Embedding Planar Graphs on the Grid

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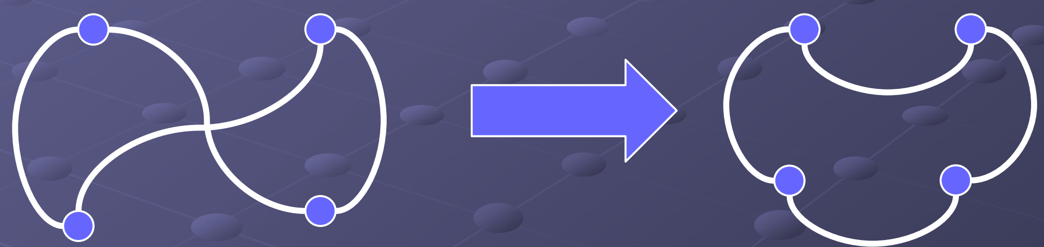
Review of Planarity

- A Graph G=(V,E) is a Planar Graph if it can be drawn in the plane with no edges crossing



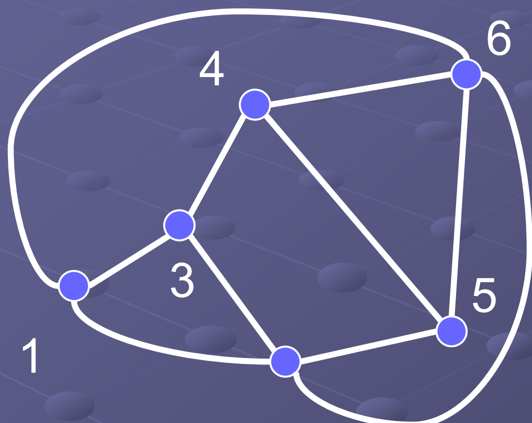
Planar Straight Line Embedding

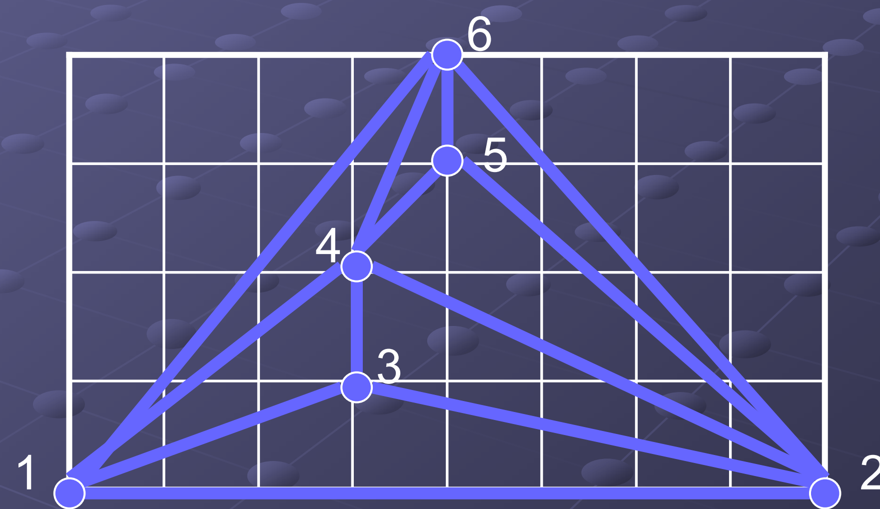
- Straight Line Representation of a planar graph in which no two edges cross.

- Also known as a Fáry Embedding.
- Used extensively in microchip layout and design, software engineering diagrams etc…

Planar Straight Line Embedding

- Planar straight line embeddings on a grid are useful for drawing applications.





- Ideally, we want the grid dimensions to be as small as possible. (lower cost microchips, manageable diagrams)

History

- Every planar graph has a straight line embedding. Fáry (1948)
- Many algorithms exist for computing straight line embeddings:
  - Tutte (1963) First known algorithm.
  - Chiba et al. (1982) First $O(n)$ algorithm.
- These algorithms tend to rely on real number coordinates and have extremely large grid sizes.

History (Cont'd)

- Open Problem:
  Does every planar graph with n vertices have a straight line embedding in an n^k x n^k grid?

- Solved in 1988 by de Fraysseix, Pach and Pollack. [$\Theta(n^2)$ grid]

- Stein had proved this same result for *Convex Maps* in 1951!

History (Cont'd)

Author(s)	Grid Size	Time	Space
de Fraysseix, Pach, Pollack (1988) [S]	(2n - 4) x (n - 2)	O(nlogn)	O(n)
Chrobak, Payne (1989) [S]	(2n - 4) x (n - 2)	O(n)	O(n)
Schnyder (1990) [R]	(n - 2) x (n - 2)	O(n)	O(n)
Chrobak, Kant (Convex Drawing), (1993) [S]	(n - 2) x (n - 2)	O(n)	O(n)

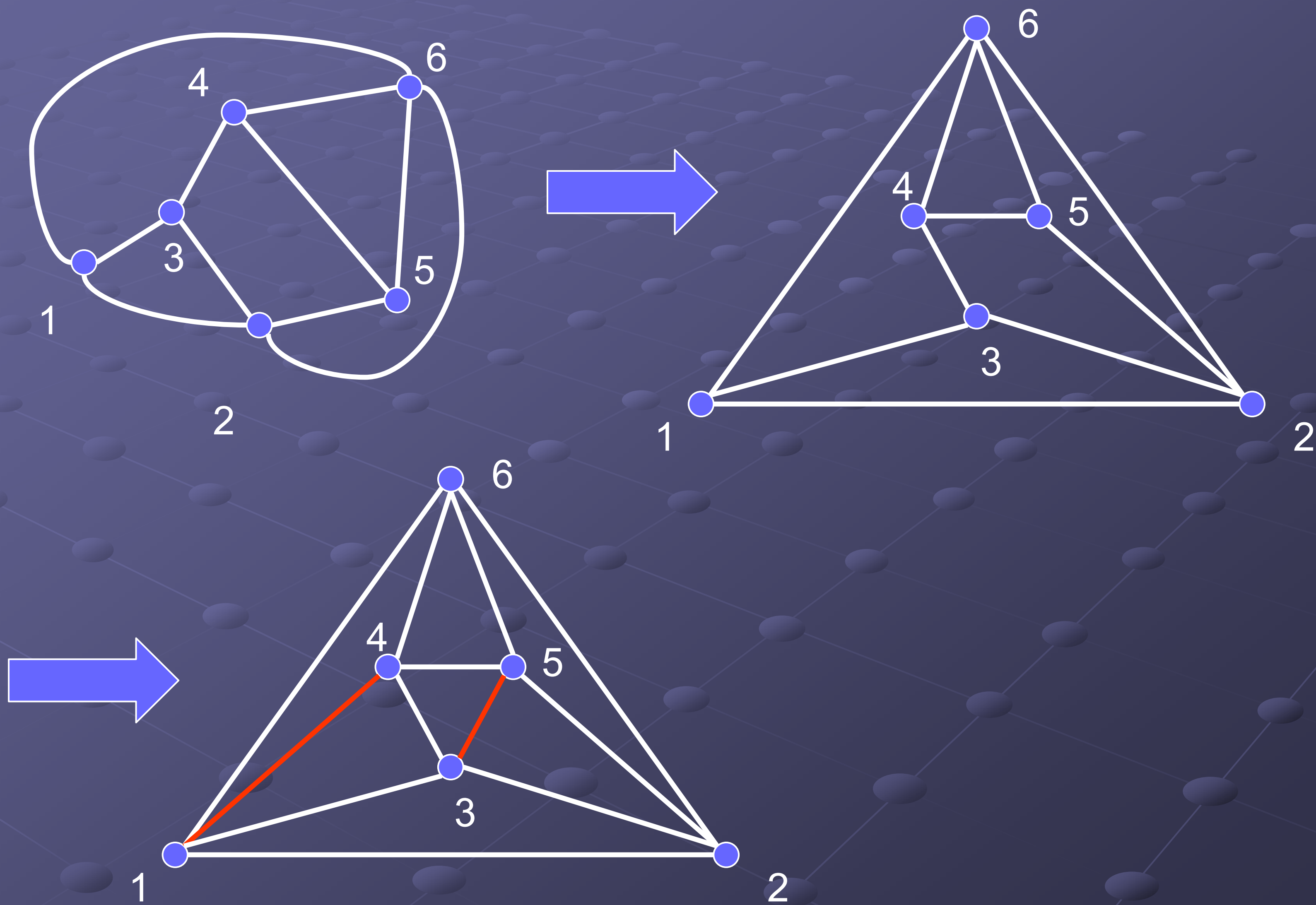
History (Cont'd)

