Kernels and some operations in edge coloured digraphs

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Abstract

Let $D$ be an edge-coloured digraph, $V(D)$ will denote the set of vertices of $D$; a set $N \subseteq V(D)$ is said to be a kernel by monochromatic paths of $D$ if it satisfies the following two conditions: For every pair of different vertices $u, v \in N$ there is no monochromatic directed path between them and; for every vertex $x \in V(D) - N$ there is a vertex $y \in N$ such that there is an $xy$-monochromatic directed path.

In this paper we consider some operations on edge-coloured digraphs, and some sufficient conditions for the existence or uniqueness of kernels by monochromatic paths of edge-coloured digraphs formed by these operations from another edge-coloured digraphs.

1 Introduction

For general concepts we refer the reader to [?]. In the paper we write digraph to mean 1-digraph in the sense of Berge [?]. In this paper $D$ will denote a possibly infinite digraph; $V(D)$ and $A(D)$ will denote the sets of vertices and arcs of $D$, respectively. If $S$ is a nonempty subset of $V(D)$ then the subdigraph $D[S]$ induced by $S$ is the digraph with vertex set $S$ and whose arcs are those arcs of $D$ which join vertices of $S$.

A directed path is a finite or infinite sequence $(x_1, x_2, ..., )$ of distinct vertices of $D$ such that $(x_i, x_{i+1}) \in A(D)$ for each $i$. When $D$ is infinite and the sequence is infinite we call the directed path an infinite outward path. If $T$ is a directed path and $a, b \in V(T)$, $(a, T, b)$ will denote the $ab$-directed path contained in $T$. (When $a$ appears before $b$ in $T$).
A set $I \subseteq V(D)$ is independent if $A(D[I]) = \emptyset$. A kernel $N$ of $D$ is an independent set of vertices such that for each $z \in V(D) - N$ there exists a $zN$-arc in $D$.

We call the digraph $D$ an $m$-coloured digraph if the arcs of $D$ are coloured with $m$ colours. A directed path or a directed cycle is called monochromatic if all of its arcs are coloured a like.

If $D$ is an $m$-coloured digraph then the closure of $D$, denoted $\mathcal{C}(D)$ is the digraph defined as follows: $V(\mathcal{C}(D)) = V(D)$ and $A(\mathcal{C}(D)) = \{(u, v) \mid$ there exists an $uv$-monochromatic directed path contained in $D\}$.

Notice that for any $m$-coloured digraph $D$; $N$ is a kernel by monochromatic paths of $D$ iff $N$ is a kernel of $\mathcal{C}(D)$.(Although the concept of kernel was defined in [?] for 1-digraphs, the same concept is valid and can be considered in multidigraphs).

The concept of kernel was introduced by Von Neumann and Morgenstern [?] in the context of Game Theory. The problem of the existence of a kernel in a given digraph has been studied by several authors in particular by Richardson [?, ?], Duchet and Meyniel [?], Duchet [?], Galeana-Sánchez and Neumann-Lara [?].

The existence of kernels of digraphs formed by some operations from another digraphs have been studied by several authors, namely; M.Blidia, P. Duchet, H. Jacob, F. Maffray and H. Meyniel [?], Jerzy Topp [?], Galeana-Sánchez [?], Galeana-Sánchez and Neumann-Lara [?, ?]. The concept of kernel by monochromatic paths generalizes the concept of kernel of a digraph and has been studied by several authors: Sauer, Sands and Woodrow [?], Shen Minggang [?], Galeana-Sánchez [?], Galeana-Sánchez [?, ?].

In [?] Jerzy Topp defined the digraphs $S(D)$, $Q(D)$, $T(D)$ and $L(D)$ which were called the subdivision digraph, the middle digraph, the total digraph and the line digraph of $D$ respectively; and studied some necessary or sufficient conditions for the existence or uniqueness of kernels of these digraphs.

In this paper we define the following digraphs: the subdivision $S(D)$, a generalization of the subdivision $S'(D)$, the digraph $R'(D)$, the middle digraph $Q(D)$ and the total digraph $T(D)$, for an $m$-coloured digraph $D$. Also it is proved the following results: If $D$ has no monochromatic infinite outward path, then $S(D)$ (resp. $S'(D)$ and $R'(D)$) has a kernel by monochromatic paths.

The number of kernels by monochromatic paths of $D$ is less than or equal to the number of kernels by monochromatic paths of $Q(D)$ (resp. $T(D)$).

If $D$ has no monochromatic directed cycle then the number of kernels by monochromatic paths of $D$ is equal to the number of kernels by monochromatic paths of $Q(D)$ (resp. $T(D)$).

2 Kernels by monochromatic paths in the subdivision digraph of an $m$-coloured digraph

In [?] was proved that the subdivision digraph of any digraph has a kernel, in this section we define the subdivision digraph $S(D)$ of an $m$-coloured digraph $D$ and it is
proved that if $D$ has no monochromatic infinite outward path, then $D$ has a kernel by monochromatic paths.  

Let $D = (V(D), A(D))$ be an $m$-coloured digraph, we define the funtions $\Gamma_D, \Gamma_{D,i}, \Gamma_D^-, \Gamma_{D,i}^-$ from $V(D)$ to $\mathcal{P}(V(D))$ as follows. For any $u \in V(D)$;

\[
\begin{align*}
\Gamma_D(u) &= \{v \in V(D) \mid (u, v) \in A(D)\}, \\
\Gamma_D^-(u) &= \{v \in V(D) \mid (v, u) \in A(D)\}, \\
\Gamma_{D,i}(u) &= \{v \in V(D) \mid (u, v) \in A(D) \text{ and } (u, v) \text{ is } i\text{-coloured}\}, \\
\Gamma_{D,i}^-(u) &= \{v \in V(D) \mid (v, u) \in A(D) \text{ and } (v, u) \text{ is } i\text{-coloured}\}.
\end{align*}
\]

If $U \subseteq V(D)$, we denote $\Gamma_D(U) = \bigcup_{u \in U} \Gamma_D(u), u \in U$.

**Definition 2.1** Let $D$ be an $m$-coloured digraph, we define the subdivision digraph $S(D)$ of $D$ as follows:

$V(S(D)) = V(D) \cup A(D)$ and

\[
\Gamma_{S(D),i}(x) = \begin{cases} 
\{x\} \times \Gamma_{D,i}(x) & \text{if } x \in V(D), \\
\{v\} & \text{if } x = (u, v) \in A(D) \text{ and } v \in \Gamma_{D,i}(u).
\end{cases}
\]

Notice that for a vertex $x$ of the subdivision digraph we have the following: If $x$ corresponds to a vertex of $D$ then $x$ is adjacent toward the arcs which incide from $x$ in $D$, preserving the colour of those arcs; and if $x$ corresponds to an arc of $D$ then $x$ is adjacent only toward the terminal endpoint of $x$ preserving the colour of $x$. Also notice that $S(D)$ is obtained from $D$ by changing each arc of $D$ for a directed path of length two with the same colour as the arc.

