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THE MINI-SUPERSPACE LIMIT OF THE SL(2,C)/SU(2)-WZNW MODEL

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Abstract. Many qualitatively new features of WZNW models associated to noncompact cosets are due to zero modes with continuous spectrum. Insight may be gained by reducing the theory to its zero-mode sector, the mini-superspace limit. This will be discussed in some detail for the example of SL(2,C)/SU(2)-WZNW model. The mini-superspace limit of this model can be formulated as baby-CFT. Spectrum, structure constants and fusion rules as well as factorization of four point functions are obtained from the harmonic analysis on SL(2,C)/SU(2). The issues of operator-state correspondence or the appearance of non-normalizable intermediate states in correlation functions can be discussed transparently in this context.

1. Introduction

In recent years there as been a lot of progress in the subject of rational conformal field theories (RCFT's, finite number of primary fields) or quasi-RCFT's (infinite number of primary fields but finite-dimensional fusion or braid relations). Not much is known on a class of theories that might be called non-compact CFT's: These have continuous families of primary fields and the operator product expansion of two primary field will generically involve an integral over (a subset of) the continuum of primary fields available. Such theories are more difficult to study, as i.e. there are generically no nullvectors in the relevant current algebra representations, so that most of the techniques from rational conformal field theories are not available.

One of the simplest examples for a noncompact CFT in the above sense is Liouville theory. By formal path-integral arguments [S][P] one was led there to expect more qualitatively new features as compared to (quasi)-RCFT's:

1. Loss of one-to-one correspondence between states and operators, which is a fundamental axiom in many approaches to RCFT. One aspect of this issue is that one will have to distinguish between operators corresponding to normalizable and non-normalizable states respectively.
2. Conditional nature of factorization: The set of intermediate represenations to factorize over will in general not coincide with the spectrum of the theory. Although outside the spectrum, it may happen that non-normalizable states appear in intermediate channels of correlation function.

Most of these points are related to the fact that one has zero modes with continuous spectrum. It is therefore useful to consider a limiting case in which only the zero mode dependence survives. Such a limiting case for Liouville theory has been discussed under the name of mini-superspace limit in [S]. It may be understood as considering only space-independent field configurations when spacetime is a cylinder.

Another simple example that has been studied i.e. in view of applications to the stringy euclidean 2D black hole is the WZNW-model corresponding to the coset $H_3^+ \equiv SL(2,\mathbb{C})/SU(2)$,

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cf. [Ga] and references therein. The mini-superspace limit of the 2D euclidean black hole CFT which can be easily obtained from the mini-superspace limit of H_3^+ WZNW model has been used to obtain amplitudes for reflection of strings in the 2D euclidean black hole geometry in [DVV].

My aim will be to develop the mini-superspace limit of the H_3^+-WZNW model in some detail. This turns out to be just quantum mechanics with configuration space H_3^+, mathematically nothing but harmonic analysis on that symmetric space. Some aspects of this limit have been discussed in [Ga], but the present discussion will be quite complimentary to that given in loc. cit..

One purpose is to illustrate how the above mentioned new qualitative features of noncompact CFT's are naturally understood in analogy with the harmonic analysis on symmetric spaces. To this aim a formulation will be presented that makes an analogy with the bootstrap approach of [BPZ] transparent.

However, the mini-superspace limit serves not only for illustration of qualitative features: It is expected to describe a certain semiclassical limit of the full theory, so one may use it to check the exact results for the full theory proposed in [T1]. This works both for the structure constants and the fusion rules.

Furthermore, it was proposed in [MSS] that in the context of 2D quantum gravity or noncritical string theory the mini-superspace limit may even be exact. Similar assumptions were used in [DVV] to obtain the reflection amplitudes of strings in the 2D euclidean black hole geometry. Let me however note that such reflection amplitudes can now be compared to exact results proposed in [T1], where one finds explicit quantum corrections to the mini-superspace result.

The present paper is the first in a series of papers devoted to the study of the H_3^+ and $SL(2,\mathbb{R})$-WZNW models. The next paper [T1] contains a derivation of an exact expression for the structure constants by using the methods of [TL], from which one can find a reflection amplitude as in [ZZ]. It further suggests a way to obtain spectrum and fusion rules from canonical quantization.

Some mathematical foundations on the use of non-highest or lowest weight representations to construct conformal blocks are laid in [T2], which treats the $SL(2,\mathbb{R})$ case on equal footing. This will be used to give a treatment of the $SL(2,\mathbb{R})$ case along the lines of [T1] in a forthcoming publication.

The contents of the present paper are as follows: The second section discusses quantization of the mini-superspace limit: Space of states and operators, where the momentum operators in a Schrödinger representation can be chosen to represent the Lie-algebra of the symmetry group $SL(2,\mathbb{C})$, whereas primary fields can be realized as multiplication operators. The issue of operator-state correspondence can be made quite transparent in this context.

The third section studies correlation functions. First it is described how far one can get by exploiting $SL(2,\mathbb{C})$ symmetry in a bootstrap appraoch à la [BPZ]. This treatment of the mini-superspace limit as baby-CFT makes the structural analogy to the full theory transparent. This is afterwards compared to the definition and (in some cases) calculation of correlation functions as overlaps. Fusion rules are here obtained by relating them to the spectral decomposition. It is explained how contributions of non-normalizable intermediate states arise in a well defined manner.

Two appendices contain certain technical aspects: Appendix A treats the spectral decomposition of the Laplacian of H_3^+. This could have been extracted from [GGV], but since it would also have taken some time to explain how all the results needed here follow from those given there, I preferred to give a different treatment, self-contained and adapted to the present needs.

Appendix B shows how one may explicitely calculate fusion relations for the mini-superspace conformal blocks.

2. The mini-superspace limit of H_3^+-WZNW-model

In a Lagrangian formulation, one may formulate the H_3^+-WZNW-model by starting from an $SL(2,\mathbb{C})$ model and gauging the $SU(2)$ subgroup, see i.e. [Ga] and references therein. Equivalently one may realize the coset $SL(2,\mathbb{C})/SU(2)$ as the space of two-by-two hermitian complex matrices h with unit determinant and start from the action [Ga]

$$S[h]=\frac{1}{\pi}\int d^2z\Big((\partial_z\psi)(\partial_{\bar z}\psi)+(\partial_z+\partial_z\psi)\bar v(\partial_{\bar z}+\partial_{\bar z}\psi)v\Big) \tag{1}$$

where h was parametrized as

$$h=\begin{pmatrix} e^{\psi}(1+|v|^2) & v\\ \bar v & e^{-\psi}\end{pmatrix}. \tag{2}$$

If one considers the theory on a cylinder with periodic space and infinite time, and furthermore restricts to field configurations that are independent of the space variable, one gets the action

$$S_m[h]=\frac{\kappa}{4}\int dt\,\mathrm{Tr}\big(h^{-1}\partial_t h\big)^2 \tag{3}$$

This action is real and invariant under the $SL(2,\mathbb{C})$-symmetry $h\to g^{-1}h(g^{-1})^\dagger$. One therefore expects this symmetry to be unitarily realized in the corresponding quantum theory.