- Two principle methods for computing straight line embeddings exist:
  - The "Shift Method":
    Add each vertex, and reposition previous vertices as necessary.
  - The "Realizer Method":
    Compute the position of each vertex relative to its neighbours, then compute the actual positions.

Schnyder's Algorithm

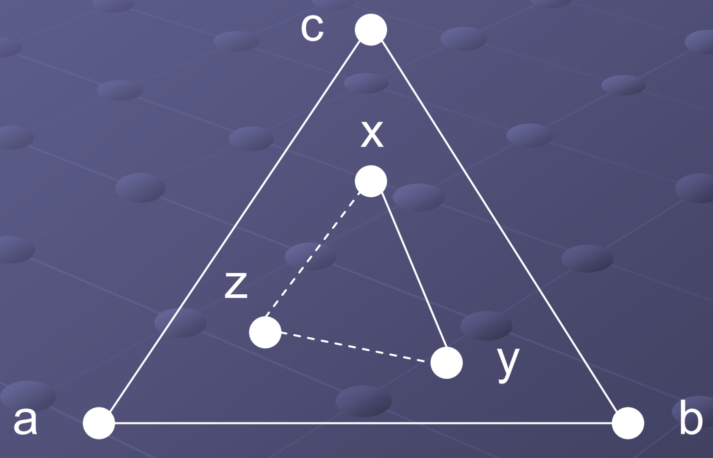
- Consists of three main stages:
  - Preprocess and triangulate the input graph G so that it is *maximal planar*, and *topologically embedded*.
  - Computation of a normal labeling and realizer T_1, T_2, T_3 of G.
  - Count of combinatorial objects (vertices or triangles) in regions of the realizer to obtain grid coordinates for every vertex of G.

Triangular supergraph



Barycentric Representations

- Barycentric representation of a graph G:
  - Every vertex v in G has 3 barycentric coordinates (v_1, v_2, v_3) and $v_1 + v_2 + v_3 = 1$



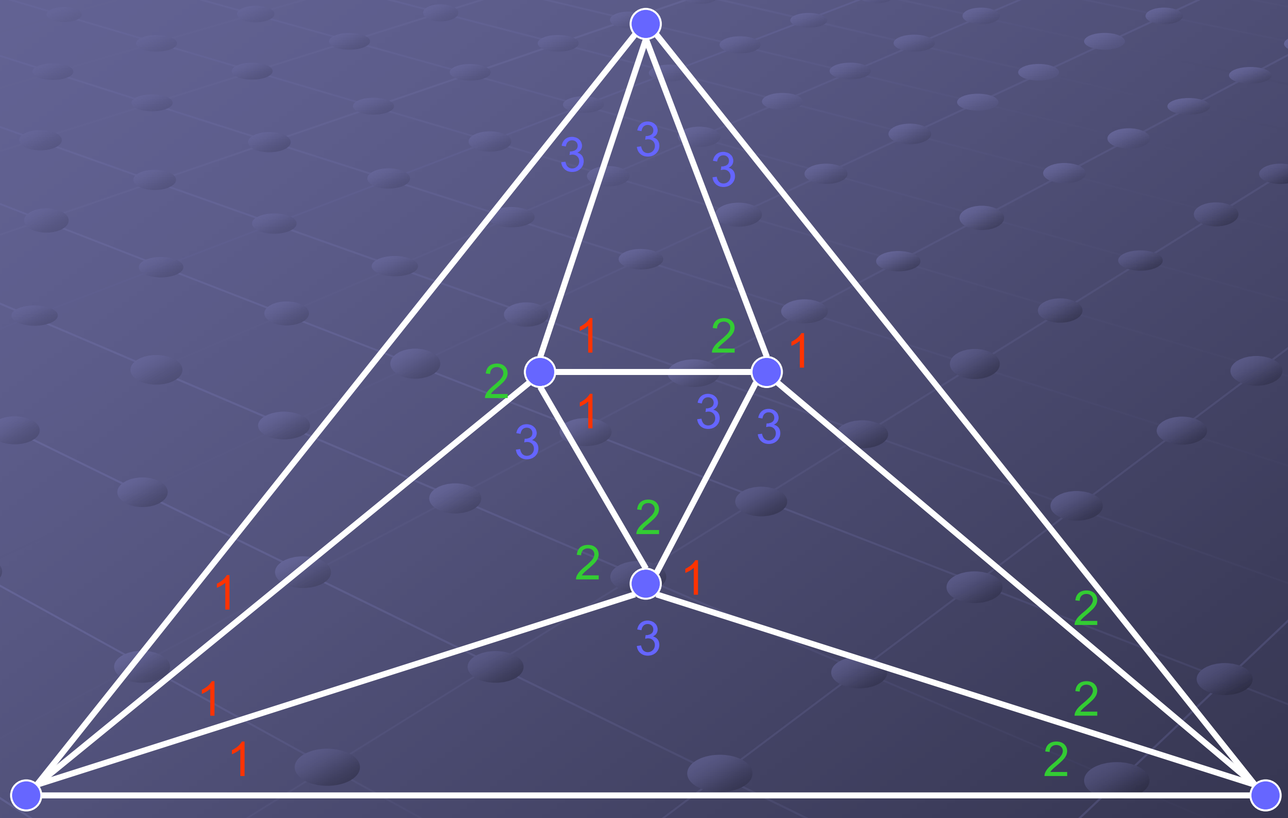
  - For every edge {x, y} and each vertex z not in {x,y}, there is some k in {1,2,3} such that $x_k < z_k$ and $y_k < z_k$.

