Combining Data-Closeness and Fourier Domain Integrability Constraints in Shape-from-Shading

Mario Castelan and Edwin R. Hancock,  
Department of Computer Science, University of York,  
York Y010 5DD, UK.

Abstract

This paper describes a shape-from-shading algorithm that combines constraint on data-closeness from Lambert’s law and Fourier domain integrability. The data closeness is ensured by constraining surface normals to fall on an irradiance cone, whose axis points in the light source direction and whose apex angle varies with iteration number. The integrability is ensured by projecting the non-integrable set of surface normals to the nearest integrable one by globally minimizing the distance among them in the Fourier domain. The combination of both data-closeness and integrability constraints is aimed to overcome the problem of high dependency on the image irradiances. Experimental results prove that the new method recovers needle maps that are both smooth and integrable and improves height surface stability.

1. Introduction

Shape-from-shading (SFS) is a problem in computer vision which has been an active topic of research for some three decades. The process was identified by Marr[6] as a key process in the computation of the 2.5D sketch, and was studied in depth by Horn[2]. The topic has also been the focus of recent research in the psychophysics literature [5]. Stated more formally, the SFS problem can be regarded as that of calculating the set of partial derivatives (\(Z_x, Z_y\)) corresponding to a surface \(Z = Z(x, y)\), assuming as an input a single intensity image whose irradiances obey the interaction of factors such as light source direction and surface material.\(^1\) In brief, we need to solve the image irradiance equation, 

\[
E(x, y) = R(p(x, y), q(x, y), s),
\]

where \(E\) is the intensity value of the pixel with position \((x, y)\), \(R\) is a function referred to as the reflectance map[3], that maps the surface gradients \(p = \frac{\partial Z(x, y)}{\partial x}\) and \(q = \frac{\partial Z(x, y)}{\partial y}\) to an intensity value and \(s\) is the light source direction. If the surface normal at the location \((x, y)\) is \(n = (p, q, -1)\) then under Lambertian reflectance model, the image irradiance equation becomes \(E(x, y) = n \cdot s\).

Unfortunately, the image irradiance equation is underconstrained, and the family of surface normals fall on a reflectance cone whose apex angle \(\alpha\) is equal to \(\cos^{-1} E(x, y)\), and whose axis points in the light source direction \(s\). Several constraints have been used to overcome the underconstrained nature of the Lambertian shape-from-shading problem. However, their main drawback is that they have a tendency to oversmooth the recovered surface slopes and result in poor data-closeness. The net result is a loss of fine surface detail. For a complete survey of most SFS methods, see [11].

In a recent series of papers [9, 10] it has been demonstrated how these problems may be overcome by constraining the surface normals to lie on the reflectance cone and allowing them to rotate about the light source direction subject to curvature consistency constraints. Unfortunately, the needle maps delivered by the method are not guaranteed to satisfy the integrability constraint, which means that the recovered partial derivatives are not independent on the path of integration (i.e. the height function may not be recoverable). Besides, these needle maps also suffer the drawback of high dependency on the image intensities, making the method prone to noisy data such as specularities, roughness and overshadowed areas.

There are a number of ways in which a surface may be recovered from a field of surface normals [4]. One approach is to use trigonometry to increment the height function along a path or a front. However, one of the most elegant approaches is that described by Frankot and Chellappa [1] which shows how the surface may be reconstructed subject to integrability constraints by performing a Fourier analysis of the field of surface normals.

The aim in this paper is to develop a shape-from-shading scheme that can be used to recover integrable needle maps subject to hard constraints on Lambertian reflectance as well as relaxing the image intensity dependance driven by

---

\(^1\) More than one image can be used, but this is an extension of SFS referred to as photometric stereo.
such constraints.

In order to demonstrate how the two techniques can be combined, in subsequent sections we will briefly explain a recently developed geometric approach for SFS as well as the algorithm proposed by Frankot and Chellappa [1] for enforcing integrability in SFS.