2.1. **Space of states.** The Schrödinger representation for the quantum mechanics with phase space $T^*H_3^+$ is obtained by taking the Hilbert space to consist of functions on H_3^+, square-integrable w.r.t. the measure $dh=d\phi d^2v$ if h is parametrized as

$$h=\begin{pmatrix} e^{\phi}\sqrt{1+|v|^2} & v\\ \bar v & e^{-\phi}\sqrt{1+|v|^2}\end{pmatrix}. \tag{4}$$

The symmetry group $SL(2,\mathbb{C})$ acts on wave-functions on H_3^+ via

$$T_g\Psi(h)=\Psi(g^{-1}h(g^{-1})^\dagger). \tag{5}$$

The point about the present choice of scalar product is that it realizes the $SL(2,\mathbb{C})$-symmetry *unitarily*:

This may be seen by noting that each $h\in H_3^+$ can be written as $h=gg^\dagger$ for some $g\in SL(2,\mathbb{C})$. The point is to observe that

$$<\Psi_2,\Psi_1>\equiv\int_{H_3^+}dh\Psi_2^*(h)\Psi_1(h)=V_{SU(2)}^{-1}\int_{SL(2,\mathbb{C})}dg\Psi_2^*(gg^\dagger)\Psi_1(gg^\dagger), \tag{6}$$

where dg is the $SL(2,\mathbb{C})$-invariant measure

$$dg=\left(\frac{i}{2}\right)^4 d^2\alpha d^2\beta d^2\gamma d^2\delta\;\;\delta^2(\alpha\delta-\beta\gamma)=\left(\frac{i}{2}\right)^3|\alpha|^{-2}d^2\alpha d^2\beta d^2\gamma\qquad\text{if}\quad g=\begin{pmatrix}\alpha&\beta\\ \gamma&\delta\end{pmatrix},$$

and $V_{SU(2)}$ is the volume of $SU(2)$. Given (6), unitarity of the $SL(2,\mathbb{C})$-action is a trivial consequence of the invariance property $d(g_0g)=dg$ of the measure dg on $SL(2,\mathbb{C})$. In order to establish (6) one may rewrite the right hand side with the help of the Iwazawa decomposition

$$g=kan=k\begin{pmatrix}e^p&0\\0&e^{-p}\end{pmatrix}\begin{pmatrix}1&z\\0&1\end{pmatrix}\qquad k\in\mathrm{SU}(2),\quad z\in\mathbb{C},\quad p\in\mathbb{R}.$$

The measure dg factorizes in this parametrization as $dg = dkdpd^2z$, as may be seen by explicit calculation of the Jacobian of the change of variables from (a,b,c) to (k,p,z). The integration over $SU(2)$ factors out since the integrand is k-independent. The remaining variables (p,z) provide an alternative parametrization of H_3^+. Changing variables from (p,z) to (ϕ,v) one finds $dpd^2z = dh = d\phi d^2v$ as required.

It will be important to know the Hilbert space decomposes into $SL(2,\mathbb{C})$ irreducible representations. This was first found in [GGV]. In order to have a self-contained account of all the results needed here, I have summarized an alternative approach based on spectral analysis of the Laplacian on H_3^+ in the appendix A. There one can find (scetches of) the proofs of all the statements made in this subsection.

Abstractly the decomposition reads

$$\mathcal{H} \equiv L^2(H_3^+, dh) = \int_{\rho>0}^{\oplus} d\rho\rho^2 \ \mathcal{H}_{-\frac{1}{2}+i\rho}, \tag{7}$$

where $\mathcal{H}_j$ is a representation of the princial series of $SL(2,\mathbb{C})$, which may i.e. be explicitely realized on $L^2(\mathbb{C})$ via

$$T_g f(z) = |\beta z+\delta|^{4j} f\left(\frac{\alpha x+\gamma}{\beta x+\delta}\right) \quad \text{if} \quad g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}. \tag{8}$$

The decomposition (7) may be realized explicitely as a kind of Fourier-transform: The Fourier-components are defined by

$$F(j;x,\bar{x}) \equiv \int_{H_3^+} dh \ \Psi(j;x,\bar{x}|h) \ f(h),$$

where the kernel $\Psi^j(x,\bar{x}|h)$ that takes the role of the plane waves e^{ikx} in the ususal Fourier-transform is given as

$$\Psi(j;x,\bar{x}|h) = \frac{2j+1}{\pi}\left((x,1)\cdot h\cdot\begin{pmatrix}\bar{x}\\1\end{pmatrix}\right)^{2j}.$$

It is easy to check from the definitions that $F(j;x,\bar{x})$ indeed transforms under $SL(2,\mathbb{C})$ as in (8) if $f(h)$ is transformed by (5).

The function $f(h)$ is recovered from its transform $F(j;x,\bar{x})$ via the inversion formula

$$f(h) = \frac{i}{(4\pi)^3}\int_{\mathcal{P}_+} dj \ (2j+1)^2 \int d^2x \ \Psi^*(j;x,\bar{x}|h) \ F(j;x,\bar{x}) \qquad \text{where } \mathcal{P}_+ = -\tfrac{1}{2} + i\mathbb{R}^{>0}$$

One may therefore consider the set of functions $\{\Psi(j;x,\bar{x}|.); x\in\mathbb{C}, j\in\mathcal{P}_+\}$ as a plane wave basis for $\mathcal{H}$. Indeed it may be shown (Appendix A) that they are δ-function normalizable:

$$< \Psi(j;x,\bar{x}), \Phi(j';x',\bar{x}') >= 2\pi\delta^{(2)}(x-x')\delta(j-j') \qquad \text{for } j,j'\in\mathcal{P}_+ \tag{9}$$

The functions $\Psi(j;x,\bar{x}|h)$ and $\Psi(-j-1;x,\bar{x}|h)$ are linearly related to each other:

$$\Psi(j;x,\bar{x}) = \frac{2j+1}{\pi}\int d^2x' \ |x-x'|^{4j}\Psi(-j-1;x',\bar{x}') \tag{10}$$

The general form of this relation (but not the j-dependent prefactor) is determined by $SL(2,\mathbb{C})$-symmetry: The integral operator with kernel $|x-x'|^{4j}$ is just the intertwining operator [GGV] between representations with spin $-j-1$ and j expressing the equivalence of these representations.

It is sometimes useful to also use an alternative basis $\Psi^j_{np}(h)$ $n \in \mathbb{Z}$, $p \in \mathbb{R}$ which is related to the $\Psi(j;x,\bar{x}|h)$ by the following Fourier-transform on $L^2(\mathbb{C})$:

$$\Psi^j_{np}(h) \equiv \int_{\mathbb{C}} d^2x \;\; e^{in\arg(x)}|x|^{-2j-2+ip}\Psi(j;x,\bar{x}|h). \tag{11}$$

The explicit expression for $\Psi^j_{np}(h)$ in terms of the hypergeometric function can be found in appendix A.

2.2. **Momentum operators: The Lie algebra of $SL(2,\mathbb{C})$.** The action of the Lie algebra of $SL(2,\mathbb{C})$ on differentiable function on H_3^+ is given by the differential operators

$$\begin{gathered} K^a f(h) \equiv \left(\frac{d}{dt} f\left(e^{-tT_a} h e^{-tT_a^\dagger}\right)\right)_{t=0} \qquad L^a f(h) \equiv \left(\frac{d}{dt} f\left(e^{-itT_a} h e^{itT_a^\dagger}\right)\right)_{t=0} \\ T_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad T_0 = \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad T_- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{gathered} \tag{12}$$

Alternatively one may use the holomorphic (resp. antiholomorphic) differential operators

$$J^a f(h) \equiv \left(\frac{\partial}{\partial \tau} f\left(e^{-\tau T_a} h e^{-\bar{\tau} T_a^\dagger}\right)\right)_{\tau=0} \qquad \bar{J}^a f(h) \equiv \left(\frac{\partial}{\partial \bar{\tau}} f\left(e^{-\tau T_a} h e^{-\bar{\tau} T_a^\dagger}\right)\right)_{\tau=0} \tag{13}$$

In terms of the parametrization (4) one finds

$$\begin{aligned} J^+ &= -e^{-\phi}\sqrt{1+|v|^2}\frac{\partial}{\partial v} - \frac{1}{2}e^{-\phi}\frac{\bar{v}}{\sqrt{1+|v|^2}}\frac{\partial}{\partial \phi} \\ J^- &= -e^{\phi}\sqrt{1+|v|^2}\frac{\partial}{\partial \bar{v}} + \frac{1}{2}e^{+\phi}\frac{v}{\sqrt{1+|v|^2}}\frac{\partial}{\partial \phi} \end{aligned} \qquad J^0 = \frac{1}{2}\left(-v\frac{\partial}{\partial v} + \bar{v}\frac{\partial}{\partial \bar{v}} - \frac{\partial}{\partial \phi}\right), \tag{14}$$

whereas the $\bar{J}^a$ are given by the complex conjugate operators. Their hermiticity properties with respect to the $L^2(H_3^+, dh)$ scalar product are $(J^a)^\dagger = -\bar{J}^a$, $a = -, 0, +$.