Barycentric Representations (Cont'd)

- All (and only) planar graphs have a Barycentric representation.
- Given any 3 non-colinear points a, b, c, the mapping $v \rightarrow v_1 a + v_2 b + v_3 c$ is a straight line embedding in the plane spanned by a,b and c.
- A barycentric representation of a graph G leads to a normal labeling of G.

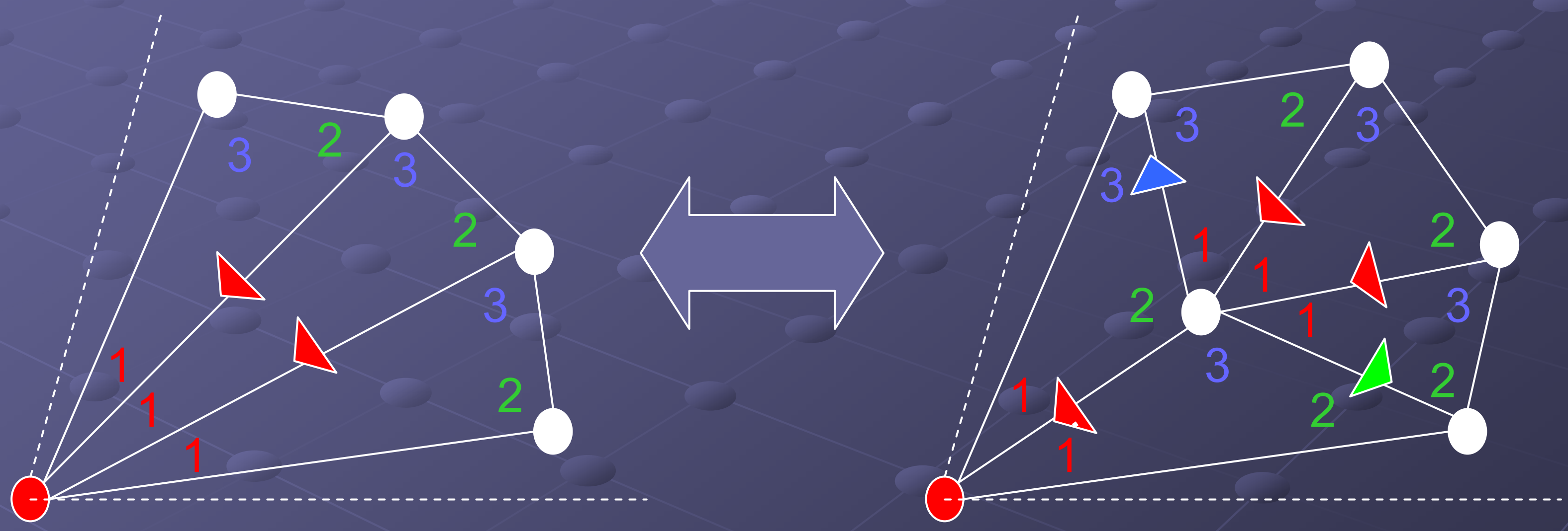
Normal Labeling

- Normal Labeling of a Triangular Graph G



Normal Labeling (Cont'd)

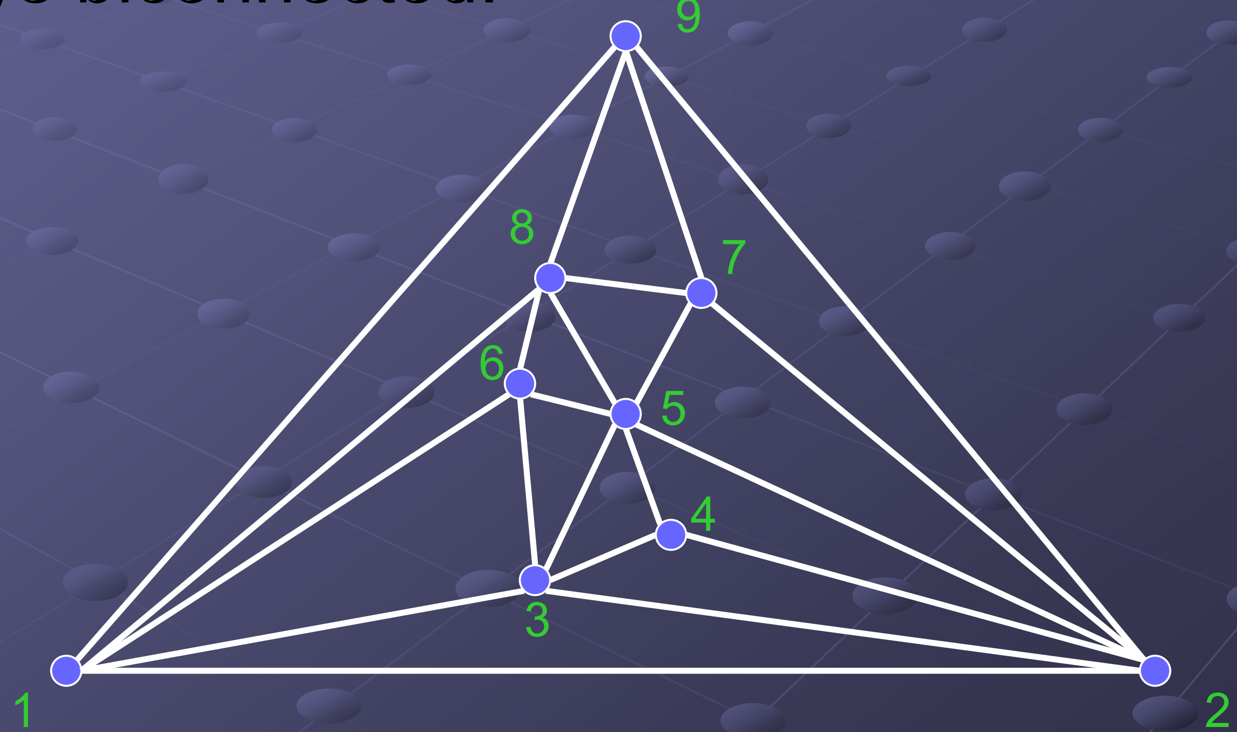
- Built incrementally using a method called ***edge contraction***.



- Relies on a *canonical ordering* of the vertices to determine the order of the contractions.

