An interpolation approach with stable polynomial interpolants applied to the simultaneous stabilization of a segment of systems.

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Abstract—This paper presents the design of a simultaneous compensator for a segment of systems based on an interpolation method with stable polynomial interpolants. This problem leads to formulate conditions of polynomial divisibility in the case of the simultaneous control as a polynomial interpolation issue. Finally, an algorithm permitting to compute a simultaneous controller stabilizing a segment of systems is given.

I. INTRODUCTION

The problem of finding a controller which simultaneously stabilizes more than one plant was essentially raised in the ring of stable rational functions $RH_\infty$ but some works have shown the limitations of this approach, see [1], [2]. Hence, the need to develop other approaches to synthesize simultaneous compensators for example in the space of stable polynomials (i.e. polynomials with no zeros in the left half plane) in order to consider problems that we can not handle with rational functions in $RH_\infty$. That is the case of the issue of the stabilization of a segment of systems with a simultaneous compensator.

The question of the simultaneous stabilization in the polynomial space has been formulated as an optimization problem see [3]. In this context, [3] gives bilinear matrix inequality (BMI) conditions for stabilizing simultaneously a finite family of $n$ systems when the order of the controller is fixed. That is the method that we have initially adopted to deal with the case of the simultaneous stabilization of a segment of systems, see [4]. Unfortunately, there does not exist off-the-shelf algorithms for solving this non-convex problem formulated as BMIs. However, an algorithm has been developed for approximating the BMI constraints as a special linear matrix inequality (LMI) problem with a rank-one constraint, we can not be pleased of this close solution. This is the reason why we have looked for another approach using no approximation methods.

Similarly to the simultaneous control formulated in the ring of stable rational functions as an unit interpolation issue in $RH_\infty$, see [5], [6], [7] and [8], we can also express the simultaneous control as a polynomial interpolation problem in the space of stable polynomials. The originality of our contribution is to give a tractable method to synthesize simultaneous compensators of a segment of systems by using a polynomial interpolation method producing stable polynomial interpolants when their even (or odd) part is fixed. The proposed approach highlights results that have been studied in [9] and [10] in regards to the polynomial interpolation and the polynomial stability. Solutions have been adapted to the context of the simultaneous stabilization of a convex family of SISO (single-input single-output) linear systems defined as a segment of systems.

This question of the simultaneous stabilization of a segment of systems was initially tackled by [11], [12] but no tractable and complete conditions to check the simultaneous stabilizability of such systems have been shown. [13] and [14] have addressed the question of the strong stabilization for this class of systems. These authors have stated existence conditions of stable compensators being able to stabilize strongly each element of this family. That does not imply existence conditions of a single controller stabilizing the whole set of systems belonging to this segment. Conditions for the simultaneous stabilizability of such plants have been given in [15]. We can note that for the stabilization of a segment of systems, [16] suggests a parameter-dependent linear controller stabilizing the range of transfer functions corresponding to the interval of parameter values.

The paper is organized as follows. In section II, the problem of the simultaneous stabilization of two plants with the same even part for each closed loop characteristic polynomial is stated. Thereafter, an application of this case of simultaneous stabilization is shown for a segment of systems. In section III, some results are studied in the polynomial space to solve this problem. Conditions for the interpolation of stable polynomials with a fixed even part (or odd part) are presented in section IV. Section V contains a brief result giving existence conditions of proper simultaneous compensators. Finally in section VI, a simple algorithm is obtained permitting to design simultaneous controllers for a segment of systems.

II. PROBLEM FORMULATION

A. Preliminaries

Notations : $P$, $H$ and $\delta(A)$ denote the set of real polynomials, the set of Hurwitz (stable) polynomials and the degree of a real polynomial $A$, respectively.

Consider $\{N_i, D_i, N_j, D_j, X, Y\} \in P$

Definition 1: Stabilization of a plant.