2. Geometric approach for SFS

A geometric approach in SFS is a method in which the image irradiance equation is treated as a hard constraint by forcing the recovered surface normals to lie on the reflectance cone. Suppose that $\hat{N}_k$ is a smoothed set of surface normals at step $k$ of the algorithm, then the update equation for the surface normal directions is

$$\hat{N}_{k+1} = \Theta \hat{N}_k$$  \hspace{1cm} (1)

where $\Theta$ is a rotation matrix computed from the apex angle $\alpha$ and the angle between the current smoothed estimate of the surface normal direction $\hat{N}_k$ and the light source direction. To restore the surface normal to the irradiance cone, it must be rotated by an angle

$$\theta = \cos^{-1}(E) - \cos^{-1}\left(\frac{\hat{N}_k \cdot s}{\|\hat{N}_k\| \cdot \|s\|}\right)$$ \hspace{1cm} (2)

about the axis $(u, v, w)^T = \hat{N}_k \times s$. Hence, the rotation matrix is

$$\Theta = \begin{pmatrix}
c + u^2c' & -sw + uv' & vs + uw' \\
ws + uv' & c + v^2c' & -us + vw' \\
-us + uv' & us + vw' & c + w^2c'
\end{pmatrix}$$ \hspace{1cm} (3)

where $c = \cos(\theta), c' = 1 - c$ and $s = \sin(\theta)$.

The needle maps delivered by this geometric framework have proved to be useful in experiments for topography-based object recognition.

3. Integrability in SFS

The integrability condition in SFS ensures that the recovered surface satisfies the following condition on the partial derivatives of the height function: $Z_{xy} = Z_{yx}$. In [1] Frankot and Chellappa proposed a method to project a gradient field to the nearest integrable solution. They suggested to use a set of integrable basis functions to represent the surface slopes so as to minimize the distance between an ideally integrable gradient field and a non integrable one.

Following [1], if the surface $Z$ is given by

$$\tilde{Z}(x, y) = \sum_{\omega \in \Omega} \hat{C}(\omega)\phi(x, y, \omega)$$ \hspace{1cm} (4)

where $\omega$ is a two dimensional index belonging to a domain $\Omega$, and $\phi(x, y, \omega)$ is a set of basis functions which are not necessarily mutually orthogonal, the partial derivatives of $\tilde{Z}$ can also be expressed in terms of this set of basis functions using the formulae

$$\tilde{Z}_x(x, y) = \sum_{\omega \in \Omega} \hat{C}(\omega)\phi_x(x, y, \omega)$$ \hspace{1cm} (5)

$$\tilde{Z}_y(x, y) = \sum_{\omega \in \Omega} \hat{C}(\omega)\phi_y(x, y, \omega)$$ \hspace{1cm} (6)

Given that $\phi_x(x, y, \omega)$ and $\phi_y(x, y, \omega)$ are integrable, then so are the mixed partial derivatives of $\tilde{Z}(x, y)$.

In the same way, the possibly non integrable gradient field (which, indeed, is the only information we have) can be represented as

$$\tilde{Z}_x(x, y) = \sum_{\omega \in \Omega} \hat{C}_1(\omega)\phi_x(x, y, \omega)$$ \hspace{1cm} (7)

$$\tilde{Z}_y(x, y) = \sum_{\omega \in \Omega} \hat{C}_2(\omega)\phi_y(x, y, \omega)$$ \hspace{1cm} (8)

Note that, as $\hat{C}_1 \neq \hat{C}_2$, then $\tilde{Z}_{xy} \neq \tilde{Z}_{yx}$.

The goal then is to find the set of coefficients that minimize the quantity

$$d\{(\tilde{Z}_x, \tilde{Z}_y), (\tilde{Z}_x, \tilde{Z}_y)\} = \int \int \|\tilde{Z}_x - \tilde{Z}_x\|^2 + \|\tilde{Z}_y - \tilde{Z}_y\|^2 dx dy$$ \hspace{1cm} (9)

In [1], details are given about how to solve this equation globally, in the Fourier domain.

4. Introducing the integrability condition in the geometric approach for SFS

The idea underpinning this paper is to calculate the nearest integrable surface following their method and obtain the apex angle of the Lambertian cone on this surface after each iteration. In this way, we ensure that the surface normals will lie on reflectance cones whose apex angles correspond to integrable surfaces.

The algorithm can be summarized as follows:

1. Calculate an initial estimate of surface normals $N = (N_x, N_y, N_z)$.
2. Obtain the nearest integrable surface $\tilde{Z}$ and its derivatives ($\tilde{Z}_x, \tilde{Z}_y$) by solving (9) using the surface normal field $N$. The result will be the projected integrable set of surface normals $\tilde{N}$, corresponding to the surface $\tilde{Z}$.
3. Get the apex angle $\alpha$ of the Lambertian cone using the values of $\tilde{Z}$, that is to say, $\alpha = \cos^{-1}(\tilde{Z})$.
4. Smooth $\tilde{N}$ to obtain $\hat{N}$.
5. Calculate \( \hat{N} \), by rotating \( \tilde{N} \), using (1).