The action of these generators on the Fourier transform $F[f](j,x,\bar{x})$ of $f(h)$ is then given by

$$\begin{gathered} \pi_j(J^a)F[f](j,x,\bar{x}) \equiv F[J^a f](j,x,\bar{x}) = \mathcal{D}^a_j F[f](j,x,\bar{x}) \\ \mathcal{D}^+_j = -x^2\partial_x - 2jx \qquad \mathcal{D}^0_j = x\partial_x - j \qquad \mathcal{D}^-_j = \partial_x \end{gathered} \tag{15}$$

2.3. **Primary fields, Operator-State correspondence.** In conformal field theory one is interested mostly in the so-called primary fields, operators that have a particularly simple transformation law under the chiral algebra, in the case of the H_3^+-WZNW model two copies of the Kac-Moody algebra $\widehat{sl_2}$ generated by the modes J^a_n, $\bar{J}^a_n$. Quite generally the primary fields $\Phi(v|z,\bar{z})$ may be labelled by vectors v in some representation V of the *zero-mode subalgebra* of the chiral algebra which is generated by the $J^a \equiv J^a_0$, $\bar{J}^a \equiv \bar{J}^a_0$:

$$[J^a_n, \Phi(v|z,\bar{z})] = z^n \Phi(\pi_V(J^a_0)v|z,\bar{z}), \tag{16}$$

and an analogous formula for the $\bar{J}^a_n$, where $\pi_V(J^a_0)$ denotes the operator that represents J^a on V. If one i.e. takes V to be the irreducible representation realized in $\mathcal{H}_j$ by means of the differential operators $\mathcal{D}^a_j$ this reads

$$[J^a_n, \Phi^j(x,\bar{x}|z,\bar{z})] = z^n \mathcal{D}^a_j \Phi^j(x,\bar{x}|z,\bar{z}), \tag{17}$$

This kind of transformation law is a sufficient condition for $\Phi(v|z,\bar{z})$ to correspond to a primary state v, i.e. to a state that satisfies $J^a_n v = 0$ for $n > 0$ via the usual operator-state correspondence

$$\lim_{z\to 0} \Phi(v|z,\bar{z})|0> . \tag{18}$$

However, a crucial difference between WZNW models corresponding to compact resp. non-compact groups is that the zero-mode representations of the latter are generically infinite-dimensional. This means that the vector obtained via (18) is by no means guaranteed to be normalizable (not even in δ-function sense!)[2].

In the presently discussed mini-WZNW model one may see rather explicitly that indeed one has to distinguish between normalizable and non-normalizable states. This fact will also be of crucial importance for understanding the fusion rules.

First note that condition (17) in the mini-WZNW case reduces just to the statement of covariance under the zero-mode subalgbra, the z-dependence disappears. It is easy to find such operators in the presently used Schrödinger-representation: These are just the multiplication operators $\Phi^j(x,\bar{x})$:

$$\Phi^j(x,\bar{x})\Psi(h) := \Psi(j;x,\bar{x}|h)\Psi(h). \tag{19}$$

In order to speak of operator-state correspondence one needs to define the "vacuum" $|0>$. Its defining property is usually taken to be the invariance under the chiral algebra, here the Lie algebra of $SL(2,\mathbb{C})$. The trivial representation of $SL(2,\mathbb{C})$ corresponds to $j=0$. The wave function of $|0>$ is $\Phi(j=0;x,\bar{x}|h)=1$, so the operator corresponding to it via (19) is just the unity operator. However, since the norm of $|0>$ is thereby the (infinite) volume of H_3^+, the state $|0>$ is clearly not contained in the spectrum. But still one has the fact that any multiplication operator that acts by multiplication with a normalizable function on $L^2(H_3^+,dh)$ does create reasonable states from the "vacuum" $|0>$. The state $<0|$ conjugate to $|0>$ is of course a functional well defined on a suitable subspace of $L^2(H_3^+,dh)$.

3. Correlation functions

Since my intention is to present the H_3^+ quantum mechanics as a baby conformal field theory, I will start by discussiong how far one can get by a strategy analogous to the conformal bootstrap of [BPZ]. As in the case of full-fledged CFT one will find that the symmetries determine the correlation functions to a large extend, but structure constants and fusion rules are not easy to determine explicitely within this approach.

The following subsection then explains how in the presently considered baby CFT all the missing information can be found by exploiting the harmonic analysis on H_3^+.

3.1. **Baby-bootstrap.**

According to the previous discussion one may try to define vacuum expectation values

$$<0|\Phi^{j_n}(x_n,\bar{x}_n)\dots\Phi^{j_1}(x_1,\bar{x}_1)|0>,$$

where the "vacuum" $|0>$ is to denote the $SL(2,\mathbb{C})$ invariant "state". The $SL(2,\mathbb{C})$ invariance of $|0>$ then results in a set of differential equations

$$\sum_{i=1}^{\infty}\mathcal{D}^a_{x_i,j} <0|\Phi^{j_n}(x_n,\bar{x}_n)\dots\Phi^{j_1}(x_1,\bar{x}_1)|0>= 0, \qquad a=-,0,+,$$

[2]This fact was first pointed out in the context of Liouville theory by Seiberg [S] and Polchinski [P].

which allow to express the correlation function in terms of its values for (say) $x_1 = 0$, $x_2 = 1$, $x_n = \infty$. In particular, this determines the two- and three point functions almost completely :

$$\begin{aligned} (20)\quad & <0|\Phi^{j_2}(x_2,\bar{x}_2)\Phi^{j_1}(x_1,\bar{x}_1)|0>= \\ & \qquad = N(j_1)\ \delta(j_2+j_1+1)\delta^{(2)}(x_2-x_1) + B(j)\ \delta(j_2-j_1)|x_2-x_1|^{4j_1} \\ & <0|\Phi^{j_3}(x_3,\bar{x}_3)\Phi^{j_2}(x_2,\bar{x}_2)\Phi^{j_1}(x_1,\bar{x}_1)|0>= \\ & \qquad = C(j_3,j_2,j_1)|x_3-x_2|^{2(j_3+j_2-j_1)}|x_3-x_1|^{2(j_3+j_1-j_2)}|x_2-x_1|^{2(j_2+j_1-j_3)} \end{aligned}$$

The only thing that may look somewhat unusual to anyone familiar with CFT à la BPZ is the term with $\delta^{(2)}(x_2 - x_1)$ in the expression for the two point function. In real CFT this would be called a contact term. Here it is just the term that gives the scalar product (9) when j_1, j_2 are restricted to $\mathcal{P}_+$.

Now the primary fields form multiplets under the symmetry algebra $\mathfrak{sl}(2,\mathbb{C})_L \oplus \mathfrak{sl}(2,\mathbb{C})_R$ generated by the holomorphic (resp. antiholomorphic generators J^a (resp. $\bar{J}^a$). In order to keep the analogy with [BPZ] as close as possible it is natural to introduce as secondary fields the derivatives

$$\Phi^{j,n,\bar{n}}(x,\bar{x}) \equiv \frac{\partial^n}{\partial x^n}\frac{\partial^{\bar{n}}}{\partial \bar{x}^{\bar{n}}}\Phi^j(x,\bar{x}).$$

The general strategy of the bootstrap amounts to construction of ($n > 3$)-point functions in terms of two- and three point functions. This will be possible if there are operator product expansions of two operators for their arguments close to each other:

$$\begin{aligned} (21)\quad & \Phi^{j_2}(x_2,\bar{x}_2)\Phi^{j_1}(x_1,\bar{x}_1) \\ & = \int dj\ \ |x_2-x_1|^{2(j_2+j_1-j)} \sum_{n,\bar{n}=0}^{\infty} \mathcal{C}_{n\bar{n}}(j;j_2,j_1)(x_2-x_1)^n(\bar{x}_2-\bar{x}_1)^{\bar{n}}\ \Phi^{j,n,\bar{n}}(x_1,\bar{x}_1) \end{aligned}$$

