to \infty} \frac{1}{M} \text{tr} \left\{ \left( I + \nu \hat{K} \right)^{-1} M \right\} \geq \lim_{K,M \to \infty} \frac{1}{M} \sum_{m=1}^{M} \lambda_{M-m+1} \left( \hat{M} \right)
$$

(7)

where step (a) follows from property $\text{tr} \{ AB \} \geq \sum_{m=1}^{M} \lambda_{m}(A)\lambda_{M-m+1}(B)$ in [18] and $F^{-1}_\text{K}(x)$ denotes the inverse function of the asymptotic eigenvalue cumulative density function (a.e.c.d.f.). The last step follows from the fact that the ordered eigenvalues can be obtained by uniformly sampling the inverse c.d.f. in the asymptotic regime [5].

To calculate the expression of (8), it suffices to derive the asymptotic densities of $\hat{K}$, $\hat{M}$, which can be achieved through the principles of free probability theory [19], [20], [21], [22] as described in sections III-1 and III-2. Free probability (FP) has been proposed by Voiculescu [19] and has found numerous applications in the field of wireless communications. More specifically, FP has been applied for capacity derivations of variance profiled [23], correlated [4] Rayleigh channels, as well as Rayleigh product channels [11]. Furthermore, it has been used for studying cooperative relays [24], interference channels [12] and interference alignment scenarios [14]. The advantage of the FP methodology compared to other techniques, such as the Stieltjes method, replica analysis and deterministic equivalents, is that the derived formulas usually require just a polynomial solution instead of fixed-point equations. However, the condition for these simple solutions is that the original a.e.p.d.f. can be expressed in polynomial form [25].

I) Fading First Hop: The first hop from users to BSs can be modelled as a Rayleigh fading channel, namely $H_1 \sim \mathcal{CN}(0,1)$.

Definition III.1. Considering a Gaussian $K \times M$ channel matrix $H_1 \sim \mathcal{CN}(0,1)$, the a.e.p.d.f. of $\frac{1}{M}H_1H_1^H$ converges almost surely (a.s.) to the non-random limiting eigenvalue distribution of the Marčenko-Pastur law [26], whose density functions are given by

$$
\forall x > 0 \quad f_{\frac{1}{M}H_1H_1^H}(x) = \frac{1}{\pi} \frac{\sqrt{\delta_+ - x} \sqrt{x - \delta_-}}{\delta_+ - \delta_-} \quad \text{for } \delta_{\pm} = \mu^2 \pm \sqrt{\mu^4 - 4\nu},
$$

(14)

where $\mu = K\mu$.

Proof. The a.e.p.d.f. can be calculated considering the transformation $z(x) = (1 + K\mu x)$, where $z$ and $x$ represent the eigenvalues of $M$ and $\frac{1}{M}H_1H_1^H$ respectively:

$$
f_{\frac{1}{M}H_1H_1^H}(z) = \frac{1}{z(z^{-1}(x))} \cdot f_{\frac{1}{M}H_1H_1^H}(x) = \frac{1}{\mu} f_{\text{MP}} \left( \frac{x - 1}{\mu} \right).
$$

Theorem III.1. The inverse $\eta$-transform of $M$ is given by (15).

Proof. Due to lack of space, only the outline is provided here. The transform can be calculated starting from the definition, following a series of change of variables and finishing with Cauchy integration.
\[ \eta_M^{-1}(x) = -x\bar{\mu} - \beta \bar{\mu} + \bar{\mu} - 1 + \sqrt{x^2\bar{\mu}^2 + 2x\bar{\mu}^2 - 2x\bar{\mu}^2 - 2x\bar{\mu} + \beta^2\bar{\mu}^2 - 2\beta\bar{\mu}^2 + 2\beta\bar{\mu} + \bar{\mu}^2 + 2\bar{\mu} + 1}. \] (15)