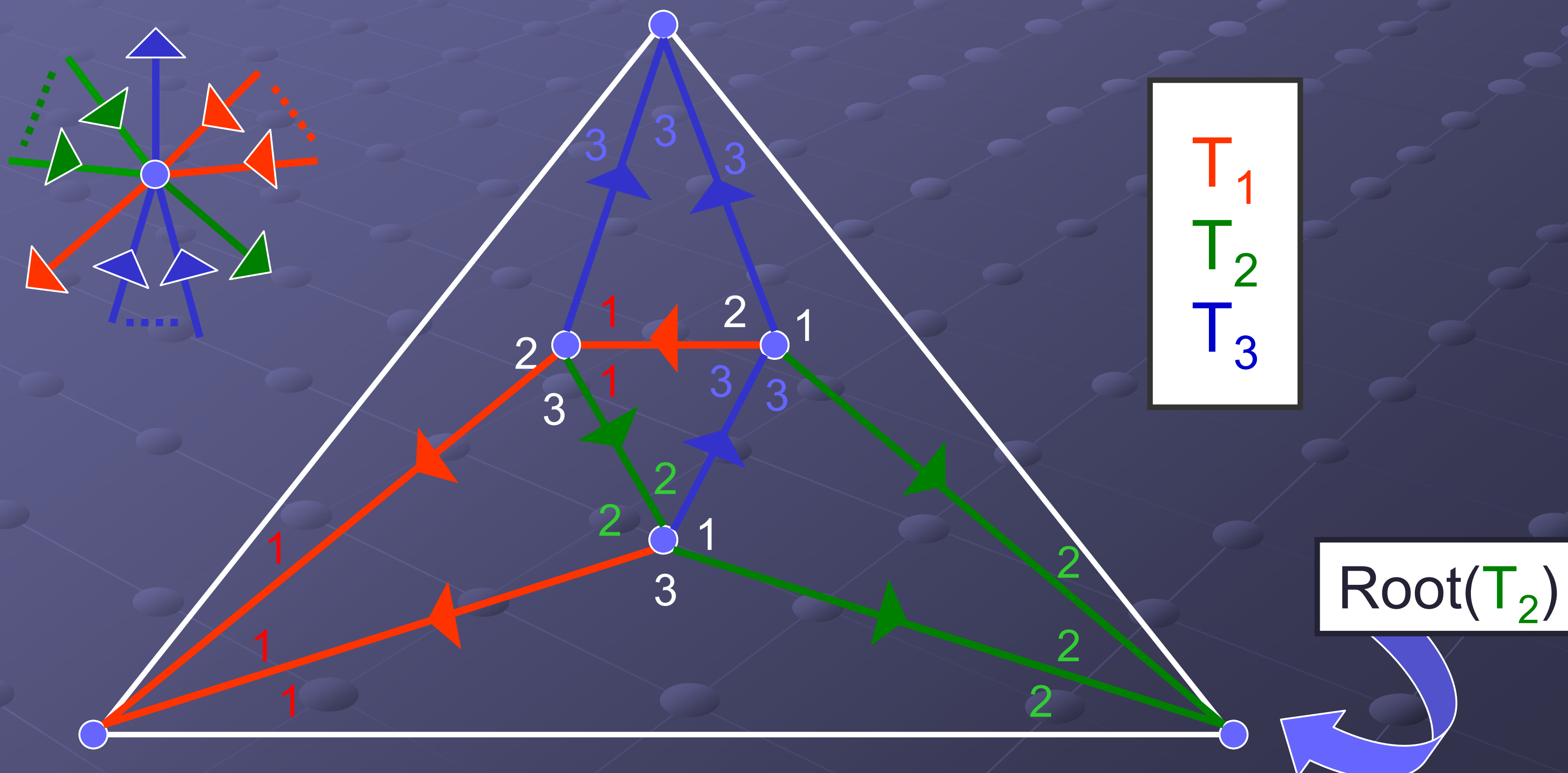
Canonical Ordering

- A canonical ordering of a maximal planar graph G specifies an order for removing the vertices one by one such that the remaining graph is always biconnected.



Computing the Realizer

- Realizer of a Triangular Graph G

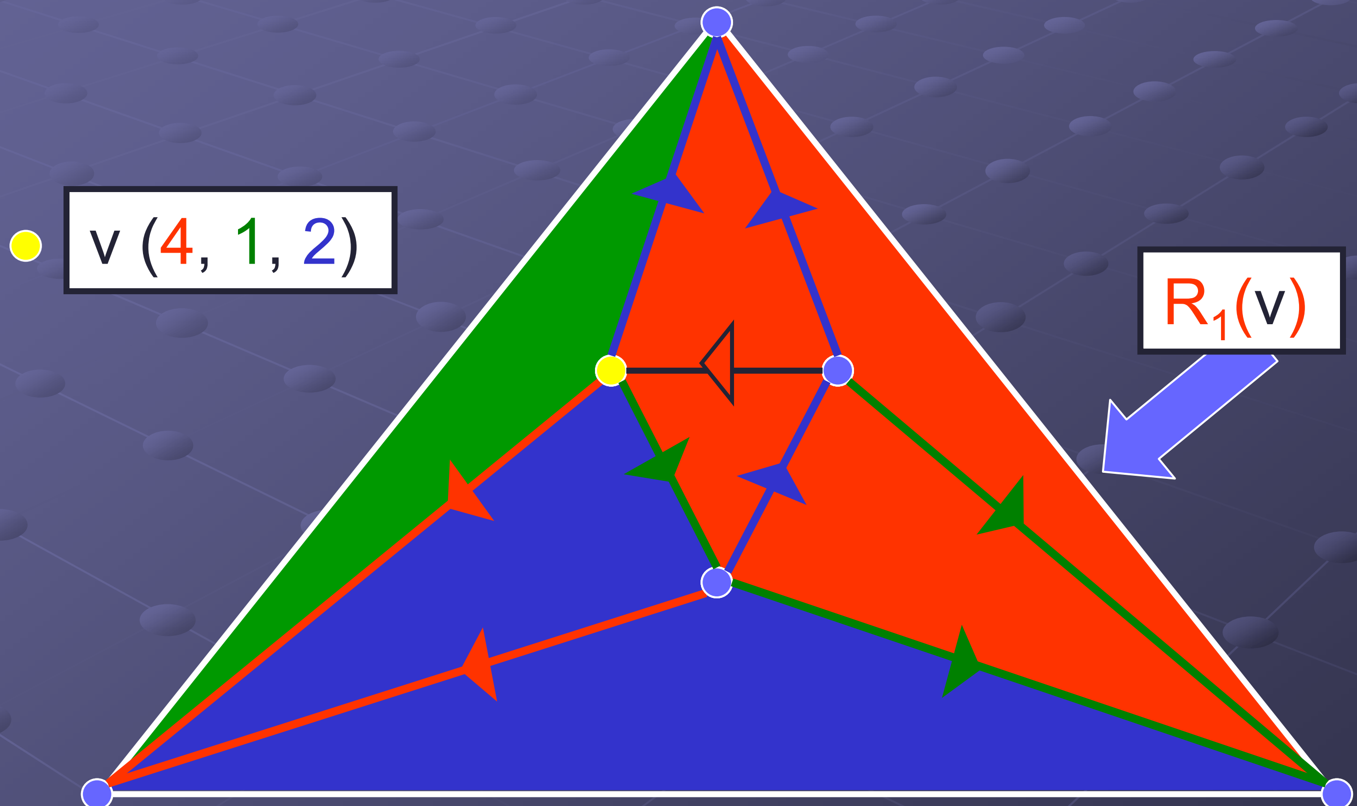


Count of combinatorial objects

- The 3 paths along T_i (i = 1,2,3) from any interior vertex v to the vertices root(T_i) divide G into 3 regions $R_i(v)$.
- Counting the triangles in each region for a vertex can give us v_1, v_2 ,v_3, which in turn yields (2n-5) x (2n-5) grid coordinates.
- Counting the vertices in each region is more complicated (boundary conditions) and yields (n-2) x (n-2) grid coordinates.