As in the case of real CFT the requirement that both sides of (21) transform the same way under the symmetry algebra allows to fix the coefficients $\mathcal{C}_{n\bar{n}}(j;j_2,j_1)$ uniquely in terms of $C(j;j_2,j_1) \equiv \mathcal{C}_{00}(j;j_2,j_1)$. In the present baby CFT it is easily possible to carry this out explicitly:

$$(22)\quad \begin{aligned} \mathcal{C}_{n\bar{n}}(j;j_2,j_1) &= R_n(j;j_2,j_1)R_{\bar{n}}(j;j_2,j_1)D(j;j_2,j_1) \\ R_n(j;j_2,j_1) &= \frac{\Gamma(j_1-j_2-j-1+n)\Gamma(-2j-1-n)}{\Gamma(j_1-j_2-j-1)\Gamma(-2j-1)n!} \end{aligned}$$

If one knew both the range of values for j in (21), i.e. the fusion rules, and the explicit expression for the structure constants $D(j;j_2,j_1)$ then one could in principle unambigously determine any $n > 3$-point function: Inserting (21) i.e. into a four point function leads to the expansion

$$\begin{aligned} (23)\quad & <\Phi^{j_4}\dots\Phi^{j_1}>= \int dj_{21}\ |x_2-x_1|^{2(j_2+j_1-j_{21})} \sum_{n,\bar{n}=0}^{\infty}(x_2-x_1)^n(\bar{x}_2-\bar{x}_1)^{\bar{n}} \\ & \qquad \mathcal{C}_{n\bar{n}}(j_{21};j_2,j_1) <0|\Phi^{j_4}(x_4,\bar{x}_4)\Phi^{j_3}(x_3,\bar{x}_3)\Phi^{j_{21},n,\bar{n}}(x_1,\bar{x}_1)|0> \end{aligned}$$

By observing that the x-dependent pieces factorize into parts depending holomorphically resp. anti-holomorphically on the x_i one may cast the expansion into the form

$$(24)\quad <\Phi^{j_4}\dots\Phi^{j_1}> \;=\; \int dj_{21}C(j_4,j_3,j_{21})D(j_{21};j_2,j_1)\left|\mathcal{F}^s_{j_{21}}\left[\begin{smallmatrix} j_4 & j_3 & j_2 & j_1 \\ x_4 & x_3 & x_2 & x_1\end{smallmatrix}\right]\right|^2$$

where the (s-channel) "conformal blocks" $\mathcal{F}^s_{j_{21}}$ are defined as

(25)
$$\mathcal{F}^s_{j_{21}}\left[\begin{smallmatrix} j_4\, j_3\, j_2\, j_1 \\ x_4 x_3 x_2 x_1\end{smallmatrix}\right] = \sum_{n=0}^{\infty} R_n(j_{21}|j_2,j_1)\;\frac{\partial^n}{\partial x^n} C\left(\begin{smallmatrix} j_4 & j_3 & j_{21} \\ x_4 & x_3 & x\end{smallmatrix}\right) C\left(\begin{smallmatrix} j_{21} & j_2 & j_1 \\ x & x_2 & x_1\end{smallmatrix}\right)$$
$$\text{where}\quad C\left(\begin{smallmatrix} j_3 & j_2 & j_1 \\ x_3 & x_2 & x_1\end{smallmatrix}\right) = (x_3-x_2)^{j_3+j_2-j_1}(x_3-x_1)^{j_3+j_1-j_2}(x_2-x_1)^{j_2+j_1-j_3}$$

The sum may be carried out explicitely in terms of the hypergeometric function: Let x be the crossratio $x=\frac{(x_2-x_1)(x_4-x_3)}{(x_3-x_1)(x_4-x_2)}$,

(26)
$$\begin{aligned}\mathcal{F}^s_{j_{21}}\left[\begin{smallmatrix} j_4\, j_3\, j_2\, j_1 \\ x_4 x_3 x_2 x_1\end{smallmatrix}\right] =& (x_4-x_3)^{j_4+j_3-j_2-j_1}(x_4-x_2)^{2j_2}(x_4-x_1)^{j_4+j_1-j_3-j_2} \\ & (x_3-x_1)^{j_3+j_2+j_1-j_4} x^{j_1+j_2-j_{21}} F(j_4-j_3-j_{21}, j_1-j_2-j_{21}; -2j_{21}; x).\end{aligned}$$

The decomposition (24) makes explicit to which extend the correlation functions are determined by the symmetry: The conformal blocks are completely determined by it, whereas one has without further input no information on structure constants $C(j_3,j_2,j_1)$ and fusion rules (range of integration over j_{21}).

The additional requirement that one may expect to determine these pieces of information also is crossing symmetry: One may use an expansion of type (21) for the product of operators $\Phi^{j_3}\Phi^{j_2}$ instead to get an expansion of the four point function into a different set of conformal blocks (t-channel):

(27)
$$<\Phi^{j_4}\ldots\Phi^{j_1}> \;=\; \int dj_{32} C(j_4,j_{32},j_1) D(j_{32};j_3,j_2)\left|\mathcal{F}^t_{j_{32}}\left[\begin{smallmatrix} j_4\, j_2\, j_3\, j_1 \\ x_4 x_2 x_3 x_1\end{smallmatrix}\right]\right|^2,$$

The result should of course be equal to the expansion (24). Equality of the two expansions (crossing symmetry) leads to restrictions for the structure constants and fusion rules:

This may be made more explicit by observing that one has fusion transformations for the conformal blocks in this baby CFT, as shown in Appendix B. They take the form

(28)
$$\mathcal{F}^s_{j_{21}}\left[\begin{smallmatrix} j_4\, j_3\, j_2\, j_1 \\ x_4 x_3 x_2 x_1\end{smallmatrix}\right] = \int d\mu(j_{32})\; F_{j_{21}j_{32}}\left[\begin{smallmatrix} j_3 j_2 \\ j_4 j_1\end{smallmatrix}\right] \mathcal{F}^t_{j_{32}}\left[\begin{smallmatrix} j_4\, j_2\, j_3\, j_1 \\ x_4 x_2 x_3 x_1\end{smallmatrix}\right]$$

Given fusion relations (28), the requirement of crossing symmetry translates itself into a system of equations for the structure constants:

(29)
$$\begin{aligned}\int_{\mathcal{S}_s} d\mu(j_{21})\; F_{j_{21}j_{32}}\left[\begin{smallmatrix} j_3 j_2 \\ j_4 j_1\end{smallmatrix}\right] \bar{F}_{j_{21}j'_{32}}\left[\begin{smallmatrix} j_3 j_2 \\ j_4 j_1\end{smallmatrix}\right]\; & C(j_4,j_3,j_{21}) D(j_{21};j_2,j_1) \\ &= \delta(j_{32},j'_{32})\; C(j_4,j_{32},j_1) C(j_{32};j_3,j_2).\end{aligned}$$

Instead of solving this horrible system of equations one has in the present baby CFT a direct way to obtain structure constants and fusion rules, which will be discussed next.

3.2. **N-point functions as overlaps.** According to the previous discussion of state-operator correspondence one may alternatively define vacuum expectation values with help of the scalar product in $L^2(H_3^+, dh)$:

(30)
$$<\Phi^{j_N}(x_N,\bar{x}_N)\ldots\Phi^{j_1}(x_1,\bar{x}_1)> = \int_{H_3^+} dh \prod_{i=1}^{N} \Psi(j_i;x_i,\bar{x}_i|h)$$

In order to discuss convergence of such integrals it is easier to consider

$$<\Phi^{j_N}_{n_N,p_N}\ldots\Phi^{j_1}_{n_1p_1}> = \delta(\textstyle\sum n_i)\delta(\sum\omega_i)\displaystyle\int_0^\infty dy \prod_{i=1}^{N}\Theta^{j_i}_{n_ip_i}(y),$$

where

$$\Theta^j_{np}(y) \;=\; B^{-1}_{np}(j) y^{|n|}(1+y)^{ip}\, F\Big(\tfrac{1}{2}(|n|+ip)-j, \tfrac{1}{2}(|n|+ip)+j+1; 1+|n|; -y\Big)$$
$$B_{np}(j) \equiv \frac{\Gamma(1+|n|)\Gamma(2j+1)}{\Gamma(\frac{1}{2}(|n|+ip)+j+1)\Gamma(\frac{1}{2}(|n|-ip)+j+1)}.$$

In order to find the conditions for convergence of the integral over y one needs the asymptotics of $\Theta^j_{m\bar{m}}(y)$ for $y \to \infty$:

$$\Theta^j_{np}(y) \sim y^{-j-1} + y^j \frac{B_{np}(j)}{B_{np}(-j-1)},$$

The overlap defining the n-point functions will therefore be convergent provided

$$\sum_{i=1}^{n} \left(\left|\Re\left(j_i + \tfrac{1}{2}\right)\right| - \tfrac{1}{2} \right) < -1$$

One observes that this condition can never be satisfied for $n \neq 2$, a case which requires special discussion. However, it is important to observe that these integrals give well defined correlation functions with $n > 2$ not only for operators corresponding to normalizable states $\Re(j_i) = -1/2$ but also for a class of operators corresponding to non-normalizable states.