**Lemma 2.1** Let $D$ be an $m$-coloured digraph, $S(D)$ its subdivision digraph and $a, b, c \in V(S(D))$ such that $b \in A(D), a \neq b$ and $b \neq c$. If $T_1$, is an ab-monochromatic directed path in $S(D)$ and $T_2$ is a bc-monochromatic directed path in $S(D)$, then $T_1$, and $T_2$ are coloured alike.

*Proof:* We may assume $b = (u, v) \in A(D)$. Clearly $\Gamma_{S(D)}^-(b) = \{u\}$ and $\Gamma_{S(D)} = \{v\}$, since $a \neq b$ and $b \neq c$ we have $\ell(T_i) \geq 1, i \in \{1, 2\}$ and thus $(u, b) \in A(T_1)$ and $(b, v) \in A(T_2)$, so from the definition of $S(D)$, $(u, b)$ and $(b, v)$ are coloured alike; so $T_1$ and $T_2$ are coloured alike. \qed

**Theorem 2.1** Let $D$ be an $m$-coloured digraph and $S(D)$ its subdivision digraph. If $D$ has no monochromatic infinite outward path, then $S(D)$ has a kernel by monochromatic paths.

*Proof:* First we prove that $S(D)$ has no monochromatic infinite outward path. Assume by contradiction that $T = (x_n)_{n \in \mathbb{N}}$ is a monochromatic infinite outward path coloured $i$, in $S(D)$. By definition of $S(D)$, $T$ is an alternating succession of vertices and arcs of $D$, hence $T$ contains a subsucceesion of vertices of $D$ and a subsucceesion of arcs of $D$. Let $J = \{n \in \mathbb{N} \mid x_n \in A(D)\}$ and $T' = (T - \{x_n \mid n \in J\})$, clearly $T'$ is
infinite and $T'$ contains $V(T) \cap V(D)$ moreover it follows from the definition of $S(D)$ that for each $n \in J$, $n \geq 2$, $x_n$ is the arc $(x_{n-1}, x_{n+1}) \in A(D)$ coloured $i$, we conclude that $T'$ is a monochromatic infinite outward path in $D$ coloured $i$, a contradiction.

Now suppose that $A(D) \neq \emptyset$. We define in $A(D)$ the following binary relations $\preceq$ and $\sim$. Let $a, b \in A(D)$ we write $a \preceq b$ if there exists a $ba$-monochromatic directed path in $S(D)$, and $a \sim b$ if $a \preceq b$ and $b \preceq a$.

Clearly $\preceq$ is reflexive and now we prove that $\preceq$ is transitive; let $a, b, c \in A(D)$ such that $a \preceq b$ and $b \preceq c$; if $a = b$ or $b = c$ then $a \preceq c$; suppose $a \neq b$ and $b \neq c$, then there exist $T$, a $ba$-monochromatic directed path in $S(D)$ and $T_2$ a $cb$-monochromatic directed path in $S(D)$; it follows from Lemma 2.1 that $T$, and $T_2$ are coloured alike, so there exists a $ca$-monochromatic directed path in $S(D)$ which means $a \preceq c$. Now it is easy to see that $\sim$ is an equivalence relation.

Denote by $A(D)/\sim$ the set of the equivalence classes of $A(D)$ module $\sim$, and by $\bar{a}$ the equivalence class of $a$ module $\sim$. We define the binary relation $\leq$ in $A(D)/\sim$ as follows: $\bar{a} \leq \bar{b}$ if and only if $a \preceq b$. It is easy to prove that $\leq$ is well defined and $(A(D)/\sim, \leq)$ is a partial ordered set. For $a, b \in A(D)/\sim$ we write $\bar{a} < \bar{b}$ if and only if $\bar{a} \leq \bar{b}$ and $\bar{a} \neq \bar{b}$. 

(Claim I). Every chain in $(A(D)/\sim, \leq)$ has a minimum element. Let $C$ be a chain in $(A(D)/\sim, \leq)$ and assume by contradiction that $C$ has no minimum element. Let $\bar{a} \in C$, thus there exists $\bar{a}_1 \in C$ such that $\bar{a}_1 < \bar{a}$; since $C$ has no minimum element, there exists $\bar{a}_2 \in C$ such that $\bar{a}_2 < \bar{a}_1$; so there exists a succession $(\bar{a}_n)_{n \in \mathbb{N}}$ in $C$ such that for each $n \in \mathbb{N}$, $\bar{a}_{n+1} < \bar{a}_n$ and $\bar{a}_1 < \bar{a}$, thus for each $n \in \mathbb{N}$, there exists an $a_n d_{n+1}$-monochromatic directed path in $S(D)$, namely $T'_{n+1}$ and an $a_1$-monochromatic directed path $T'_1$. First we prove that all the directed paths $T'_n$, $n \in \mathbb{N}$ are coloured alike. Since $\bar{a}_1 < \bar{a}$, we have $\bar{a}_1 \neq \bar{a}$ and then $a_1 \neq a$; analogously, for each $n \in \mathbb{N}, a_{n+1} \neq a_n$ moreover $a_n$ is the final vertex of $T'_n$ and is the initial vertex of $T'_{n+1}$; then it follows from Lemma ?? that $T'_n$ and $T'_{n+1}$ are coloured alike, say they are coloured $i$.

(Claim I.1). For each $n \in \mathbb{N}$, there exists $y_n \in V(S(D))$, $T_n \subseteq S(D)$ and $T'_n \subseteq S(D)$, such that:

(a) $T_n$ is an $ay_n$-directed path coloured $i$,

(b) $T'_{n+1}$ is an $yn a_{n+1}$-directed path coloured $i$, contained in $T'_n$,

(c) $y_n \in V(T_n) \cap V(T'_n)$,

(d) $T_n \cup T'_{n+1}$ is a directed path,

(e) For $n \geq 2$, $y_n \neq y_j$ for each $j$, $j \leq n - 1$ and

(f) For $n \geq 2$, $T_n$ contains $T_{n-1}$.

We proceed by induction on $n$. For $n = 1$, let $T'_2 = (a_1 = z_{1,1}, z_{1,2}, ..., z_{1,k_1} = a_2)$ be and $i_1 = \max\{j \in \{1, 2, ..., k_1\} \mid z_{1,j} \in V(T'_1)\}$, we take $y_1 = z_{1,i_1}$, $T_1 = (a, T'_1, y_1)$ and $T'_2 = (y_1, T'_2, a_2)$, clearly they satisfy properties (a) to (f). For $n = 2$, let $T'_3 = \ldots$
(a₂ = z₂,1, z₂,2, ..., z₂,k₂ = a₃) be and i₂ = max{j ∈ {1, 2, ..., k₂}}{z₂,j ∈ V(T''₂)}; we take y₂ = z₂,i₂, T₂ = T₁ ∪ (y₁, T''₂, y₂) and T'''₃ = (y₂, T'''₃, a₃)

(a) follows directly from the fact T₂ ⊆ T₁ ∪ T''₂ which is a directed path coloured i.