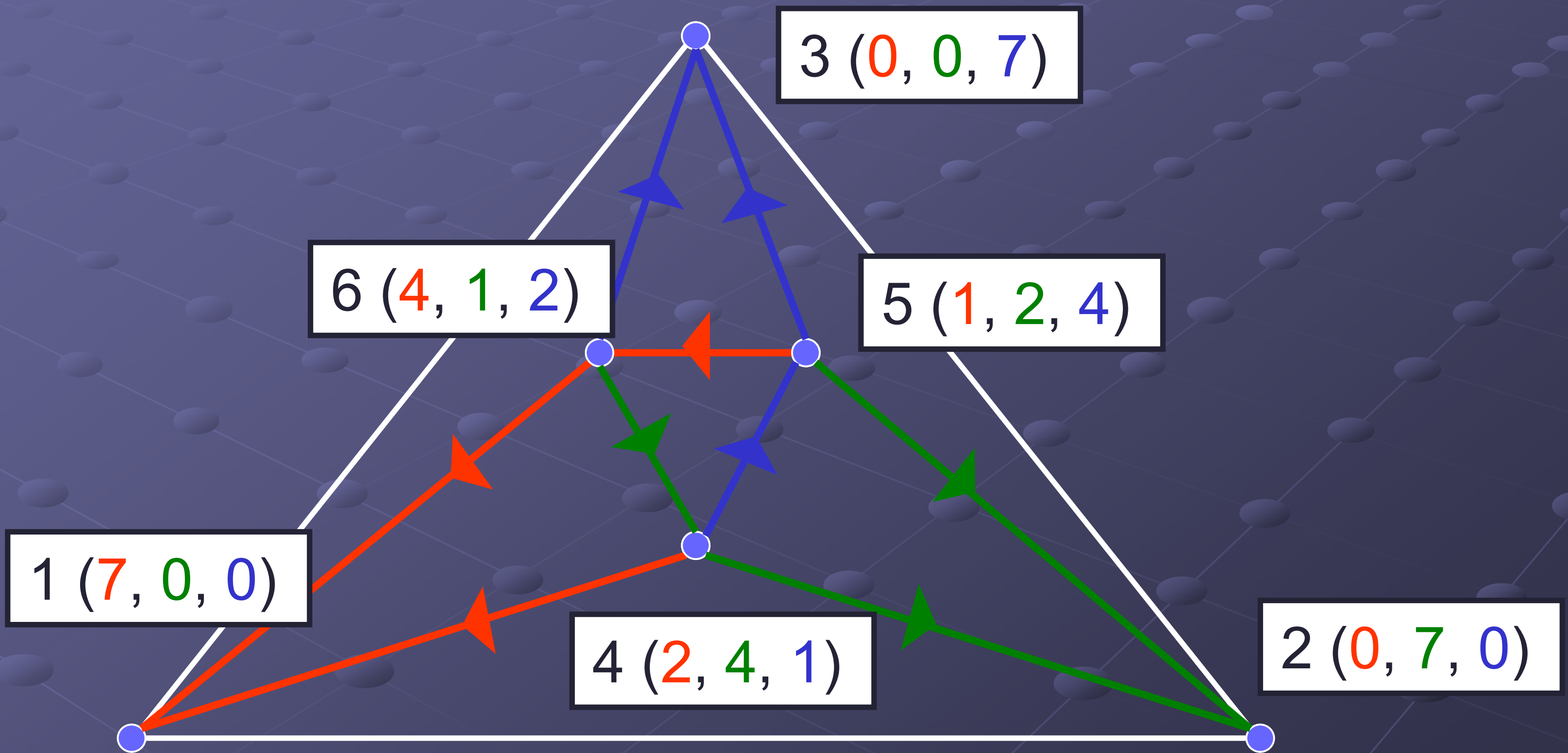
Count of combinatorial objects

- Coordinates that Count Triangles:



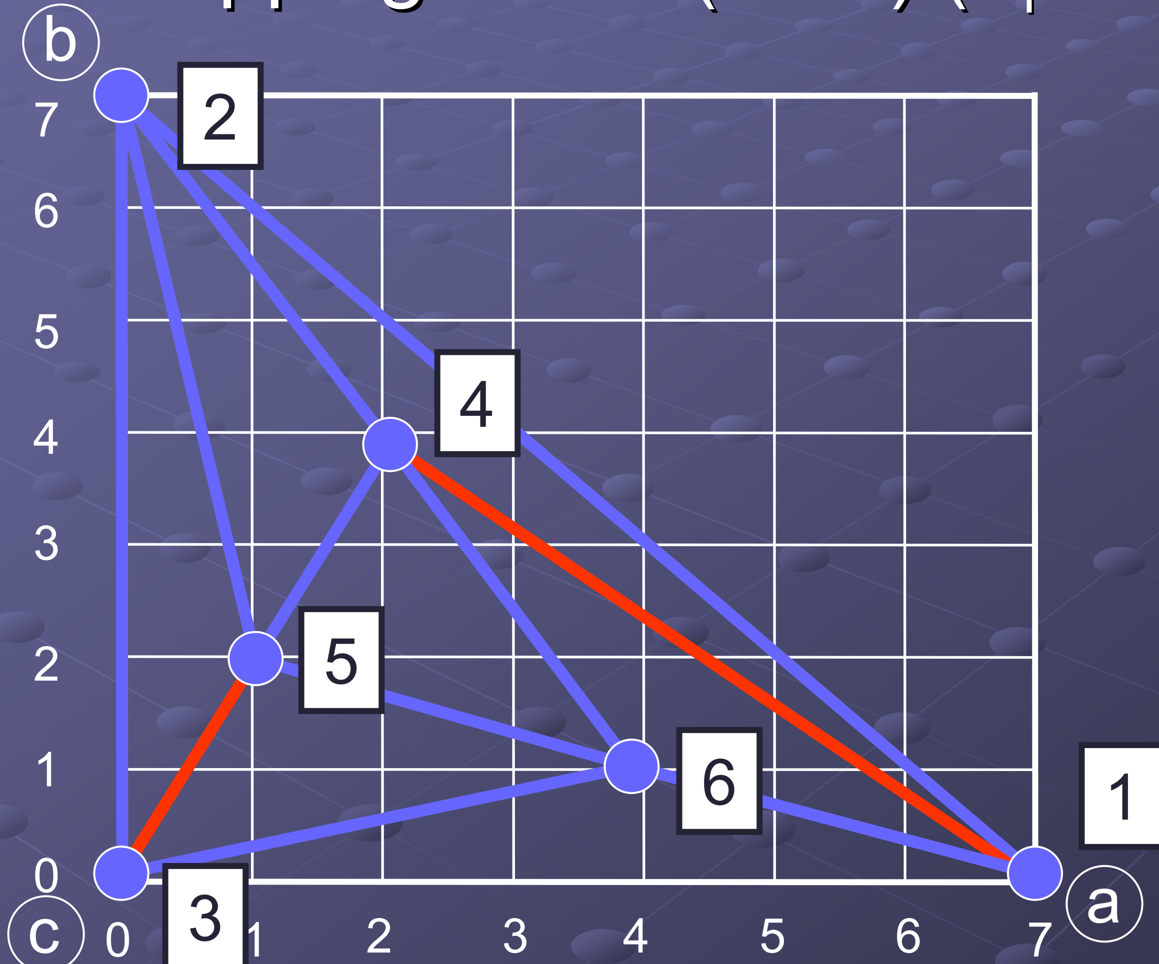
Count of combinatorial objects

- Coordinates that Count Triangles:



Mapping the Coordinates to a Grid

- Mapping v -> 1/(2n-5) (v_1a + v_2b + v_3c)



a = (2n-5, 0)
b = (0, 2n-5)
c = (0, 0)

n = 6, 2n-5 = 7

<u>Vertex 5</u>
(1* (7, 0) +
2 * (0 ,7) +
4 * (0, 0)) / 7
= (1, 2)

Conclusion

- This concludes the summary of three main stages of Schnyder's algorithm.
  - Triangulation of the input (planar) graph.
  - Computation of a normal labeling and realizer of the triangulated graph.
  - Count of the triangles (vertices) in the three regions of each vertex to obtain grid coordinates.

Comments

- To run in linear time, Schnyder's algorithm requires linear time planarity testing, topological embedding, triangulation, and canonical ordering algorithms.
- These algorithms exist, but are not provided by Schnyder and make implementing his algorithm in linear time much more challenging.
- Aesthetic properties of the grid drawing with Schnyder's algorithm leave a lot to be desired (skinny, non-convex faces frequently result)

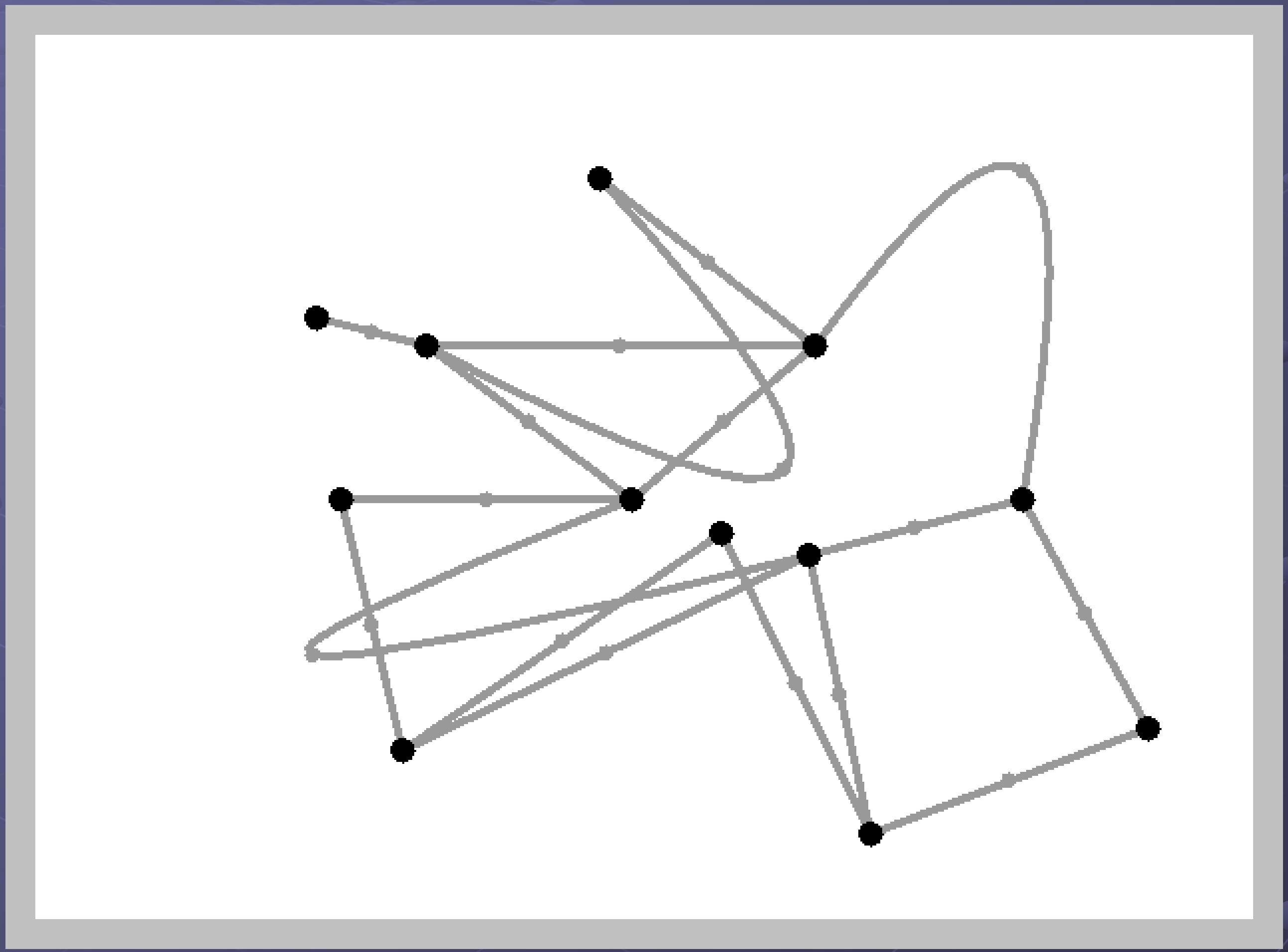
Interactive Demo

- A visual implementation of Schnyder's algorithm called JGraphEd is available as a Java Applet at:

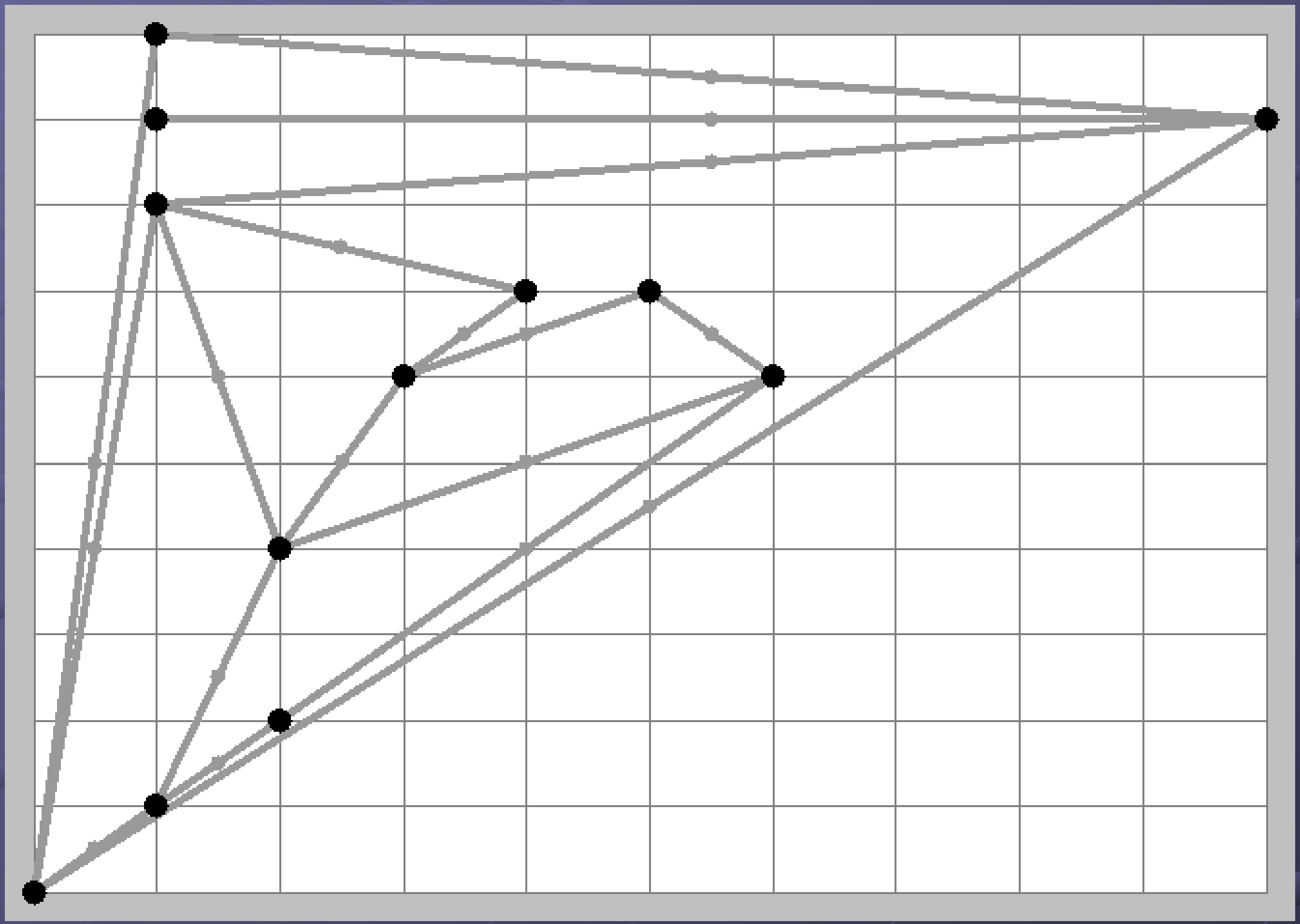
  http://www.jharris.ca/JGraphEd/

- Includes linear time: Planarity Testing, Canonical Ordering, Normal Labeling and n-2 by n-2 Straight Line Grid Embedding.

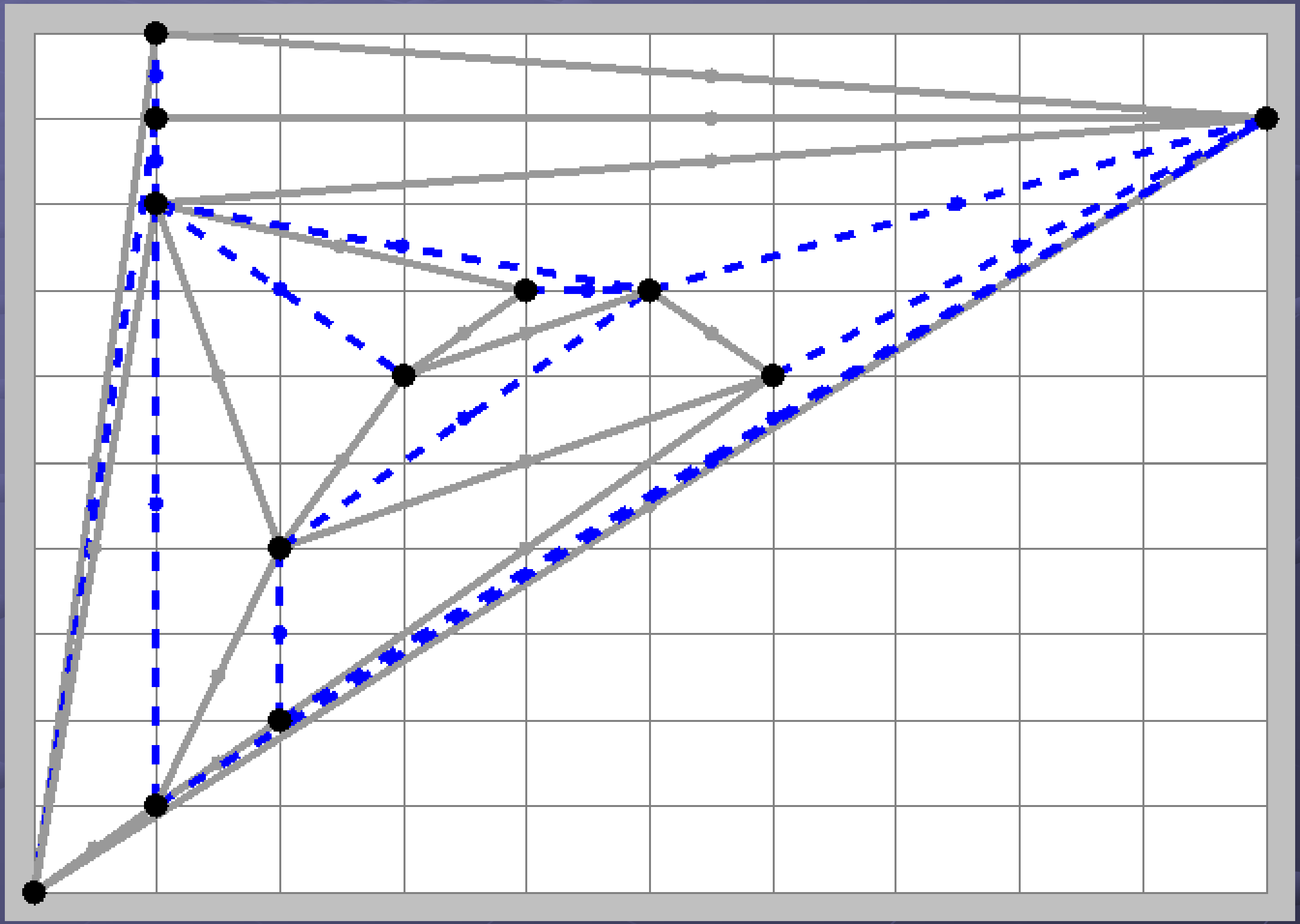
Example (12 vertices)



