Some 3-Connected 4-Edge-Critical Non-Hamiltonian Graphs

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Abstract: Let $\gamma(G)$ be the domination number of graph $G$, thus a graph $G$ is $k$-edge-critical if $\gamma(G) = k$, and for every nonadjacent pair of vertices $u$ and $v$, $\gamma(G + uv) = k - 1$. In Chapter 16 of the book “Domination in Graphs—Advanced Topics,” D. Sumner cites a conjecture of E. Wojcicka under the form “3-connected 4-critical graphs are Hamiltonian and perhaps, in general (i.e., for any $k \geq 4$), $(k - 1)$-connected, $k$-edge-critical graphs are Hamiltonian.” In this paper, we prove that the conjecture is not true for $k = 4$ by constructing a class of 3-connected 4-edge-critical non-Hamiltonian graphs.

Keywords: edge-critical graph; domination number; hamiltonian

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1. INTRODUCTION

We consider only finite undirected graphs without loops or multiple edges.

The neighborhood and the closed neighborhood of a vertex \( x \) are denoted by
\[
N(x) = \{ y \in V(G) \mid xy \in E(G) \},
\]
and
\[
N[x] = N(x) \cup \{ x \},
\]
respectively. For a set \( S \subseteq V(G) \),
\[
N[S] = \bigcup_{x \in S} N(x).
\]

A set \( S \) is a dominating set if and only if \( N(S) = V(G) \). The domination number \( \gamma(G) \) is the minimum cardinality of dominating sets.

A graph \( G \) is \( k \)-edge-domination-critical (or just \( k \)-edge-critical) if \( \gamma(G) = k \), and for every nonadjacent pair of vertices \( u \) and \( v \), \( \gamma(G + uv) = k - 1 \).

Wojcicka [11] showed that every connected, 3-edge-critical graph on more than 6 vertices has a Hamiltonian path. She conjectured that every connected 3-edge-critical graph with no endpoints has a Hamiltonian cycle. This conjecture has been proved in [2, 7, 10].

Some researches on hamiltonicity and connectivity of 3-edge-critical graphs are discussed in [3, 4, 5, 6, 8, 12].

In Chapter 16 of [9], D. Sumner cites a conjecture of E. Wojcicka:

**Conjecture 1.** 3-connected 4-critical graphs are Hamiltonian and perhaps, in general (i.e., for any \( k \geq 4 \)), \((k - 1)\)-connected, \( k \)-edge-critical graphs are Hamiltonian.

In this paper, we prove that Conjecture 1 is not true for \( k = 4 \) by constructing a class of 3-connected 4-edge-critical non-Hamiltonian graphs.

2. SOME COUNTEREXAMPLES OF CONJECTURE 1 FOR \( k = 4 \)

We construct a class of graphs \( G_t = \{ V(G_t), E(G_t) \} \) as the one shown in Figure 1, where
\[
V(G_t) = \{ w_i, u_j, v_j, p_k \mid 0 \leq i \leq t - 1; \ 0 \leq j \leq 2; \ 0 \leq k \leq 5 \},
\]
\[
E(G_t) = \{ w_iw_{i+1}, w_iu_j, u_jp_{2j}, u_jp_{2j+1}, u_jp_{(2j+4) \mod 6}, u_jp_{(2j+5) \mod 6}, v_jp_{2j}, v_jp_{(2j+1) \mod 6}, v_jp_{(2j+4) \mod 6}, v_jp_{(2j+5) \mod 6} \mid 0 \leq i_1 < i_2 \leq t - 1; \ 0 \leq i_1 \leq t - 1; \ 0 \leq j \leq 2 \}.
\]

We will prove that \( G_t \) are 3-connected, 4-edge-critical, non-Hamiltonian, i.e., the graphs \( G_t \) are counterexamples to Conjecture 1 for \( k = 4 \).

For \( j = 0, 1 \) and 2, let
\[
V_j = \{ u_j, v_j, p_{2j}, p_{2j+1} \}.
\]

For a dominating set \( S \) of \( G_t \), let
\[
b_j = b_j(S) = |S \cap V_j|.
\]
Lemma 2.1. \( \gamma(G_t) = 4 \).

**Proof.** Since \( \{ w_0, v_0, v_1, v_2 \} \) is a dominating set of \( G_t \), we have \( \gamma(G_t) \leq 4 \). Now, we prove that for any domination set \( S \) of \( G_t \), \(|S| \geq 4\).

**Case 1.** \( w_i \in S \) for some \( i, 0 \leq i \leq t - 1 \).

**Case 1.1.** \( b_j \geq 1 \) for all \( j, j = 0, 1, 2 \).

Hence, \(|S| = 1 + b_0 + b_1 + b_2 \geq 4\).

**Case 1.2.** \( b_j = 0 \) for some \( j, j = 0, 1, 2 \).

By symmetry, we can suppose that \( b_0 = 0 \). To dominate \( v_0 \), \( S \) must contain \( p_4 \) or \( p_5 \), say \( p_4 \in S \). Since \( p_5 \) is dominated, \( S \cap \{ u_2, v_2, p_5 \} \neq \emptyset \), then \(|S \cap V_2| \geq 2\), i.e., \( b_2 \geq 2 \). To dominate \( p_0 \), \( S \) must contain \( u_1 \) or \( v_1 \), i.e., \( b_1 \geq 1 \). Therefore, \(|S| = 1 + b_0 + b_1 + b_2 \geq 4\).