(b) follows from the definition of T''₂. To prove (c) observe that from the definition of y₂ we have y₂ ∈ V(T''₂), and T''₂ ⊆ T₂ (see (b) of case n = 1); on the other hand we have y₂ ∈ V(T₂) by the definition of T₂, we conclude y₂ ∈ V(T₂) ∩ V(T''₂). Now we prove that T₂ ∪ T'''₃ is a directed path ((d)), the choice of y₂ implies V(y₁, T''₂, y₂) ∩ V(T'''₃) = {y₂} when V(T₁) ∩ V(T'''₃) = ∅ (taking z ∈ V(T₁) ∩ V(T'''₃) we have that (a₂, T''₃, y₂) ∪ (y₂, T'''₃, z) ∪ (z, T₁, y₁) ∪ (y₁, T₁, a₁) contains an a₂a₁-monochromatic directed path coloured i, contradicting a₂ < a₁, hence V(T₁) ∩ V(T'''₃) = ∅ and T₂ ∪ T'''₃ is a monochromatic directed path coloured i. To prove (e) (y₂ ≠ y₁), suppose by contradiction y₁ = y₂, then (a₂, T''₃, y₂ = y₁) ∪ (y₁, T''₁, a₁) contains an a₂a₁-directed path coloured i, contradicting that a₂ < a₁, so y₂ ≠ y₁. Finally (f) (T₂ contains T₁) follows from the definition of T₂.

Suppose that if n ≥ 2, then for each j ∈ {1, ..., n} there exists, yⱼ ∈ V(S(D)), Tⱼ ⊆ S(D) and T''ₙ₊₁ ⊆ S(D) such that: (a) Tⱼ is an aⱼyⱼ-directed path coloured i, (b) T''ₙ₊₁ is a yⱼaⱼ₊₁-directed path coloured i contained in T''ₙ₊₁, (c) yⱼ ∈ V(Tⱼ) ∩ V(T''ₙ₊₁), (d) Tⱼ ∪ T''ₙ₊₁ is a directed path, (e) yⱼ ≠ yⱼ whenever k ≤ j < j, j ≥ 2, and (f) Tⱼ contains Tⱼ₋₁ for each j ≥ 2.

We will prove properties (a) to (f) for n + 1.

Let T''ₙ₊₂ = (aₙ₊₁ = zₙ₊₁,₁, zₙ₊₁,₂, ..., zₙ₊₁,kₙ₊₁,aₙ₊₂) and

\[ i_{n+1} = \max \{j \in \{1, 2, ..., k_{n+1}\} \mid z_{n+1,j} \in V(T''_{n+1})\} \]

take yₙ₊₁ = zₙ₊₁,iₙ₊₁, Tₙ₊₁ = Tₙ ∪ (yₙ, T''ₙ₊₁, yₙ₊₁) and T''ₙ₊₂ = (yₙ₊₁, T''ₙ₊₂, aₙ₊₂).

(a) Tₙ₊₁ is an aₙ₊₁-directed path coloured i. It follows from the inductive hypothesis (d) on j = n that Tₙ ∪ T''ₙ₊₁ is a directed path coloured i, thus Tₙ₊₁ is an aₙ₊₁-directed path coloured i.

(b) T''ₙ₊₂ is an yₙ₊₁aₙ₊₂-directed path coloured i. It follows from the definition of T''ₙ₊₂.

(c) yₙ₊₁ ∈ V(Tₙ₊₁) ∩ V(T''ₙ₊₁). It follows from the choice of yₙ₊₁ that yₙ₊₁ ∈ V(T''ₙ₊₁) and from (b) in the inductive hypothesis Tₙ₊₁ ⊆ T''ₙ₊₁; on the other hand it follows from the definition of Tₙ₊₁ that yₙ₊₁ ∈ Tₙ₊₁.

(d) Tₙ₊₁ ∪ T''ₙ₊₂ is a directed path. The definition of yₙ₊₁ implies that

\[ V(yₙ, T''ₙ₊₁, yₙ₊₁) \cap V(T''ₙ₊₂) = \{yₙ₊₁\}. \]

If there exists z ∈ V(Tₙ) ∩ V(T''ₙ₊₂) then (aₙ₊₁, T''ₙ₊₂, yₙ₊₁) ∪ (yₙ₊₁, T''ₙ₊₂, z) ∪ (z, Tₙ, yₙ) ∪ (yₙ, Tₙ, aₙ) contains an aₙ₊₁aₙ-directed path coloured i, contradicting that aₙ₊₁ < aₙ; thus V(Tₙ) ∩ V(T''ₙ₊₂) = ∅ and Tₙ₊₁ ∪ T''ₙ₊₂ is a directed path (and clearly coloured i).

(e) yₙ₊₁ ≠ yⱼ for each j ≤ n. We proceed by contradiction; assume that yₙ₊₁ = yⱼ for some j ≤ n. We have from (f) on the inductive hypothesis that Tⱼ ⊆ Tₙ; and form (c) on the inductive hypothesis yⱼ ∈ V(Tⱼ) then yⱼ ∈ V(Tₙ) and yⱼ ∈ V(Tₙ), so {yⱼ, yₙ} ⊆ V(Tₙ). Thus (aₙ₊₁, T''ₙ₊₂, yₙ₊₁ = yⱼ) ∪ (yⱼ, Tₙ, yₙ) ∪ (yₙ, Tₙ, aₙ) contains an aₙ₊₁aₙ-directed path coloured i, contradicting that aₙ₊₁ < aₙ.

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(f) \( T_{n+1} \) contains \( T_n \). It follows from the definition of \( T_{n+1} \); and Claim I.1 is proved.

Now, let \( T = \bigcup_{n=1}^{\infty} T_n \) be, clearly it follows from Claim I.1 that \( T \) is a monochromatic infinite outward path contained in \( S(D) \), a contradiction. So Claim I is proved.

It follows from Claim I and the Zorn’s Lemma that \( (A(D)/\sim, \leq) \) has minimal elements.

Consider the minimal elements of \( (A(D)/\sim, \leq) \) and let \( N' \) be a set obtained by taking a representant of each one of these classes.

Denote by \( N = \{v \in V(D) | \Gamma_D(v) = \emptyset \} \cup \{a \in N' | \Gamma_D(\Gamma_{S(D)}(a)) \neq \emptyset \} \). Clearly \( \emptyset \neq N \subseteq V(S(D)) \).

(Claim II). If \( a \in A(D) \) and \( \Gamma_D(\Gamma_{S(D)}(a)) = \emptyset \), then \( \bar{a} = \{a\} \) and \( \bar{a} \) is a minimal element of \( (A(D)/\sim, \leq) \).

Clearly if \( a \in A(D) \) with \( \Gamma_D(\Gamma_{S(D)}(a)) = \emptyset \), then there is no monochromatic directed path of length at least two in \( S(D) \) starting in \( a \); thus for any \( b \in A(D) \), \( b \neq a \) there is no \( ab \)-monochromatic directed path.