**Theorem III.2.** The a.e.c.d.f. of \( M \) for \( \beta = 1 \) is given by:

\[ F_M(x) = \frac{\sqrt{(x - 1)(4\bar{\mu} - x + 1)} - 2\arcsin\left(\frac{2\bar{\mu} + 1}{\sqrt{2\bar{\mu}}}\right)}{2\bar{\mu}}. \] (16)

**Proof.** The c.d.f. follows from (14) after integration for \( \beta = 1 \).

**Theorem III.3.** The inverse \( \eta \)-transform of \( K \) is given by:

\[ \eta_K^{-1}(x) = \Sigma_N(x - 1)\eta_M^{-1}(x) \] (17)

**Proof.** Given the asymptotic freeness between deterministic matrix with bounded eigenvalues \( \tilde{N} \) and unitarily invariant matrix \( M \), the \( \Sigma \)-transform of \( K \) is given by multiplicative free convolution:

\[ \Sigma_K(x) = \Sigma_N(x - 1)\Sigma_M(x) \]

\[ \left( -\frac{x + 1}{x} \right) \eta^{-1}_K(\bar{\mu} = 1) = \Sigma_N(x) \left( -\frac{x + 1}{x} \right) \eta^{-1}_M(\bar{\mu} = 1) \]

where step (a) combines \( \Sigma \)-transform definition and eq. (11). The variable substitution \( y = x + 1 \) yields eq. (17).

2) Ill-conditioned Second Hop: Matrix \( H_2 \) is modelled as a deterministic matrix with power normalization \( \text{tr}(H_2^T H_2) = K \). Due to line of sight and lack of scattering, this matrix may appear ill-conditioned. We consider the tilted semicircular law distribution which can accommodate a variable condition number and more importantly its \( \Sigma \)-transform is given by a first degree polynomial [27].

**Theorem III.4.** In the asymptotic regime preserving the power normalization, the tilted semicircular law converges to the following distribution:

\[ f^\infty_{\tilde{N}} = \frac{2\zeta}{\pi(\zeta - 1)^2} \frac{1}{x^2} \sqrt{\left(\frac{x - 1}{\zeta}\right)^+ \left(\frac{x - 1}{\zeta}\right)^+} \] (18)

with support \([\zeta^{-1}, \zeta]\). In this case, the transforms of the tilted semicircular law are given by:

\[ \eta_{\tilde{N}}(x) = 1 + 2\zeta x + \zeta^2 - 2\zeta \sqrt{\zeta^2 x + \zeta x^2 + \zeta^2 x - \zeta^2 x} \] (19)

\[ S_{\tilde{N}}(x) = -x + 2\zeta - \zeta^2 x + 2\zeta \sqrt{\zeta^2 - x + \zeta^2 x + \zeta^2 x - \zeta^2 x} \] (20)

\[ R_{\tilde{N}}(x) = \frac{2\zeta - \sqrt{\zeta^2 + 2\zeta x - x - \zeta^2 x}}{\zeta^2 - 1} \] (21)

\[ \Sigma_{\tilde{N}}(x) = 1 - \frac{(x - 1)^2}{4\zeta - x} \] (22)

**Proof.** The closed-form expressions for the transforms are derived by integrating over the aepdf (18) using the transform definitions.

**Theorem III.5.** The Stieltjes transform of \( K \) is given by the solution of the cubic polynomial in (23).

Proof. The first step is to substitute eq. (15) and (21) into (17). Using \( S_X(x) = -\eta_X(-1/x)/x \) and applying suitable change of variables:

\[ x\eta_K^{-1}(-xS_K(x)) + 1 = 0. \] (24)

The final form of the polynomial is derived through algebraic calculations.

**Remark III.1.** For \( M = K \), the eigenvalues of \( K \) and \( \tilde{K} \) are identical. Thus, the a.e.p.d.f. of \( K \) is given by eq. (23) and \( f_K(x) = \lim_{y \to 0^+} \frac{1}{\pi} \left\{ S_X(x) + jy \right\} \) for \( \beta = 1 \).