**Case 2.** \( w_i \notin S \) for all \( i, 0 \leq i \leq t - 1 \).

**Case 2.1.** \( b_j \geq 1 \) for all \( j, j = 0, 1, 2 \).

If \( b_0 = b_1 = b_2 = 1 \), then since \( w_0 \) is dominated, \( S \cap \{ u_0, u_1, u_2 \} \neq \emptyset \). By symmetry, we can suppose that \( S \cap V_0 = \{ u_0 \} \). To dominate \( v_0 \), \( S \) must contain \( p_4 \) or \( p_5 \), say \( S \cap V_2 = \{ p_4 \} \). But the set \( V_1 \) cannot be dominated by only one vertex in \( V_1 \) and this case is impossible. Therefore, \( b_j \geq 2 \) for some \( j = 0, 1, 2 \) and \(|S| = b_0 + b_1 + b_2 \geq 4\).

**Case 2.2.** \( b_j = 0 \) for some \( j, j = 0, 1, 2 \).

By symmetry, we can suppose that \( b_0 = 0 \). Since \( w_0 \) is dominated, \( S \cap \{ u_1, u_2 \} \neq \emptyset \), say \( u_1 \in S \). To dominate \( v_1 \), \( S \cap \{ v_1, p_2, p_3 \} \neq \emptyset \), then \( b_1 \geq 2 \). To dominate \( v_0 \), \( S \) must contain \( p_4 \) or \( p_5 \), say \( p_4 \in S \). Since \( p_5 \) is dominated, \( S \cap \{ u_2, v_2, p_5 \} \neq \emptyset \), then \(|S \cap V_2| \geq 2\), i.e., \( b_2 \geq 2 \) and \(|S| = b_0 + b_1 + b_2 \geq 4\).

From Cases 1–2, we have \(|S| \geq 4\), i.e., \( \gamma(G_t) \geq 4 \), hence, \( \gamma(G_t) = 4 \).

Lemma 2.2. For all \( e \notin E(G_t) \), \( \gamma(G_t + e) = 3 \).
Proof. We can show that there is a dominating set $S$ with $|S| = 3$ in every $G_t + e$, $e \notin E(G_t)$. By symmetry, it is sufficient to prove the property for $e \in \{w_0p_0, w_0v_0, u_0u_1, u_0v_0, u_0v_1, u_0v_2, p_0p_1, p_0p_2, p_0v_2, v_0v_1\}$.

If $e = w_0p_0$ or $p_0v_2$, then let $S = \{w_0, p_1, v_2\}$;
If $e = w_0v_0$ or $v_0v_1$, then let $S = \{w_0, v_1, v_2\}$;
If $e = u_0u_1$, then let $S = \{u_0, p_4, v_1\}$;
If $e = u_0p_2$, then let $S = \{u_0, p_3, v_0\}$;
If $e = u_0v_0$, then let $S = \{u_0, p_2, p_3\}$;
If $e = u_0v_1$, then let $S = \{u_0, u_1, p_4\}$;
If $e = u_0v_2$, then let $S = \{u_0, u_2, p_0\}$;
If $e = p_0p_1$, then let $S = \{w_0, p_0, v_2\}$;
If $e = p_0p_2$, then let $S = \{u_0, p_0, p_3\}$.

Hence, we have, $\gamma(G_t + e) = 3$, for all $e \notin E(G_t)$. □

For a set $S \subset V(G)$, $\omega(G - S)$ denotes the number of components of $G - S$. From [1], we have the following theorem:

Theorem 2.3. If $G$ is Hamiltonian, then for any nonempty set $S \subset V(G)$, $\omega(G - S) \leq |S|$.

Theorem 2.4. $G_t$ are 3-connected 4-edge-critical non-Hamiltonian.

Proof. Since the graphs obtained by deleting any pair of vertices from $G_t$ are still connected, $G_t$ are 3-connected. From Lemmas 2.1 and 2.2, $G_t$ are 4-edge-critical. Since $G_t - \{u_0, u_1, u_2, v_0, v_1, v_2\}$ contains 7 components, by Theorem 2.3, $G_t$ is non-Hamiltonian. □

3. FURTHER RESULTS

Let $S_n$ be the set of 3-connected 4-edge-critical graphs with order $n$. By the help of computer, we get $S_n$ for $n$ up to 12, where

$$S_n = \emptyset, \quad n \leq 10,$$
$$S_{11} = \{G_{11,j} \mid 1 \leq j \leq 3\},$$
$$S_{12} = \{G_{12,j} \mid 1 \leq j \leq 610\}.$$
Graphs $G_{11,1}, G_{11,2}, G_{11,3},$ and $G_{12,1}$ are shown in Figure 2. Graphs $G_{12,j}, 2 \leq j \leq 610$ are available for reader on require. Since all graphs in $S_{11} \cup S_{12}$ are Hamiltonian, we have the following.

**Theorem 3.1.** $G_0$ is a counterexample with the smallest order of Conjecture 1 for $k = 4$.

For $k > 4$, Conjecture 1 is still open.

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