(Claim III). For any \( x, y \in N \) with \( x \neq y \), there is no \( xy \)-monochromatic directed path in \( S(D) \).

(Case III.a). \( x, y \in A(D) \). In this case \( x, y \in N' \), so \( \bar{x}, \bar{y} \) are different minimal classes of \( (A(D)/\sim, \leq) \). If there exists an \( xy \)-monochromatic directed path in \( S(D) \) then \( \bar{y} \leq \bar{x} \) and since \( \bar{x} \) is minimal if follows \( \bar{x} = \bar{y} \), contradiction.

(Case III.b). \( x \in V(D) \). From the definition of \( N \), we have \( \Gamma_D(x) = \emptyset \), so \( \Gamma_{S(D)}(x) = \emptyset \) and clearly Claim III holds.

(Case III.c). \( x \in A(D) \) and \( y \in V(D) \). Since \( y \in N \cap V(D) \) we have \( \Gamma_D(y) = \emptyset \). Suppose by contradiction that there exists an \( xy \)-monochromatic directed in \( S(D) \) and let \( T = (x = x_1, x_2, \ldots, x_k = y) \) be such a path. It follows from the definition of \( S(D) \) that \( x_{k-1} \in A(D) \), \( \Gamma_{S(D)}(x_{k-1}) = \{y\} \) and thus \( \Gamma(\Gamma_{S(D)}(x_{k-1}))) = \emptyset \); now it follows from Claim II that \( \bar{x}_{k-1} = \{x_{k-1}\} \) and \( \bar{x}_{k-1} \) is a minimal element of \( (A(D)/\sim, \leq) \). Since \( (x, T, x_{k-1}) \) is an \( xx_{k-1} \)-monochromatic directed path, we have \( \bar{x}_{k-1} \leq \bar{x} \) and then \( \bar{x}_{k-1} = \bar{x} \) (\( \bar{x} \) is minimal in \( (A(D)/\sim, \leq) \) as \( x \in N \cap A(D) \)). It follows that \( x_{k-1} = x \) and \( \Gamma_D(\Gamma_{S(D)}(x)) = \Gamma_D(\Gamma_{S(D)}(x_{k-1}))) = \emptyset \), contradicting that \( x \in N \).

(Claim IV). For each \( z \in V(S(D)) - N \) there exists a \( zN \)-monochromatic directed path in \( S(D) \).

Take \( z \in V(S(D)) - N \), and consider the two following cases:

(Case IV.a). \( z \in A(D) \).

If \( \bar{z} \) is a minimal element of \( (A(D)/\sim, \leq) \), then there exists \( a \in \bar{z} \cap N' \) (by the definition of \( N' \)). When \( a \in N' \), we have \( a \neq z \), and there exists a \( za \)-monochromatic directed path (as \( z \sim a \)) so there exists \( zN \)-monochromatic directed path. When \( a \notin N \) we obtain \( \Gamma_D(\Gamma_{S(D)}(a)) = \emptyset \) and \( \bar{a} = \{a\} \) (Claim II and definition of \( N \)); hence \( z = a \); let \( v \in V(D) \) be such that \( \Gamma_{S(D)}(a) = \{v\} \); so \( v \in N \) and \((z = a, v)\) is a monochromatic directed path in \( S(D) \).
If \( \bar{z} \) is not a minimal element of \((A(D)/\sim, \leq)\) then it follows from the Zorn’s Lemma that there exists \( a \in A(D) \), \( a \neq z \) such that \( \bar{a} \) is a minimal element of \((A(D)/\sim, \leq)\) and \( \bar{a} \leq \bar{z} \) and \( a \in N' \), which implies that there exists a \( z\alpha \)-monochromatic directed path, namely \( T \); when \( a \in N \) we obtain that \( T \) is a \( zN \)-monochromatic directed path in \( S(D) \); when \( a \notin N \) we have that \( T' = (a, v) \) is an \( aN \)-monochromatic directed path in \( S(D) \), where \( \Gamma_{S(D)}(a) = \{v\} \) (recall that \( \Gamma_D(\Gamma_{S(D)}(a)) = \emptyset \) as \( a \notin N \)); now, from Lemma ?? we have that \( T \) and \( T' \) are coloured alike and then \( T \cup T' \) contains a \( zN \)-monochromatic directed path in \( S(D) \).

\((\text{Case IV.b). } z \in V(D)\).

Since \( z \notin N \), we have \( \Gamma_D(z) \neq \emptyset \) and \( \Gamma_{S(D)}(z) \neq \emptyset \). Let \( a \in \Gamma_{S(D)}(z) \) be it follows from the definition of \( S(D) \) that \( a \in A(D) \). If \( a \in N \), then \( (z, a) \) is a \( zN \)-monochromatic directed path contained in \( S(D) \). If \( a \notin N \) then it follows from Case IV.a that there exists an \( ax \)-monochromatic directed path in \( S(D) \), namely \( T \), for some \( x \in N \), clearly \( a \neq x \) and then from Lemma ?? we have that \( T \) and \( (z, a) \) are coloured alike; thus \( (z, a) \cup T \) contains a \( zN \)-monochromatic directed path.

We conclude that \( N \) is a kernel by monochromatic paths of \( S(D) \).

When \( A(D) = \emptyset \); clearly \( V(D) = V(S(D)) \) is a kernel by monochromatic paths of \( S(D) \). \( \square \)

**Remark 2.1** The method developed in the proof of Theorem ?? allows to construct some kernels by monochromatic paths of \( S(D) \); the number of these kernels can be very different of the number of kernels by monochromatic paths of \( S(D) \); for example consider \( D \) a monochromatic directed cycle; in this case we obtain \( n \) different kernels by monochromatic paths (one for each arc of \( D \)), however \( S(D) \) has \( 2n \) kernels by monochromatic paths (as each vertex of \( S(D) \) is a kernel by monochromatic paths of \( S(D) \)).

The following result asserts that if \( D \) has no monochromatic directed cycles then the kernel by monochromatic paths of \( S(D) \) obtained with the method developed in the proof of Theorem ?? is the unique kernel by monochromatic paths of \( S(D) \).

**Theorem 2.2** Let \( D \) be an \( m \)-coloured digraph which has no monochromatic infinite outward path, and \( S(D) \) its subdivision digraph. If \( D \) has no monochromatic directed cycles, then \( S(D) \) has an unique kernel by monochromatic paths.

**Proof:** Consider the relations \( \sim \) and \( \leq \) defined in the proof of Theorem ???. First we prove that each equivalence class of \( A(D)/\sim \) has exactly one element. Assume by contradiction that there exist \( a, b \in A(D) \) such that \( a \neq b \) and \( a \sim b \); thus we have that there exists in \( S(D) \) an \( ab \)-monochromatic directed path, namely \( T_1 \), and a \( ba \)-monochromatic directed path, namely \( T_2 \); it follows form Lemma ?? that \( T_1 \) and \( T_2 \) are coloured alike; so \( T_1 \cup T_2 \) contains a monochromatic directed cycle and clearly this implies that \( D \) contains a monochromatic directed cycle, a contradiction.
The previous assertion implies that the method developed in the proof of Theorem ?? allow us to construct an unique kernel by monochromatic paths for $S(D)$, let $N$ be such kernel by monochromatic paths.