To finish this subsection, I would like to make the following remark: Since the general classical solution of Liouville theory in the case of only "weak" insertions is given by

$$e^{-j\phi_L(z,\bar{z})} = \left((z,1)\cdot gg^\dagger \cdot \begin{pmatrix} \bar{z} \\ 1 \end{pmatrix} \right)^{2j_i} \qquad g \in SL(2,\mathbb{C})$$

one finds correspondence between WZNW and Liouville semiclassical correlation function via $x \to z$.

3.2.1. *Two-point function.* The two-point function of operators Φ^{j_2}, Φ^{j_1} with $j_i \in -\frac{1}{2} + i\mathbb{R}$, $i = 1,2$ may be read off from the orthogonality relations found in the Appendix as

$$< \Phi^{j_2}_{n_2p_2}\Phi^{j_1}_{n_1p_1} >= (2\pi)^3 \delta_{n_1+n_2}\delta(p_2+p_1)\left(i\delta(j_1+j_2+1) + \frac{B_{n_1p_1}(j_1)}{B_{n_1p_1}(-j_1-1)} i\delta(j-j') \right).$$

In terms of the $\Phi^j(x,\bar{x})$ it reads

$$< \Phi^{j_2}(x_2,\bar{x}_2)\Phi^{j_1}(x_1,\bar{x}_1) >= 2\pi \left(\delta^{(2)}(x_2-x_1) i\delta(j_2+j_1+1) + |x_2-x_1|^{4j_1}\frac{2j_1+1}{\pi} i\delta(j_1-j_2) \right).$$

There does not seem to be a way to extend this result to j_i with $\Re(2j_i+1) \neq 0$.

3.2.2. *Three point function.* The integral defining the three point function of operators $\Phi_i = \Phi^{j_i}(x_i,\bar{x}_i)$ $i = 1,2,3$ has been calculated in [ZZ] with the result

$$< \Phi_3\Phi_2\Phi_1 > \;=\; \pi^{-3}\frac{\Gamma(-j_1-j_2-j_3-1)\Gamma(j_3-j_2-j_1)\Gamma(j_2-j_1-j_3)\Gamma(j_1-j_2-j_3)}{\Gamma(-2j_1-1)\Gamma(-2j_2-1)\Gamma(-2j_3-1)}$$
$$\times |x_3-x_2|^{2(j_3+j_2-j_1)}|x_3-x_1|^{2(j_3+j_1-j_2)}|x_2-x_1|^{2(j_2+j_1-j_3)}$$

Note that in contrast to the two point function it is obviously possible to meromorphically continue the result to general values of the j_i.

3.3. **Operator product expansions.** According to the previous discussion on operator-state correspondence the operator obtained by taking the product of two operators $\Phi^{j_2}(x_2,\bar{x}_2)$ and $\Phi^{j_1}(x_1,\bar{x}_1)$ is just represented by the product of the corresponding wave functions. Operator product expansion therefore corresponds to expanding the wave function $\Psi(j_2;x_2,\bar{x}_2|h)\Psi(j_1;x_1,\bar{x}_1|h)$ in terms of the basis given by the $\Psi(j;x,\bar{x}|h)$. The crucial observation in this context is that there exists a range of values for j_1, j_2 given by

$$|\Re(j_1+j_2+1)|<\tfrac{1}{2} \qquad |\Re(j_1-j_2)|<\tfrac{1}{2} \tag{31}$$

where the product wave function $\Psi(j_2;x_2,\bar{x}_2|h)\Psi(j_1;x_1,\bar{x}_1|h)$ is normalizable. This range evidently includes the case where Φ^{j_2} and Φ^{j_1} correspond to the spectrum ($\Re(j_i)=-1/2$) and may be visualized as some "strip" around the axis $\Re(j_i)=-1/2$. By the completeness of the basis spanned by the $\Psi(j;x,\bar{x}|h)$ one may therefore expand

$$\Phi^{j_2}(x_2,\bar{x}_2)\Phi^{j_1}(x_1,\bar{x}_1)=\int_{\mathcal{P}_+}dj\int d^2x\;D\left(\begin{smallmatrix} j & j_2 & j_1\\ x & x_2 & x_1\end{smallmatrix}\right)\Phi^j(x,\bar{x}). \tag{32}$$

Of course, taking the two point function with Φ^{-j-1}, $j\in\mathcal{P}_+$ identifies the coefficients $D(\dots)$ with the three point function:

$$D\left(\begin{smallmatrix} j & j_2 & j_1\\ x & x_2 & x_1\end{smallmatrix}\right)=<\Phi^{-j-1}(x,\bar{x})\Phi_2(x_2,\bar{x}_2)\Phi_1(x_1,\bar{x}_1)>, \qquad j\in\mathcal{P}_+$$

In order to establish the relation to the bootstrap formalism one should expand the coefficient C for x_2 near x_1. The integration over x is then carried out by means of formula (10). One indeed recovers (21) and (22).

One way to go beyond the considered region of values for the j_i is by analytic continuation of (32). This requires extending the integration in (32) to an integration over the whole axis $\mathcal{P}\equiv -1/2+i\mathbb{R}$, which may be done by using (10):

$$\Phi^{j_2}(x_2,\bar{x}_2)\Phi^{j_1}(x_1,\bar{x}_1)=\frac{1}{2i}\int_{\mathcal{P}}dj\int d^2x\;D\left(\begin{smallmatrix} j & j_2 & j_1\\ x & x_2 & x_1\end{smallmatrix}\right)\;\Psi^j(x,\bar{x}), \tag{33}$$

Analytically continuing (33) in the parameter j_1+j_2 from $j_1+j_2<-1/2$ around the pole at $j_1+j_2=-1/2$ to $-1/2<j_1+j_2<0$ one encounters the situation that a pole of the integrand hits the integration contour. Suitably deforming it, one rewrites the integral as the sum of an integral over the original contour plus a residue contribution:

(34)

$$\Phi^{j_2}(x_2,\bar{x}_2)\Phi^{j_1}(x_1,\bar{x}_1)=\int d^2xE\left(\begin{smallmatrix} j_2 & j_1\\ x_2 & x_1\end{smallmatrix};x\right)\Phi^{-j_2-j_1-1}(x,\bar{x})+\int_{\mathcal{P}}dj\int d^2x\;D\left(\begin{smallmatrix} j & j_2 & j_1\\ x & x_2 & x_1\end{smallmatrix}\right)\;\Phi^j(x,\bar{x}),$$

where

$$E\left(\begin{smallmatrix} j_2 & j_1\\ x_2 & x_1\end{smallmatrix};x\right)=\mathrm{Res}_{j=-j_1-j_2-1}\left(\begin{smallmatrix} j & j_2 & j_1\\ x & x_2 & x_1\end{smallmatrix}\right)=(2j_2+1)(2j_1+1)|x-x_2|^{4j_2}|x-x_1|^{4j_1}$$

There are two further ways to understand this pattern of decomposition: First one may note that the product $\Psi_2\Psi_1$ is no longer normalizable for $-1/2<j_{21}<0$, due to the leading asymptotics given by $|v|^{2(j_1+j_2)}$. However, the first term on the right hand side of (34) can be seen[6] to have the same leading asymptotics, substracting it from the left hand side will then yield something normalizable that can be expanded in terms of the Φ^j with $j\in\mathcal{P}$. One learns that even certain non-normalizable states possess a well-defined expansion if the basis $\{\Phi^j;j\in\mathcal{P}_+\}$ is suitably extended.