**Remark III.2.** The average MMSE \( \text{mse}_{\text{avg}} \) is given by eq. (8) where \( F_{\tilde{M}}(x) \) can be calculated using Theorem III.2 and \( F_M(x) \) using integration and inversion over the a.e.p.d.f. in Remark III.1.

IV. NUMERICAL RESULTS

In order to assess the accuracy of the derived closed-form expressions and gain some insights on the system performance of the considered model, a number of numerical results are presented in this section.

Figure 2 depicts the effect of condition number \( \zeta^2 \) and second hop power \( \nu \) on the average MMSE. As expected, the average MMSE increases with \( \zeta^2 \) but decreases with \( \nu \). It can be seen that performance can be improved using stronger amplification but for high \( \nu \) there is a saturation threshold which is governed by the first hop performance. Figures IV and IV depict the accuracy of the proposed lower bound. The solid plots were calculated through Monte Carlo simulations of (2), whereas the dashed plots represent our lower bound which was calculated using Remark III.2. It can be seen that the proposed bound is tight for low values of \( \zeta^2 \), but it progressively diverges as \( \nu \) and \( \zeta^2 \) grow large.

In this section, the performance of the proposed system is compared to the conventional system (as described in section II-B) by fixing the user and BS power at 10 dBs. As it can be seen in Fig. 5, while the condition number increases, the performance of the proposed system degrades and even falls below conventional performance for extremely ill-conditioned BS-CP channels. There is a crossing point in 160 dBs for the MMSE throughput. However, a two-fold performance gain can still be harnessed for condition numbers up to 120 dBs, which is well beyond the dynamic range of actual receivers.

V. CONCLUSION

In this paper, we have investigated the performance of BS cooperation scenario with microwave backhauling to a CP, where multiple users and BSs share the same channel resources. The user signals are forwarded by the BSs to an antenna array connected to a CP which is responsible for joint MMSE filtering followed by single user decoding. This
\[
\begin{align*}
\frac{1}{\bar{\mu}} x^3 + 4 \zeta (\zeta - 1)^2 \bar{\mu} x^2 \right) S_K(x)^3 \\
+ \left( (\zeta - 1)^2 (3 \zeta^2 + 3 + 2 \zeta) \bar{\mu} x - (\zeta (\zeta - 1)^2 \bar{\mu} x - (4 \mu \zeta)^2 \right) S_K(x)^2 \\
+ \left( (\zeta + 1)^2 (3 \zeta^2 + 3 - 2 \zeta) \bar{\mu} x - 4 (\zeta + 1)^2 \zeta ((\beta - 1) \bar{\mu}^2 + \bar{\mu}) \right) S_K(x) \\
(\zeta + 1)^4 \bar{\mu} 
\end{align*}
\]

(23)

The system has been modelled as a DH AF SIMO MAC with a ill-conditioned or rank-deficient second hop due to line of sight and lack of scattering in the microwave links. Its performance has been analysed through a large-system free-probabilistic analysis. It can be concluded that a performance gain can be achieved compared to conventional resource partitioning even for highly ill-conditioned second hop.

Fig. 2. Average MMSE scaling vs. condition number \(\zeta^2\) and second hop power \(\nu\) in dBs. Parameters: \(\mu = 10dB, \beta = 1\). For high amplification, first hop performance acts as bottleneck.

Fig. 3. Average MMSE performance (solid line) and proposed lower bound (dashed line) vs second hop power \(\nu\). Parameters: \(\mu = 10dB\).

Fig. 4. Average MMSE performance (solid line) and proposed lower bound (dashed line) vs condition number \(\zeta^2\). Parameters: \(\mu = 10dB\).

Fig. 5. Throughput comparison between proposed and conventional system vs. condition number \(\zeta^2\) in dBs. Parameters: \(\mu = \nu = 10dB, \beta = 1\). The proposed system is preferable for condition numbers up to 120 dBs.