Let $N'$ be a kernel by monochromatic paths of $S(D)$; we will prove that $N' = N$.

Let $x \in N$. If $x \in V(D)$ then by definition of $N$, $\Gamma_D(x) = \emptyset$ which implies $\Gamma_{S(D)}(x) = \emptyset$ and clearly $x$ belongs to each kernel by monochromatic paths of $S(D)$, in particular $x \in N'$. If $x \in A(D)$, then the definition of $N$ implies that $\bar{x}$ is a minimal element of $(A(D)/\sim, \leq)$. Now suppose by contradiction that $x \notin N'$; since $N'$ is a kernel by monochromatic paths of $S(D)$, there exists an $xx'$-monochromatic directed path in $S(D)$ for some $x' \in N'$; this implies that $x = x'$ (as $\bar{x} \leq \bar{x}$, $\bar{x}$ is minimal, and $|\bar{x}| = 1$), a contradiction. We conclude that $N \subseteq N'$.

Now let $x' \in N'$ be. Assume by contradiction that $x' \notin N$, it follows that there exists an $x'x$-monochromatic directed path for some $x \in N$; so there exists an $x'N'$-monochromatic directed path with $x' \in N'$ (as $N \subseteq N'$); contradicting that $N'$ is a kernel by monochromatic paths. Thus, $N' \subseteq N$. \hfill $\square$

**Remark 2.2** Figure 1 shows a digraph $D$ which has a monochromatic directed cycle and whose subdivision digraph has an unique kernel by monochromatic paths, namely $\{(v, x), (x, v)\}$. So the converse assertion of those of Theorem ?? is not true.

## 3 A generalization of the subdivision digraph of an $m$-coloured digraph

In this section we define a generalization of the subdivision digraph of an $m$-coloured digraph $D$; and we prove that it has a kernel by monochromatic paths whenever $D$ has no monochromatic infinite outward path.

**Definition 3.1** Let $D$ be an $m$-coloured digraph and $S(D)$ its subdivision digraph. We define the $m$-coloured digraph $S'(D)$ as follows:

$$
S'(D) = S(D) - \{(u, a) \mid a \in A(D) \text{ and } u \text{ is the initial vertex of } a\} \cup \beta_a_{a \in A(D)}
$$

where $\{\beta_a \mid a \in A(D)\}$ is a set of monochromatic directed paths such that; if $a \in A(D)$ and $a = (u, v)$ then:

(i) $\beta_a$ is a $ua$-monochromatic directed path of the same colour as $(u, a)$ in $S(D)$.

(ii) $V(\beta_a) \cap V(S(D)) = \{u, a\}$, and

(iii) $V(\beta_a) - \{u\} \cap V(\beta_b) = \emptyset$ for $a \neq b$

$S'(D)$ is obtained from $S(D)$ by changing (for each $a \in A(D)$) the arc $a'$ which incides toward $a$, for a monochromatic directed path of the same colour as $a'$. 

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Theorem 3.1 Let $D$ be an $m$-coloured digraph. If $D$ has no monochromatic infinite outward path, then $S'(D)$ has a kernel by monochromatic paths.

Proof: It follows from Theorem ?? that the subdivision digraph $S(D)$ has a kernel by monochromatic paths. Let $N$ be a kernel by monochromatic paths of $S(D)$. We will prove that $N$ is a kernel by monochromatic paths of $S'(D)$. If $a = (u, v) \in A(D)$ and $\beta_a$ has length at least two, then we denote $\beta'_a = \beta_a - \{u, a\}$, notice that $V(\beta'_a) \cap V(S(D)) = \emptyset$.

(Claim I). For $x, y \in N$, $x \neq y$; there is no $xy$-monochromatic directed path in $S'(D)$.

Let $x, y \in N$ and suppose by contradiction that there exists an $xy$-monochromatic directed path in $S'(D)$, namely $T'$. Since $N$ is a kernel by monochromatic paths in $S(D)$, then $T'$ is not contained in $S(D)$; so, there exists $a = (u, v) \in A(D)$, such that the length of $\beta_a$, $\ell(\beta_a)$ is at least two and $V(T') \cap V(\beta'_a) \neq \emptyset$; moreover $(V(T') - \{x, y\}) \cap V(\beta'_a) \neq \emptyset$ (as $\{x, y\} \subseteq V(S(D))$); and clearly this implies $\beta_a \subseteq T'$ (from the definition of $S'(D)$). Let

$$A' = \{a \in A(D) \mid \ell(\beta_a) \geq 2 \text{ and } V(T') \cap V(\beta'_a) \neq \emptyset\}$$

and $T = (T' - (\bigcup_{a \in A'} V(\beta'_a))) \cup \{(u, a) \text{ coloured } i \mid a = (u, v) \in A' \text{ and } \beta_a \text{ is } i \text{ coloured}\}$; clearly $T$ is an $xy$-monochromatic directed path in $S(D)$, contradicting that $N$ is a kernel by monochromatic paths of $S(D)$.

(Claim II). For each $z \in V(S'(D)) - N$, there exists a $zN$-monochromatic directed path in $S'(D)$.

Let $z \in V(S'(D)) - N$ be. If $z \in V(S(D))$ then there exists a $zN$ monochromatic directed path in $S(D)$ and clearly this implies that there exists a $zN$-monochromatic directed path in $S'(D)$. So we may assume $z \notin V(S(D))$; then there exists $a = (u, v) \in A(D)$ with $\ell(\beta_a) \geq 2$ and $z \in V(\beta'_a)$. If $a \in N$, then $(z, \beta_a, a)$ is a $zN$-monochromatic directed path in $S'(D)$. If $a \notin N$, then there exists an $aN$-monochromatic directed path in $S(D)$, namely $T$, now let $T' = T - \{(u, b) \mid b \in A(D) \cap V(T)\} \cup \beta_b$ be; clearly $(z, \beta_a, a) \cup T'$ is a $zN$-monochromatic directed path in $S'(D)$.

\[\square\]

Theorem 3.2 Let $D$ be an $m$-coloured digraph which has no monochromatic infinite outward path. If $D$ has no monochromatic directed cycle the $S'(D)$ has an unique kernel by monochromatic paths.

Proof: Notice that since $D$ has no monochromatic directed cycle, then $S'(D)$ has no monochromatic directed cycle.

(Claim I). Every kernel by monochromatic paths of $S'(D)$ also is of $S(D)$.

Let $N'$ be a kernel by monochromatic paths of $S'(D)$.