Second, one may note that if $j_i\in\mathbb{R}$, $-1<j_i<0$ then the operators Ψ_i transform as unitary representations of the supplementary series under $SL(2,\mathbb{C})$. The found pattern of decomposition

[6] Again most easily by expanding in x_2-x_1 and using (10)

is precisely that predicted by Naimark's theorem on the decomposition of tensor products of $SL(2,\mathbb{C})$ representations [N].

A few remarks are in order:

1. Needless to say that the decomposition for general j_1, j_2 can be found by further analytic continuation, picking up more and more residue terms. In some cases the OPE reduces to a sum over finitely many j: This will happen iff either $2j_1+1 \in \mathbb{Z}^{>0}$ or $j_2+1 \in \mathbb{Z}^{>0}$, which is the case where one of the Ψ^{j_i} $i=1,2$ corresponds to a finite dimensional representation of $SL(2,\mathbb{C})$.
2. Nonvanishing of the three-point function does not imply appearance of any one of the three fields in the operator product expansion of the other two.

3.4. **Four point function; factorization.** Start by considering the four point function $<\Psi_4 \dots \Psi_1>$ in the case that

$$
|\Re(j_1+j_2+1)| < \tfrac{1}{2} \qquad |\Re(j_1-j_2)| < \tfrac{1}{2}
$$
$$
|\Re(j_3+j_4+1)| < \tfrac{1}{2} \qquad |\Re(j_3-j_4)| < \tfrac{1}{2} \tag{35}
$$

Under these conditions the product $\Psi_2\Psi_1$ is square-integrable, and can be expanded according to (32). This yields the following representation for the four point function:

$$
< \Phi_4 \dots \Phi_1 >= -i \int_{\mathcal{P}_+} dj \int d^2x < \Phi_4\Phi_3\Phi_j(x,\bar{x}) >< \Phi^{-j-1}(x,\bar{x})\Phi_2\Phi_1 > .
$$

The integral over x can be performed with the result

$$
\begin{aligned}
&\int d^2x < \Phi_4\Phi_3\Phi_j(x,\bar{x}) >< \Phi^{-j-1}(x,\bar{x})\Phi_2\Phi_1 >= \\
&= \ |x_{43}|^{2(j_4+j_3-j_2-j_1)}|x_{42}|^{4j_2}|x_{41}|^{2(j_4+j_1-j_3-j_2)}|x_{31}|^{2(j_3+j_2+j_1-j_4)} \\
&\quad \Big(D_j\ |x|^{2(j_1+j_2+j+1)}\ F_{-j-1}(x)F_{-j-1}(\bar{x}) + E_j\ |x|^{2(j_1+j_2-j)}\ F_j(x)F_j(\bar{x})\Big),
\end{aligned}
$$

where $x_{ij} = x_i - x_j$, $x = \frac{(x_2-x_1)(x_4-x_3)}{(x_3-x_1)(x_4-x_2)}$,

$$
\begin{aligned}
D_j =& C(j_4,j_3,j)C(-j-1,j_2,j_1)\frac{\gamma(j+1+j_4-j_3)}{\gamma(j_4-j_3-j)\gamma(2j+2)} \\
=&\frac{1}{2j+1}C(j_4,j_3,-j-1)C(-j-1,j_2,j_1), \\
E_j =& C(j_4,j_3,j)C(-j-1,j_2,j_1)\frac{\gamma(-j+j_2-j_1)}{\gamma(j+1+j_2-j_1)\gamma(-2j)} \\
=& -\frac{1}{2j+1}C(j_4,j_3,j)C(j,j_2,j_1) = D_{-j-1} \\
F_j(z) =& F(j_4-j_3-j, j_1-j_2-j; -2j; z)
\end{aligned} \tag{36}
$$

One thereby finds an expansion into conformal blocks of the form (23).

Of course one can treat the case of more general values for the j_i by analytic continuation, which will again lead to a sum over residue terms. However, if one considers analytic continuation to a region where the conditions on j_3, j_4 in (35) are violated, one will get residue terms that correspond to non-normalizable operators in the product $\Phi_4\Phi_3$. Proper description of such contributions will be somewhat problematic in a canonical operator formalism.

4. Appendix A: Spectral decomposition

A (plane wave) basis for $L^2(H_3^+, dh)$ may be found by diagonalizing a complete set of commuting differential operators. A convenient choice is to take the operators

$$K^0 \equiv \frac{1}{2}(J^0 - \bar{J}^0) = -v\frac{\partial}{\partial v} + \bar{v}\frac{\partial}{\partial \bar{v}}, \qquad F^0 \equiv \frac{1}{2}(J^0 + \bar{J}^0) = -i\frac{\partial}{\partial \phi}$$

$$\begin{aligned} Q &= \frac{1}{2}(2(J^0)^2 + J^+J^- + J^-J^+) = \frac{1}{2}(2(\bar{J}^0)^2 + \bar{J}^+\bar{J}^- + \bar{J}^-\bar{J}^+) \\ &= (1+|v|^2)\frac{\partial^2}{\partial v \partial \bar{v}} + \frac{1}{4}\left(v\frac{\partial}{\partial v} - \bar{v}\frac{\partial}{\partial \bar{v}}\right)^2 + \frac{1}{2}\left(v\frac{\partial}{\partial v} + \bar{v}\frac{\partial}{\partial \bar{v}}\right) + \frac{1}{4}\frac{1}{1+|v|^2}\frac{\partial^2}{\partial \phi^2} \end{aligned} \tag{37}$$

By writing v as $v = e^{i\varphi}\sqrt{y}$, $\varphi \in [-\pi, \pi]$, $y \in \mathbb{R}$ one has $K^0 = -i\frac{\partial}{\partial \varphi}$, so K^0 has spectrum $\mathbb{Z}$, whereas F^0 has spectrum $\mathbb{R}$. On an eigenspace of K^0, F^0 with eigenvalues n, p the operator Q reduces to

$$Q_{np} = \frac{\partial}{\partial y}y(1+y)\frac{\partial}{\partial y} - \frac{n^2}{y} - \frac{p^2}{1+y} \tag{38}$$

4.1. **Self-adjointness of Q_{np}.** One has

$$\int_0^\infty dy(Q_{np}f(y))^* g(y) - \int_0^\infty dy(f(y))^* Q_{np}g(y) = \left[y(1+y)\left(f\frac{\partial}{\partial y}g - g\frac{\partial}{\partial y}f\right)\right]_0^\infty, \tag{39}$$

so that Q_{np} is symmetric on the dense subspace D of $L^2(\mathbb{R}^{\geq 0})$ that consists of functions regular at 0 and ∞. According to the general theory of self-adjoint extensions of unbounded symmetric operators [AG] one needs to know whether there exist normalizable eigenfunctions to eigenvalues with strictly positive or strictly negative imaginary part. The eigenvalue equation $Q_{np}f = j(j+1)f$ is transformed into the hypergeometric differential equation by means of $f = y^n(1+y)^{ip}g$. The solutions with specified asymptotic behavior near $y = \infty$ are $g(y;j)$ and $g(y, -j-1)$, where

$$g(y;j) = y^j(1+y^{-1})^{ip}F(\tfrac{1}{2}(|n|+ip) - j, \tfrac{1}{2}(-|n|+ip) - j; -2j; -y).$$

Necessary for $g(y;j)$ to be square-integrable is $\Re(j) < -\frac{1}{2}$. However, $g(y;j)$ behaves for $y \to 0$ as $a_n y^n + b_n y^{-n}$ with $a_n, b_n \neq 0$ for $n \neq 0$, cf. [E], p.109, eqn. (7). It is therefore not square-integrable for $n \neq 0$. It remains to consider $n = 0$. In that case one should observe that $g(y;j) \sim \log(y)$ for $y \to 0$, which leads to a nonzero contribution on the r.h.s.of (39) if $f \in D$. It is therefore not possible to include $g(y;j)$ in an extension D' of the domain D such that Q_{0p} becomes selfadjoint on D'. By the general theory of [AG] one therefore has a unique selfadjoint extension of (Q_{np}, D).