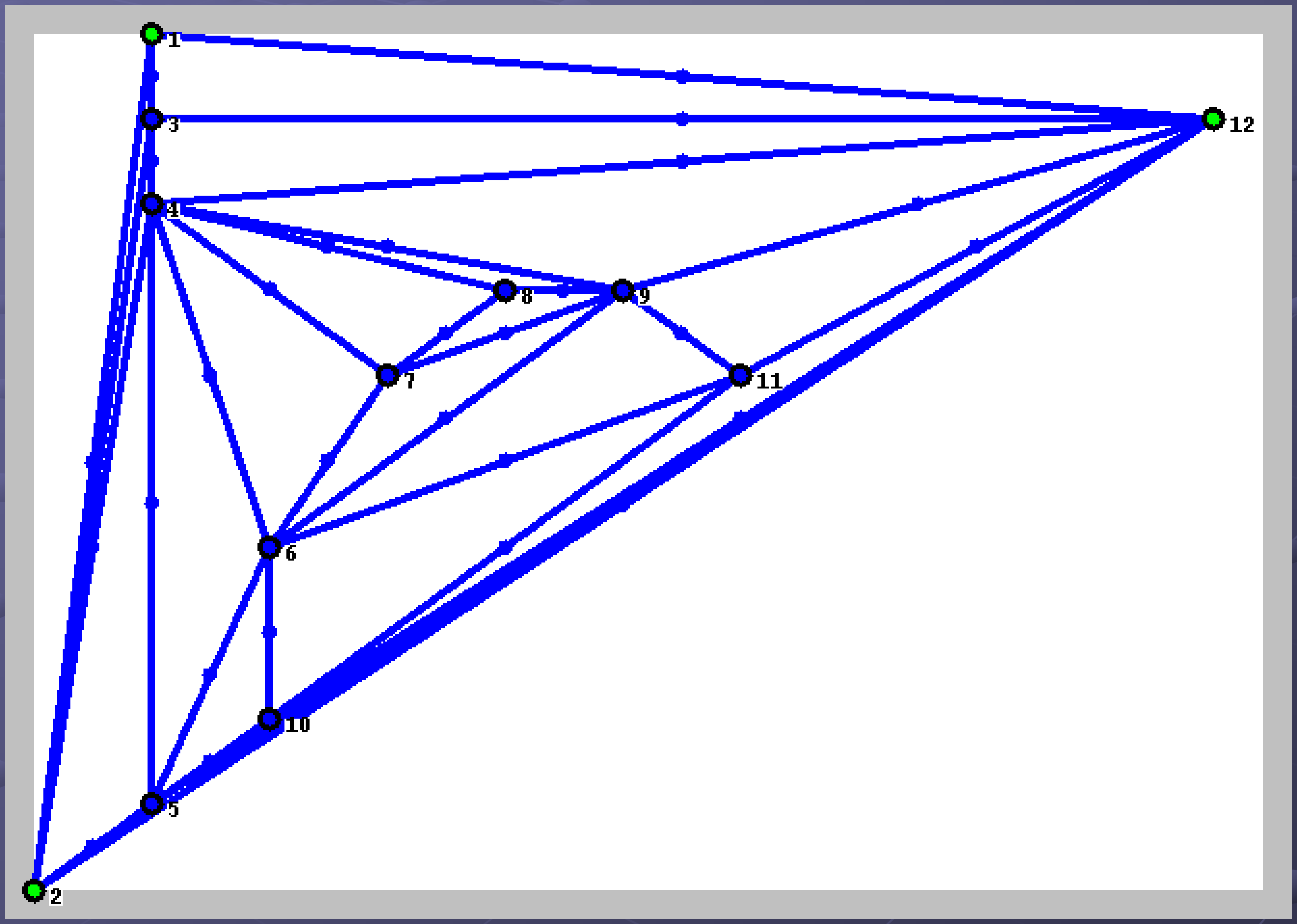
Example (10 x 10 grid embedding)



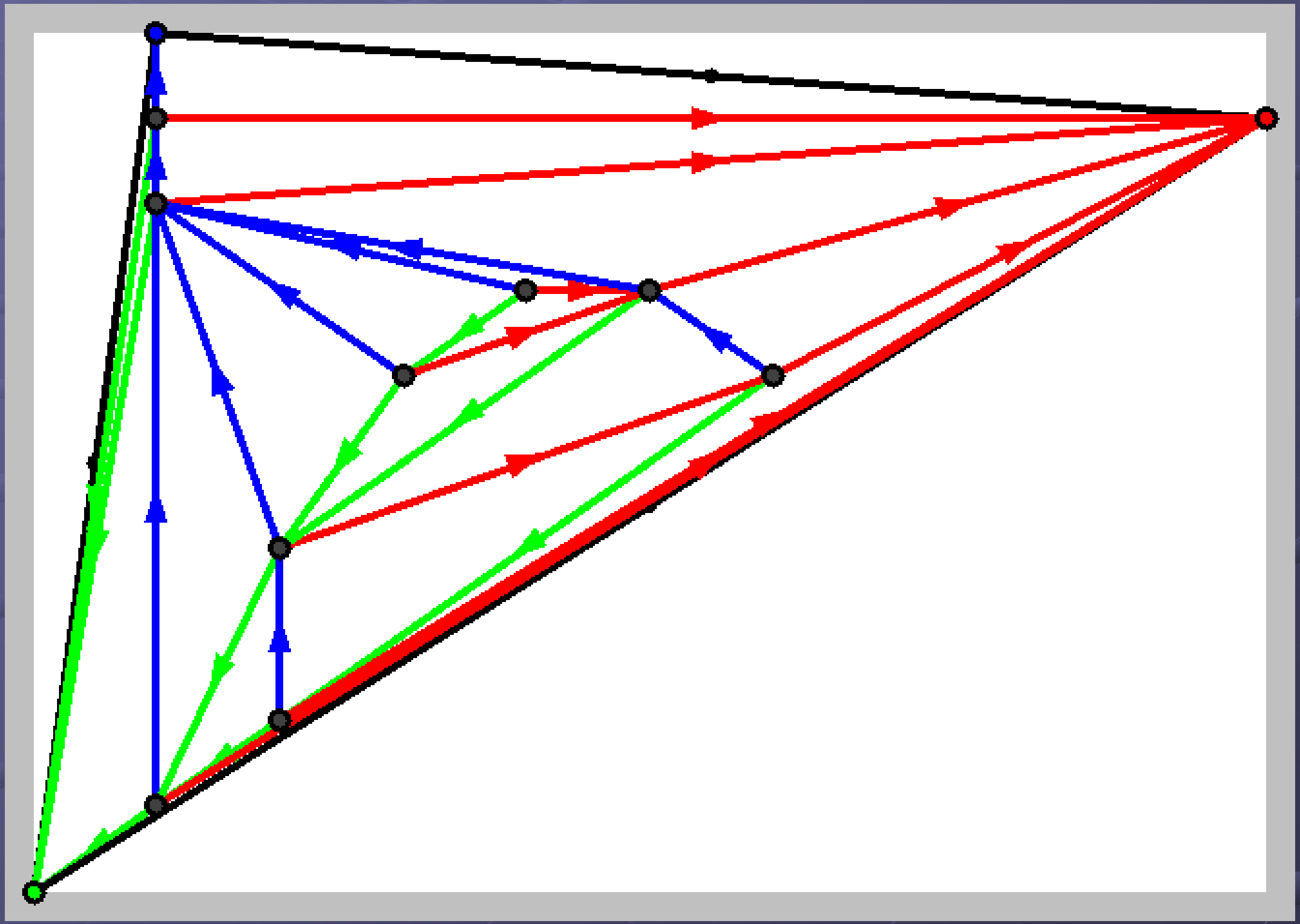
Example (Triangulation)



Example (Canonical Ordering)



Example (Normal Labeling)



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