(Claim I.1). $N' \subseteq V(S(D)) = V(D) \cup A(D)$. Assume by contradiction that there exists $x \in (N' - (V(D) \cup A(D)))$, then there exists $a = (u, v) \in A(D)$ such
that $x \in (V(\beta_a) - \{u, a\})$. Let $i$ be the colour of $a$ in $D$; thus $\beta_a$ is $i$ coloured. Let $T_1 = (x, \beta_a, a)$; since $x \in N'$ and $T_1$ is a monochromatic directed path, we have $a \notin N'$ (recall $N'$ is a kernel by monochromatic paths). Since $a \notin N'$, there exists $x' \in N'$ and an $ax'$-monochromatic directed path in $S'(D)$, namely $T_2$; observe that $T_2$ is $i$ coloured ($(a, v) \in A(T_2)$ as $\Gamma_{S'(D)}(a) = \{v\}$; and $(a, v)$ is $i$ coloured). If $x = x'$ then $T_1 \cup T_2$ contains a monochromatic directed cycle, a contradiction. If $x \neq x'$, then $T_1 \cup T_2$ contains an $xx'$-monochromatic directed path with $\{x, x'\} \subseteq N'$, a contradiction.

**Claim I.2.** For $x, y \in N'$, $x \neq y$; there is no $xy$-monochromatic directed path in $S(D)$.

Let $x, y \in N'$, $x \neq y$ and assume by contradiction that there exists an $xy$-monochromatic directed path $T$ contained in $S(D)$, then $T' = T - \{(u, a) | a \in A(D) \cap V(T)\} \cup \bigcup_{a \in A(D) \cap V(T)} \beta_a$ is a $xy$-monochromatic directed path in $S'(D)$, a contradiction.

**Claim I.3.** For each $z \in V(S(D)) - N'$, there exists a $zN'$-monochromatic directed path in $S(D)$.

By the definition of $N'$, there exists a $zN'$-monochromatic directed path in $S'(D)$ and clearly it follows that there exists a $zN'$-monochromatic directed path in $S(D)$.

It follows directly from Theorem ?? and Claim I that $S'(D)$ has an unique kernel by monochromatic paths.

\[\Box\]

4 The digraph $R'(D)$

In this section we consider an $m$-coloured digraph $D$, and define the digraph $R'(D)$ which is nearly related to $D$ and $S'(D)$. As a consequence of Theorem ?? we prove that $R(D)$ has a kernel by monochromatic paths whenever $D$ has no monochromatic infinite outward path.

**Definition 4.1** Given a generalized subdivision $S'(D)$ of an $m$-coloured digraph $D$, we define the digraph $R'(D)$ as follows: $R'(D) = S'(D) \cup D \cup \bigcup_{a \in A(D)} \Lambda_a$.

For each $a = (u, v) \in A(D)$ coloured $i$, $\Lambda_a$ is a set of $Bv$-arcs coloured $i$, where $B = V(\beta_a) \setminus \{u, a\}$.

**Theorem 4.1** Let $S'(D)$ be a generalized subdivision of an $m$-coloured digraph $D$. If $R'(D)$ is obtained from $S'(D)$ then $\mathcal{C}(S'(D)) = \mathcal{C}(R'(D))$.

**Proof:** Since $S'(D) \subseteq R'(D)$; we have $\mathcal{C}(S'(D)) \subseteq \mathcal{C}(R'(D))$. Now we will prove that $R'(D)$ is a subdigraph of $\mathcal{C}(S'(D))$. From the definition of $R'(D)$, we only need to prove that $\Delta \cup \bigcup_{a \in A(D)} \Lambda_a \subseteq A(\mathcal{C}(S'(D)))$, where $\Delta = \{a \in A(R'(D)) | a \in A(D)\}$. Let $a \in A(R'(D))$ such that $a \in A(D)$, we may assume that $a = (u, v)$ is coloured $i$ and $\beta_a = (u = x_0, \ldots, x_k = a)$; so, for each $j \in \{0, \ldots, k - 1\}$ we have that $(x_j, \beta_a, a) \cup (a, v)$ is an $x_jv$-directed path coloured $i$ in $S'(D)$, thus $(x_j, v) \in A(\mathcal{C}(S'(D)))$; in particular
\{a\} \cup \Lambda_a \subset A(\mathcal{C}(S'(D))).\) Clearly, for any \(m\)-coloured digraph \(D\); \(\mathcal{C}(\mathcal{C}(D)) \cong \mathcal{D};\) in particular \(\mathcal{C}(\mathcal{C}(S'(D))) \cong \mathcal{C}(S'(D))\) and then \(\mathcal{C}(R'(D)) \subseteq \mathcal{C}(S'(D)).\)

**Corollary 4.1** Let \(S'(D)\) be a generalized subdivision of an \(m\)-coloured digraph \(D\). If \(R'(D)\) is obtained from \(S'(D)\), then \(S'(D)\) and \(R'(D)\) have the same number of kernels by monochromatic paths.

*Proof:* Recall that if \(D\) is an \(m\)-coloured digraph, then \(N\) is a kernel by monochromatic paths of \(D\) iff \(N\) is a kernel of \(\mathcal{C}(D)\). Then the assertion follows from the fact \(\mathcal{C}(S'(D)) \cong \mathcal{C}(R'(D)).\)

**Corollary 4.2** If \(D\) is an \(m\)-coloured digraph which has no monochromatic infinite outward path, then \(R'(D)\) has a kernel by monochromatic paths.

It follows directly from Corollary ?? and Theorem ??.

**Corollary 4.3** Let \(S'(D)\) be a generalized subdivisions of an \(m\)-coloured digraph \(D\) which has no monochromatic infinite outward path, and \(R'(D)\) obtained from \(S'(D)\). If \(D\) has no monochromatic directed cycles, then \(R'(D)\) has an unique kernel by monochromatic paths.

It follows directly from Theorem 3.2 and Corollary ??.

## 5 The middle digraph of an \(m\)-coloured digraph

In \[?]\ J. Topp proves that the middle digraph of any digraph has a kernel. In this section we define the middle of an \(m\)-coloured digraph \(D\), and we prove that if \(D\) has a kernel by monochromatic paths then its middle digraph has a kernel by monochromatic paths. In \[?]\ is defined the inner colouration of the line digraph \(L(D)\) of an \(m\)-coloured digraph \(D\) and it is proved that if \(D\) has no monochromatic directed cycles then the number of kernels by monochromatic paths of \(D\) is equal to the number of kernels by monochromatic paths of an inner colouration of \(L(D)\).

**Definition 5.1** Let \(D\) be a digraph, the line digraph \(L(D)\) of \(D\) is defined as follows: \(V(L(D)) = A(D)\) and for \(h, k \in V(L(D)), (h, k) \in A(L(D))\) iff the terminal endpoint of \(h\) is the initial endpoint of \(k\).