4.2. **The resolvent.** In order to determine the spectrum of Q_{np} it is useful to construct its resolvent $R(q) = (Q_{np} - q)^{-1}$; $q \in \mathbb{C}$: The spectrum is essentially encoded in the analyticity properties of $R(q)$ [Yo]. Poles on the real axis correspond to eigenvalues, cuts to the continuous spectrum.

For any differential operator of the form $\mathcal{D}_y \equiv \partial_y p(y)\partial_y - r(y)$ one may construct the kernel of the resolvent $\mathcal{R} \equiv (\mathcal{D} - q)^{-1}$ in the form

$$R(y, y'; q) = N^{-1}\Big(\Theta(y - y')g(y;q)f(y';q) + \Theta(y' - y)f(y;q)g(y';q)\Big),$$

where $f(y;q)$ and $g(y,q)$ are two linearly independent solutions of $(\mathcal{D}_y - q)F = 0$ and $\Theta(x) = 0$ for $x < 0$, $\Theta(x) = 1$ for $x > 0$. Indeed, straightforward calculation shows that

$$(\mathcal{D}_y - q)R(y, y'; q) = N^{-1}p(f\partial_y g - g\partial_y f)\delta(y - y').$$

Moreover, the combination $p(f\partial_y g - g\partial_y f)$ is constant as a consequence of $(\mathcal{D}_y - q)F = 0$, $F = f, g$, so that the choice $N = p(f\partial_y g - g\partial_y f)$ indeed gives the resolvent kernel.

In the present case it is useful to parameterize the eigenfunctions in terms of the variable j, which from now on will have to be considered as a function of the eigenvalue q defined by

$$(40)\qquad j \equiv -\frac{1}{2} + \sqrt{\frac{1}{4} + q} \text{ for } q \in \mathbb{C} \setminus (\infty, -1/4], \quad j \equiv -\frac{1}{2} + i\sqrt{-\frac{1}{4} - q} \text{ for } q \in (\infty, -1/4].$$

$f(y;q)$ and $g(y,q)$ will now be chosen as

$$\begin{aligned} f(y;j) &= y^{|n|}(1+y)^{ip}\, F\Big(\tfrac{1}{2}(|n|+ip) - j, \tfrac{1}{2}(|n|+ip)+j+1; 1+|n|; -y\Big) \\ g(y;j) &= B_{np}(j)\, y^{j}(1+y^{-1})^{ip}\, F\Big(\tfrac{1}{2}(|n|+ip) - j, \tfrac{1}{2}(-|n|+ip) - j; -2j; -\tfrac{1}{y}\Big) \\ B_{np}(j) &\equiv \frac{\Gamma(1+|n|)\Gamma(2j+1)}{\Gamma(\frac{1}{2}(|n|+ip)+j+1)\Gamma(\frac{1}{2}(|n|-ip)+j+1)}. \end{aligned}$$

The factor $B_{np}(j)$ has been chosen for convenience since now the standard connection formula for the hypergeometric functions ([E], p. 108, eqn. (1)) reads $f(y,j) = g(y,j) + g(y,-j-1)$. The normalization is therby evaluated as

$$N = \lim_{y\to\infty} p(f\partial_y g - g\partial_y f) \;=\; \frac{1}{2j+1} B_{np}(j) B_{np}(-j-1)$$

4.3. **The spectrum.** Having explicitely constructed the resolvent it is easy to determine the spectrum: According to [Yo], chap. XI, sect.9 the discrete part of the spectrum would show up as poles of the resolvent $R(q)$ on the real $q-$axis. There are none in the present case. Furthermore, the continuous spectrum corresponds to jumps of $\mathcal{R}(q)$ on the real axis as is manifest in the formula for the spectral projection onto the interval $[a,b]$ given in ([Yo], loc. cit.):

$$(41)\qquad \mathcal{P}_{[a,b]}v = \lim_{\epsilon\to 0+} \frac{1}{2\pi i}\left[\int_a^b dq\; \mathcal{R}(q+i\epsilon)v - \int_a^b dq\; \mathcal{R}(q-i\epsilon)v\right].$$

If and only if the resolvent $\mathcal{R}$ has a jump at a certain value $q \in \mathbb{R}$ one will get a contribution to (41), so q belongs to the continuous spectrum. Here the jump arises from the square-root branch cut in (40): One has $\lim_{\epsilon\to 0+} j(q+i\epsilon) = j(q)$ and $\lim_{\epsilon\to 0+} j(q-i\epsilon) = -j(q) - 1$ for $q \in (-\infty, -1/4]$ and no jump otherwise. The continuous spectrum is therefore found to be $(-\infty, -1/4]$.

This may be reformulated as a completeness relation by noting that $\mathcal{P}_{(-\infty,-1/4]} = Id$, so rewriting (41) in terms of the corresponding kernels gives

$$\begin{aligned} \delta(y-y') = &\frac{1}{2\pi i}\int_{-\infty}^{-\frac{1}{4}} \frac{dq}{2j+1}\frac{1}{|B_{np}(j)|^2}\Big\{ \Theta(y-y')\Big(g(y;j) + g(y,-j-1)\Big) f(y';j) \\ &\qquad\qquad \Theta(y'-y)\Big(g(y';j) + g(y',-j-1)\Big) f(y;j)\Big\} \\ = &\frac{1}{2\pi i}\int_{\mathcal{P}_+} dj\; \frac{1}{|B_{np}(j)|^2}\; f(y,j) f(y',j) \qquad\qquad \mathcal{P}_+ \equiv -\tfrac{1}{2} + i\mathbb{R}^{\geq 0} \end{aligned}$$

4.4. **Basis I.** The results of the previous subsections show that the set of functions

$$(42)\qquad \Psi^j_{np}(h) = B^{-1}_{np}(-j-1) e^{in\varphi} e^{ip\phi} f(y,j) \qquad n \in \mathbb{Z}, p \in \mathbb{R}, j \in \mathcal{P}_+$$

is complete in $L^2(H_3^+)$. One may also check that it is δ-function orthonormalized: The scalar product may be evaluated by means of the formula

$$\begin{aligned}(j(j+1)-j'(j'+1))&\int_0^\infty dy\;\; f(y;j)f(y;j')\\ =\;\;&\lim_{y\to\infty} y(1+y)\left(f(y;j')\partial_y f(y;j)-f(y;j)\partial_y f(y;j')\right),\end{aligned}$$

which follows from the fact that $f(y;j)$ and $f(y;j')$ are eigenfunctions of the differential operator Q_{np} with eigenvalues $j(j+1)$ and $j'(j'+1)$ respectively. The result is

$$< \Psi^j_{np},\Psi^{j'}_{n'p'} >= \int_{H_3^+} dh\left(\Psi^j_{np}(h)\right)^*\Psi^{j'}_{n'p'}(h) = (2\pi)^3\delta_{n,n'}\delta(p-p')i\delta(j-j'), \tag{43}$$

where j, j' are assumed to be in $\mathcal{P}_+$.