6. Make \( N = \hat{N} \) and return to step 2. Repeat until a desired number of iterations has been reached.

It is worth remarking some aspects of the new algorithm. First, the non integrable estimation of surface normals is projected to the nearest integrable one in the Fourier domain. Second, a mid-processing smoothing step is required since the new integrable set of normals correspond exactly to the recovered surface, if such normals are not modified, it would be redundant to carry on the use of the rotation matrix. This smoothing step was carried out using a robust regularizer\(^2\). Finally, the rotation matrix does not remain static through the iterative process, since the changes in \( \alpha \) depend on the recovered surface after each iteration. As a consequence, the new scheme satisfies a combination of integrability, smoothness and data-closeness constraints. Note how the image irradiance equation is still treated as a hard constraint, since at each iteration the surface normals are projected back to lie on the lambertian cone. However, the hardness of this constraint is relaxed when making the reflectance cone to be based on continuous surfaces calculated after each iteration instead of making them lie on the irradiances of the image along the whole iterative process.

5. Experiments

The algorithm was tested on real world images. The evaluation criteria was based on the squared height difference and degree of gradient consistency (i.e. the percentage of pixels of every image whose differences \( Z_{xy} - Z_{yx} \) are less than or equal to a certain threshold\(^3\)). In our experiments we have compared the results obtained with the original geometric approach and the new integrable-geometric approach.

Twenty-eight real world images (ten of these with corresponding height data, taken from the range database in [12], and the rest taken from [7]\(^4\)).

The results of the experiments for degree of gradient consistency are summarized in Figure 1 (top). The figure shows that the combined algorithm (solid line) gives more consistent results than the original one (dotted lined), as the percentage of gradient consistency is always greater and more stable for the new approach, since at least 95% of the pixels, in all the cases, observe integrability. For the original algorithm, the unsteady behavior shown by the dotted line is evident. This suggests that the new method, as expected, is enforcing integrability in the original method.

Figure 2 shows the recovered needle maps for each method. A visual examination of the results suggests that the new method delivers needle maps that are both smoother and also contain fine topographic detail. The inclusion of integrability constraint leads to a less dependency on the image irradiance equation, therefore avoiding biasing the surface normals to highlighted points, as seen, for example, on the bird’s head case.

Figure 1 (bottom) shows the results for the squared height differences. The original approach is represented by the dotted line, while the new one is represented by the solid line. The plot reveals that the new method minimizes the error in a better way than the original method. This improvement is not significant though.

In Figure 3, the cross sections of the ground truth and recovered surfaces for each methods are shown (at row \( Y = 100 \), exactly the middle row of the images). From top to bottom, frog, bird and lobster. The single dotted plot on the left side corresponds to the ground truth surface for each case. The interpolated plots on the right side correspond to the recovered surface for the original method (dotted) and the new method (solid). By analyzing the interpolated plots we can notice how the new method tends to stabilize the recovered surface. In all of the cases, the high peaks on the surfaces seem to be regularized and more continuous, which can be interpreted as a consequence of the integrability constraint. We can also note that, despite the high difference among

---

\(^2\) In a recent research [9] it has been proved how robust regularizers can be included in iterative SFS schemes, leading to helpful results.

\(^3\) For all the experiments this threshold was set to 0.1.

\(^4\) For all the tests, the light source direction was assumed to be [0,0.1].
the recovered surfaces and the ground truth, the new algorithm seems to deliver more similar heights than the original one, this effect is more evident for the case of the lobster (third row).

**Figure 2. Recovered needle maps (see text).**

**Figure 3. Cross section plots (see text).**

6. Conclusions

In this paper we have demonstrated how to impose integrability constraints on a geometric approach for SFS. We follow Frankot and Chellappa and impose the constraints in the Fourier domain. Experiments reveal that the resulting method exhibits improved robustness and gradient consistency. However, although the height difference statistics do not reveal any systematic improvement in algorithm performance, both the recovered height surfaces and the needle maps delivered by the new algorithm appear to be better behaved and also preserve fine surface detail. It is important to comment that in this new method the calculation of surface orientations is less constrained by the irradiances of the image, as the rotation matrix changes through the iterative process. This is a way of relaxing the original method’s problem of hard constraints on data-closeness with the image irradiance equation. Our future plans include using alternative basis functions and in particular the discrete cosine transform, as well as comparing the output needle maps for local integration tests.

**References**