**Definition 5.2** ([?]) Let \(D\) be an \(m\)-coloured digraph and \(L(D)\) its line digraph; the inner \(m\)-colouration of \(L(D)\) is the colouration of the arc of \(L(D)\) defined as follows: If \(h\) is an arc of \(D\), coloured \(c\); then any arc of \(L(D)\) of the form \((x, h)\) is colored \(c\).

**Definition 5.3** Given an \(m\)-coloured digraph \(D\), we define the middle digraph \(Q(D)\) of \(D\) as follows: \(V(Q(D)) = V(D) \cup A(D)\) and,

\[
\Gamma_{Q(D),i}(x) = \begin{cases} 
\{x\} \times \Gamma_{D,i}(x) & \text{if } x \in V(D); \\
\{v\} & \text{if } x = (u, v) \in A(D) \text{ and } v \in \Gamma_{D,i}(u), \\
\{v\} \times \Gamma_{D,i}(v) & \text{if } x = (u, v) \in A(D).
\end{cases}
\]
Notice that $Q(D)$ is obtained from the subdivision digraph $S(D)$ of $D$, by adding on the vertices of $S(D)$ which correspond to arcs of $D$, a copy of the line digraph of $D$ with the inner colouration.

**Lemma 5.1** Let $D$ be an $m$-coloured digraph. If $a, b \in A(D)$ and $T$ is an ab-monochromatic directed path (say coloured $i$) of minimum length in $Q(D)$, then $V(T) \subseteq A(D)$.

**Proof**: Let $a, b$ and $T$ as in the hypothesis and assume by contradiction that $T = (a = x_1, x_2, \ldots, x_k = b)$ and $x_2 \notin A(D)$ for some $j \in \{2, \ldots, k - 1\}$. So, from the definition of $Q(D)$ we have; $x_j \in V(D)$, $x_{j-1}$ and $x_{j+1}$ correspond to arcs of $D$ coloured $i$; for some $u, v \in V(D)$, $x_{j-1} = (u, x_j)$ and $x_{j+1} = (x_j, v)$; which (again from the definition of $Q(D)$) implies $(x_{j-1}, x_{j+1}) \in A(Q(D))$ and is coloured $i$. Thus $T' = (T - \{x_j\} \cup (x_{j-1}, x_{j+1})$ is an ab-directed path coloured $i$ with $\ell(T') < \ell(T)$, a contradiction. \qed

**Lemma 5.2** Let $D$ be an $m$-coloured digraph and $T$ an ab-directed path coloured $i$ in $Q(D)$. If $V(T) \subseteq A(D)$, $a = (u, s)$ and $b = (t, v)$, then there exists an sv-directed path coloured $i$ in $D$.

**Proof**: Let $T = (a = a_1, a_2, \ldots, a_k = b)$, then for each $j \in \{2, \ldots, k\}$ we have that $a_j \in \Gamma_{Q(D),i}(a_{j-1})$ and from the definition of $Q(D)$, $a_j$ is $i$ coloured; also for each $j \in \{2, \ldots, k - 1\}$ there exists $v_j \in V(D)$ such that $a_j = (v_{j-1}, v_j)$. Thus $(s = v_1, v_2, \ldots, v_{k-1} = t, v)$ contains an sv-directed path coloured $i$. \qed

**Lemma 5.3** Let $D$ be an $m$-coloured digraph. If there exists an uv-directed path coloured $i$ in $D$ then there exists an uv-directed path coloured $i$ in $Q(D)$.

**Proof**: Suppose that $T = (u = v_1, v_2, \ldots, v_k = v)$ is an uv-directed path coloured $i$ in $D$; for each $j \in \{2, \ldots, k\}$ let $a_j = (v_{j-1}, v_j)$ clearly $a_j \in A(D)$ and is $i$ coloured. Thus $(a_2, \ldots, a_k)$ is a directed path coloured $i$ in $Q(D)$; now, since $(u, a_2), (a_k, v) \in A(Q(D))$ and are coloured $i$, we conclude that $(u, a_2, a_3, \ldots, a_k, v)$ is an uv-directed path coloured $i$ in $Q(D)$. \qed

**Lemma 5.4** Let $D$ be an $m$-coloured digraph $u, v \in V(D)$. If there exists an uv-directed path coloured i in $Q(D)$, then there exists an uv-directed path coloured i in $D$.

**Proof**: Let $T = (u = x_1, x_2, \ldots, x_k = v)$ be an uv-directed path coloured $i$ in $Q(D)$. Thus $x_2$ and $x_{k-1}$ are arcs coloured $i$ of $D$, and there exist $s, t \in V(D)$ such that $x_2 = (s, u)$ and $x_{k-1} = (t, v)$. Since $(x_2, T, x_{k-1})$ is a directed path coloured $i$, we may consider $T'$ an $x_2x_{k-1}$ directed path coloured $i$ of minimum length in $Q(D)$; from Lemma ?? $V(T') \subseteq A(D)$ and from Lemma ?? there exists an sv-directed path coloured $i$ in $D$, namely $T''$, so $(u, s) \cup T''$ contains an uv-directed path coloured $i$. \qed

**Theorem 5.1** Let $D$ be an $m$-coloured digraph and $Q(D)$ its middle digraph. Every kernel by monochromatic paths of $D$ is a kernel by monochromatic paths of $Q(D)$. 

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Proof: Let $N$ be a kernel by monochromatic paths of $D$.

(Claim I). For each $u, v \in N$ there is no $uv$-monochromatic directed path in $Q(D)$.
Assume by contradiction there exists an $uv$-monochromatic directed path in $Q(D)$, then it follows from Lemma ?? that there exists an $uv$-monochromatic directed path in $D$, a contradiction.

(Claim II). For each $z \in V(Q(D)) - N$, there exists a $zN$-monochromatic directed path.

(Case II.a). $z \in V(D)$; since $N$ is kernel by monochromatic paths of $D$; there exists $v \in N$ and a $zv$-monochromatic directed path in $D$; thus it follows from Lemma ?? that there exists a $zv$-monochromatic directed path in $Q(D)$.

(Case II.b). $z \in A(D)$; let $u, v \in V(D)$ such that $z = (u, v)$. If $v \in N$ then $(z, v)$ is a monochromatic directed path in $Q(D)$. If $v \notin N$ then if follows from Case II.a that there exists a $vw$-monochromatic directed path $T$ for some $w \in N$, in $Q(D)$; say $T$ is $i$ coloured and let $T = (v = x_1, x_2, \ldots, x_k = w)$. Thus: $x_2 \in A(D)$, $x_2 = (v, s)$ for some $s \in V(D)$, $x_2$ is coloured $i$, and $(z, x_2) \in A(Q(D))$ coloured $i$. Hence $(z, x_2) \cup (x_2, T, w)$ contains a $zw$-directed path coloured $i$ contained in $Q(D)$.

\[ \square \]

Corollary 5.1 The number of kernels by monochromatic paths of $D$ is less than or equal to the number of kernels by monochromatic paths of $Q(D)$.