We say that a proper compensator $C = X/Y$ stabilizes a proper SISO plant $G_i = N_i/D_i$ iff $\Phi(G_i,C) \in H$ where $\Phi(G_i,C) = N_iX + D_iY$.

Definition 2: Simultaneous stabilization of two plants.

If there exists a proper compensator $C$ such that $\Phi(G_i,C) \in H$ and $\Phi(G_j,C) \in H$ where $G_i = N_i/D_i$ and $G_j = N_j/D_j$ are two proper SISO plants then $G_i$ and $G_j$ are said simultaneously stabilizable.
Consider a controller \( C \) defined by relation (1) where \( Q_n \in \mathbb{P} \), \( Q_d \in \mathbb{H} \) and \( C_i = X_i/Y_i \) is a proper compensator satisfying \( \Phi(G_i, C) \in \mathbb{H} \). Sufficient conditions are given in Lemma 1 of Section V that does not depend of parameters \( Q_n \) and \( Q_d \) to test the causality of the compensator \( C_i \).

Let us express a simple condition to stabilize \( G_j = N_j/D_j \) with the compensator \( C \) given in (1).

**Theorem 1:** The compensator \( C \) stabilizes \( G_j \) iff there exists a polynomial \( Q_n \) satisfying (2)

\[
\left( \Phi(G_j, \widetilde{C}_i) + Q_n \Delta \right) \in \mathbb{H},
\]

where \( \Delta = N_j D_i - D_j N_i \) and \( \widetilde{C}_i = Q_d X_i/Q_d Y_i \).

**Proof:** According to Definition 1, we write \( \Phi(G_j, C) \) as

\[
\Phi(G_j, C) = N_j (Q_d X_i + Q_n D_i) + D_j (Q_d Y_i - Q_n N_i)
\]

If the compensator \( C \) satisfies \( \Phi(G_j, C) \in \mathbb{H} \) then relation (2) holds. Conversely, if relation (2) is true, then there exists a compensator \( C \) defined by (1) such that \( \Phi(G_j, C) \in \mathbb{H} \).

**Definition 3:** Simultaneous stabilization of two plants with a compensator \( C \).

Consider two proper plants \( G_i \) and \( G_j \). If there exists a compensator \( C \) such that \( \Phi(G_i, C) \in \mathbb{H} \) and \( \Phi(G_j, C) \in \mathbb{H} \) then \( G_i \) and \( G_j \) are said simultaneously stabilizable.

**Corollary 1:** The compensator \( C \) given in (1) is a simultaneous compensator for the systems \( G_i \) and \( G_j \) iff relation (2) holds.

**Proof:** By hypothesis the controller \( \widetilde{C}_i \) stabilizes \( G_i \), then we have \( \Phi(G_i, C) \in \mathbb{H} \). Moreover, if relation (2) is true then \( \Phi(G_j, C) \in \mathbb{H} \). Consequently \( C \) is a simultaneous compensator for the systems \( G_i \) and \( G_j \). Conversely, if \( C \) is a simultaneous compensator for the systems \( G_i \) and \( G_j \) then \( \Phi(G_j, C) \in \mathbb{H} \) and relation (2) holds.

**B. Problem statement**

In this paper, we study the existence of a simultaneous compensator \( C \) for two plants \( G_i \) and \( G_j \) in the case where the two closed loop characteristic polynomials \( \Phi(G_i, C)(s) = \Phi(G_i, C)^e(s^2) + s \Phi(G_i, C)^o(s^2) \) and \( \Phi(G_j, C)(s) = \Phi(G_j, C)^e(s^2) + s \Phi(G_j, C)^o(s^2) \) satisfy \( \Phi(G_i, C)^e = \Phi(G_j, C)^e \) or \( \Phi(G_i, C)^o = \Phi(G_j, C)^o \), i.e., \( \Phi(G_i, C) \) and \( \Phi(G_j, C) \) have the same even part (or the same odd part).