Finally note that alternatively one might take the Ψ^j with $j\in -1/2+i\mathbb{R}^{\leq 0}$ as basis as they differ only by a phase factor

$$\Psi^j_{np}(h) = \frac{B_{np}(j)}{B_{np}(-j-1)}\Psi^{-j-1}_{np}(h). \tag{44}$$

4.5. **Basis II.** A second useful basis may be introduced as the following Fourier transform of the basis (42)

$$\Psi(j,x,\bar{x}|h) = \frac{1}{(2\pi)^2}\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}} dp\;\; e^{-in\arg(x)}|x|^{2j-ip}\Psi^j_{np}(h).$$

In order to see that the result is simply

$$\Phi^j(x,\bar{x}|h) = \frac{2j+1}{\pi}\left((x,1)\cdot h\cdot\begin{pmatrix}\bar{x}\\1\end{pmatrix}\right)^{2j}$$

one may i.e. use the following reasoning: First note that $\Psi^j(x,\bar{x}|h)$ satisfies an intertwining property with respect to $SL(2,\mathbb{C})$-transformations:

$$\Psi^j(x,\bar{x}|ghg^\dagger) = |\beta x+\delta|^{4j}\Psi\left(\frac{\alpha x+\gamma}{\beta x+\delta}, c.c.|h\right) \quad \text{if} \quad g = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}. \tag{45}$$

On the infinitesimal level this amounts to a couple of differential equations relating derivatives w.r.t. h to those w.r.t. x, in particular ($x = e^{i\xi}r$)

$$\left(\frac{\partial}{\partial\varphi}+\frac{\partial}{\partial\xi}\right)\Psi^j = 0 \qquad \left(\frac{\partial}{\partial\phi}+r\frac{\partial}{\partial r}-2j\right)\Psi^j = 0 \qquad Q\Psi^j = j(j+1)\Phi^j$$

By the inverse transform

$$\tilde{\Psi}^j_{np}(h) \equiv \int_{\mathbb{C}} d^2x\;\; e^{in\arg(x)}|x|^{-2j-2+ip}\Psi^j(x,\bar{x}|h). \tag{46}$$

these differential equations turn into the eigenvalue equations for Q, F^0, K^0. Since moreover $\Psi(j;x,\bar{x}|h)$ is regular for $y\to 0$ one finds that $\tilde{\Psi}^j_{np}(h)$ must be proportional to $\Psi^j_{np}(h)$. To find the constant of proportionality one may consider the integral (46) in the limit $y\to\infty$ where it can be reduced to

$$\frac{2j+1}{\pi}\int d^2x e^{in\arg(x)}|x|^{-2j-2+ip}|x-1|^{4j} = \frac{B_{np}(j)}{B_{np}(-j-1)}, \tag{47}$$

so that indeed $\tilde{\Psi}^j_{np}(h) = \Psi^j_{np}(h)$.

The relations expressing orthogonality and completeness of the $\Psi^j(x,\bar{x}|h)$ are now easily obtained from those of the $\Psi^j_{np}(h)$:

$$\int_{H_3^+} dh\ (\Psi(j;x,\bar{x}|h))^*\Psi(j';x',\bar{x}'|h) \;=\; 2\pi\delta^{(2)}(x-x')i\delta(j-j') \tag{48}$$

$$\frac{1}{i}\int_{\mathcal{P}_+} dj\int_{\mathbb{C}} d^2x(\Psi(j;x,\bar{x}|h))^*\Psi(j;x,\bar{x}|h') \;=\; (2\pi)^3\delta(\varphi-\varphi')\delta(\phi-\phi')\delta(y-y') \tag{49}$$

The relation between $\Psi(j;x,\bar{x}|h)$ and $\Psi(-j-1;x,\bar{x}|h)$ is obtained by Fourier transform of (44) and again using (47):

$$\Psi^j(x,\bar{x}) = \frac{2j+1}{\pi}\int d^2x'\ |x-x'|^{4j}\Psi^{-j-1}(x',\bar{x}'). \tag{50}$$

5. Appendix B: Fusion relations

I will now present a direct calculation inspired by [BM] of the semi-classical fusion matrix. This approach will have the additional advantage to work for a larger range of the j_i-values, so I will consider complex j_i with certain restrictions to be specified below only on their real parts.

The basic ingredient is the following formula of Burchnall and Chaundy:

$$(1-x)^d = \sum_{n=0}^{\infty}(-)^n\frac{(a)_n(b)_n}{(c+n-1)_n n!}\;{}_3F_2\begin{pmatrix}-d, c+n-1, -n\\ a\quad b\end{pmatrix} x^n F(a+n,b+n;c+2n;x)$$

This formula holds for any value of a,b,c on the right hand side. It can be turned into an expansion into eigenfunctions of P if one identifies

$$\begin{aligned} a+n &= j_1-j_2-j & \qquad b+n &= j_4-j_3-j\\ c+2n &= -2j & n &= j_1+j_2-j. \end{aligned}$$

Summation over n may be traded for integration over j's corresponding to principal series intermediate representations by identifying the right hand side as the sum over the residues at $j-j_1-j_2=-n$ of

$$\begin{aligned} H(j;j_4,\dots,j_1;d;x) \;\equiv\; & \Gamma(j-j_1-j_2)\frac{\Gamma(j_1-j_2-j)\Gamma(j_4-j_3-j)\Gamma(-j_1-j_2-j-1)}{\Gamma(-2j_2)\Gamma(j_4-j_3-j_2-j_1)\Gamma(-2j-1)}\times\\ & \times{}_3F_2\begin{pmatrix}-d, -j-j_1-j_2-1, j-j_1-j_2\\ -2j_2\quad j_4-j_3-j_2-j_1\end{pmatrix}\\ & \times x^{j_1+j_2-j}F(j_1-j_2-j, j_4-j_3-j;-2j;x). \end{aligned}$$

In fact, considered as function of j one finds that $H(j_4,\dots,j_1;j;x)$ has the following poles:

$$\begin{aligned} j &= j_1+j_2-n_1 & \qquad j &= -j_2-j_2-1+n_2 & \qquad & n_1,\dots,n_4\in\mathbb{Z}^{\geq 0}\\ j &= j_1-j_2+n_3 & j &= j_4-j_3+n_4 \end{aligned}$$

One finds only the "wanted" poles $j=j_1+j_2-n; n=0,1,2,\dots$ within the half-plane $\{j;\Re(j)<-\frac{1}{2}\}$ iff

$$\Re(j_1+j_2)<-\frac{1}{2}\qquad \Re(j_1-j_2)>-\frac{1}{2}\qquad \Re(j_4-j_3)>-\frac{1}{2}$$

In this case one may rewrite the sum over residues as limit of the integrations over the closed contour

$$\mathcal{C}_r\equiv\big\{j=-\tfrac{1}{2}+i\sigma;\sigma\in\mathbb{R}\big\}\cup\big\{j;\big|j+\tfrac{1}{2}\big|=r,\Re(j+\tfrac{1}{2})<0\big\}$$

for $r \to \infty$. However, by using the estimates on the integrand given in [BM], one recognizes that the contributions from the semi-circle vanish for $r \to \infty$. One is left with

$$(1-x)^d = \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dj\; H(j;j_4,\dots,j_1;d;x). \tag{51}$$

Now consider the semi-classical t-channel conformal block

$$F^t_{j_{32}} = (1-x)^{j_2+j_3-j_{32}} F(j_3-j_2-j_{32}, j_4-j_1-j_{32}; -2j_{32}; 1-x)$$

By expanding the hypergeometric function as power series in $1-x$, applying (51) and exchanging summation with integration one arrives at

$$F^t_{j_{32}}(x) = \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dj_{21} F_{j_{32}j_{21}}\begin{bmatrix} j_3 j_2 \\ j_4 j_1 \end{bmatrix} F^s_{j_{21}}(x),$$

where

$$\begin{aligned} F_{j_{32}j_{21}}\begin{bmatrix} j_3 j_2 \\ j_4 j_1 \end{bmatrix} \;=\; & \frac{\Gamma(j_{21}-j_1-j_2)\Gamma(j_4-j_3-j_{21})\Gamma(-j_{21}-j_1-j_2-1)\Gamma(j_1-j_2-j_{21})}{\Gamma(-2j_{21}-1)\Gamma(-2j_2)\Gamma(j_4-j_3-j_2-j_1)} \\ & \times \sum_{n=0}^{\infty} \frac{\Gamma(j_3-j_2-j_{32}+n)\Gamma(j_4-j_1-j_{32}+n)\Gamma(-2j_{32})}{\Gamma(j_3-j_2-j_{32})\Gamma(j_4-j_1-j_{32})\Gamma(-2j_{32}+n)\, n!} \\ & \qquad \times {}_3F_2\begin{pmatrix} j_{32}-j_3-j_2-n, -j-j_1-j_2-1, j-j_1-j_2 \\ -2j_2 \quad j_4-j_3-j_2-j_1 \end{pmatrix} \end{aligned}$$

The summands have large n asymptotics $n^{j_{21}+j_3+j_4}$, so if also $\Re(j_3+j_4) < -1/2$ one has absolute convergence, which justifies the exchange of summation with integration that had been performed.

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