Observation 5.1 Notice that if $D$ is a monochromatic directed cycle of length $n$; then $D$ has $n$ kernels by monochromatic paths, whereas $Q(D)$ has $2n$ kernels by monochromatic paths.

Lemma 5.5 Let $D$ be an $m$-coloured digraph and $(D)$ its middle digraph. If $C$ is a directed cycle coloured $i$ of minimum length in $Q(D)$, then $V(C) \subseteq A(D)$.

Proof: We proceed by contradiction. Suppose that $C = (x_0, x_1, \ldots, x_{k-1}, x_0)$ is a monochromatic directed cycle (say, coloured $i$) and that $V(C) \not\subseteq A(D)$.

If $k = 2$, we may assume $x_0 \notin A(D)$; so $x_0 \in V(D)$ and $x_1 \in A(D)$ coloured $i$. Since $\{(x_0, x_1), (x_1, x_0)\} \subseteq A(Q(D))$, there exist $u, v \in V(D)$ such that, $x_1 = (x_0, u)$ and $(v, x_0) = x_1$ and then $x_1 = (x_0, x_0)$, a contradiction.

If $k \geq 3$, let $j \in \{0, 1, \ldots, k - 1\}$ such that $x_j \notin A(D)$. Thus $x_j \in V(D); x_{j+1}$ and $x_{j-1}$ are arcs of $D$ coloured $i$; for some $u, v \in V(D), x_{j-1} = (u, x_j)$ and $x_{j+1} = (x_j, v)$; and $(x_{j-1}, x_j)$ is an arc of $Q(D)$ coloured $i$. Hence $C' = (C - \{x_j\}) \cup (x_{j-1}, x_j)$ is a directed cycle coloured $i$, with $\ell(C') < \ell(C)$; a contradiction.

\[ \square \]

Lemma 5.6 Let $D$ be an $m$-coloured digraph and $Q(D)$ its middle digraph. If $D$ has no monochromatic directed cycle then $Q(D)$ has no monochromatic directed cycle.

Proof: Let $D$ and $Q(D)$ be as in the hypothesis and assume by contradiction that $Q(D)$ has a monochromatic directed cycle and let $C$ be one of minimum length, say $C$
is i coloured, $C = (a_0, a_1, \ldots, a_{k-1}, a_0)$; it follows from Lemma 5.2 that $V(C) \subseteq A(D)$. From the definition of $Q(D)$, there exists $v_j \in V(D)$ for each $j \in \{0, \ldots, k\}$ such that $a_j = (v_j, v_{j+1})$, $v_k = v_0$, and $a_{j+1}$ is i coloured in $D$. Thus $(v_0, v_1, \ldots, v_k, v_0)$ contains an i coloured directed cycle of $D$; a contradiction. \hfill \Box

**Lemma 5.7** Let $D$ be an $m$-coloured digraph and $Q(D)$ its middle digraph. If $D$ has no monochromatic directed cycle and $N$ is a kernel by monochromatic paths of $Q(D)$, then $N \subseteq V(D)$.

**Proof:** Let $D$, $Q(D)$ and $N$ be as in the hypothesis and assume by contradiction that there exists $a = (u, v) \in A(D) \cap N$. From definition of $Q(D)$ we have $(a, v) \in A(Q(D))$ and then $v \notin N$. From the definition of $N$, there exists $z \in N$ and a monochromatic directed path $T = (v = x_1, x_2, \ldots, x_k = z)$ (say, coloured i). The definition of $Q(D)$ implies $x_2 = (v, w) \in A(D)$ and is i coloured; $(a, x_2) \in A(Q(D))$ and is i-coloured. If $z \neq a$ then $(a, x_2) \cup (x_2, T, z)$ contains an az-directed path coloured i in $Q(D)$ with $a, z \in N$; a contradiction. If $z = a$, then $(a = z, x_2) \cup (x_2, T, z)$ contains a directed cycle coloured i, contained in $Q(D)$; a contradiction. \hfill \Box

**Theorem 5.2** Let $D$ be an $m$-coloured digraph which has no monochromatic directed cycle and $Q(D)$ its middle digraph. A set $N$ is a kernel by monochromatic paths of $D$ if and only if $N$ is a kernel by monochromatic paths of $Q(D)$.

**Proof:** In view of Theorem 5.1, we only need to prove that if $N$ is a kernel by monochromatic paths of $Q(D)$, then $N$ is a kernel by monochromatic paths of $D$. Let $N$ be a kernel by monochromatic paths of $Q(D)$. From Lemma 5.7 $N \subseteq V(D)$.

(Claim I). For $u, v \in N$; $u \neq v$; there is no uv-monochromatic directed path contained in $D$. If there exists an uv-monochromatic directed path in $D$, then it follows from Lemma 5.2 that there exists an uv-monochromatic directed path in $Q(D)$; which is impossible.

(Claim II). For each $v \in V(D) - N$ there exists an uN-monochromatic directed path contained in $D$. From the definition of $N$; there exists an uN-monochromatic directed path contained in $Q(D)$ and it follows from Lemma 5.2 that there exists an uN-monochromatic directed path contained in $D$. \hfill \Box

**Corollary 5.2** Let $D$ be an $m$-coloured digraph which has no monochromatic directed cycle, and $Q(D)$ its middle digraph. The number of kernels by monochromatic paths of $D$ is equal to the number of kernels by monochromatic paths of $Q(D)$.

### 6 The total digraph of an $m$-coloured digraph

**Definition 6.1** Let $D$ be an $m$-coloured digraph, the total digraph of $D$, $T(D)$ is defined as follows: $T(D) = Q(D) \cup D$. 


Theorem 6.1 If $D$ is an $m$-coloured digraph, then $\mathcal{C}(T(D)) = \mathcal{C}(Q(D))$.

Proof: Since $Q(D) \subseteq T(D)$ we have $\mathcal{C}(Q(D)) \subseteq \mathcal{C}(T(D))$.

Now let $a = (u, v) \in A(D)$ coloured $i$, then $(u, a, v)$ is a directed path coloured $i$ in $Q(D)$ and so $a = (u, v) \in A(\mathcal{C}(Q(D)))$, hence $A(D) \subseteq A(\mathcal{C}(Q(D)))$. It follows that $T(D) \subseteq \mathcal{C}(Q(D))$ (Clearly $Q(D) \subseteq \mathcal{C}(Q(D))$) and then $\mathcal{C}(T(D)) \subseteq \mathcal{C}(\mathcal{C}(Q(D))) = \mathcal{C}(Q(D))$. □

Corollary 6.1 Let $D$ be an $m$-coloured digraph. The number of kernels by monochromatic paths of $D$ is less than or equal to the number of kernels, by monochromatic paths of $T(D)$; and if $D$ has no monochromatic directed cycle then $D$ and $T(D)$ have the same number of kernels by monochromatic paths.

Proof: Theorem ?? implies that $T(D)$ and $Q(D)$ have the same number of kernels by monochromatic paths, so Corollary ?? follows directly from Corollary ?? and Corollary ??.

Figure 1:

References