In the sequel of this paper, the two polynomials \( \Phi(G_i, C) \) and \( \Phi(G_j, C) \) are assumed to be of same degree. Let us remark that \( \Phi(G_i, C) \) and \( \Phi(G_j, C) \) have the same even parts (and the same odd parts) thus \( \Phi(G_i, C) = \Phi(G_j, C) \).

In the following theorem and the next corollaries, we show the interest of dealing with this issue. So, we prove that if there exists a simultaneous compensator \( C \) that stabilizes the plants \( G_i \) and \( G_j \) such that \( \Phi(G_i, C)^e = \Phi(G_j, C)^e \) then this compensator stabilizes the segment of systems \( G_\lambda \) given by (3) where \( \lambda \in [0, 1] \).

\[
G_\lambda = \frac{\lambda N_i + (1-\lambda) N_j}{\lambda D_i + (1-\lambda) D_j}
\]

For that, consider the closed loop system \( G_\lambda \) with the controller \( C \) defined in (1) and study the characteristic polynomial \( \Phi(G_\lambda, C) \) given by (4) where \( \lambda \in [0, 1] \).

\[
\Phi(G_\lambda, C) = \lambda \Phi(G_i, C) + (1-\lambda)\Phi(G_j, C)
\]

**Theorem 2:** [10]. Let be two polynomials \( \Phi(G_i, C) \) and \( \Phi(G_j, C) \) with the leading coefficients of same sign and with the same even part or the same odd part (i.e \( \Phi(G_i, C)^e = \Phi(G_j, C)^e \) or \( \Phi(G_i, C)^o = \Phi(G_j, C)^o \)). Then the polynomial segment \( \Phi(G_\lambda, C) \) defined by (4) is stable iff \( \Phi(G_i, C) \) and \( \Phi(G_j, C) \) are stable.

Note that the leading coefficient of a polynomial is defined as its coefficient of the highest degree.

**Corollary 2:** If \( C \) is a simultaneous compensator for the two plants \( G_i \) and \( G_j \) satisfying \( \Phi(G_i, C)^e = \Phi(G_j, C)^e \) (or \( \Phi(G_i, C)^o = \Phi(G_j, C)^o \)) and such that their leading coefficient has the same sign then \( C \) stabilizes the segment of systems \( G_\lambda \).

**Proof:** That is a consequence of Theorem 2.

**Corollary 3:** Consider a simultaneous compensator given by (1) with \( Q_n \) unknown and a segment of systems \( G_\lambda \). If there exists a polynomial \( Q_n \) satisfying (5) with the leading coefficients of \( \Phi(G_i, C) \) and \( \Phi(G_j, C) \) of same sign then \( C \) is a simultaneous compensator for the segment of systems \( G_\lambda \).

\[
\left\{ \begin{array}{l}
\left( \Phi(G_j, \widetilde{C}_i) + Q_n \Delta \right) \in \mathbb{H}, \\
\Phi(G_j, C)^e = \Phi(G_j, \widetilde{C}_i)^e
\end{array} \right.
\]

**Proof:** That is a direct consequence of Corollary 2.

Our contribution in this paper is principally concerned with the theoretical study of existence conditions of a simultaneous linear time invariant controller stabilizing the segment of systems (3) by considering Corollary 3.

**III. Existence Conditions of a Compensator \( C \).**

From Corollary 3, the issue of existence of a simultaneous compensator \( C \) stabilizing \( G_i \) and \( G_j \) such that \( \Phi(G_i, C)^e = \Phi(G_j, C)^e \) is equivalent to that of existence of a polynomial \( Q_n \) satisfying relations (5).

It is interesting to note that if \( \Delta \in \mathbb{H} \) then an obvious solution at this problem is given by choosing for instance \( Q_d = \Delta \) and \( \Phi(G_j, C) = Q_d \Phi(G_j, C_i) \) where \( \Phi(G_j, C_i) \) is any polynomial in \( \mathbb{H} \). These conditions yield to \( \Phi(G_i, C)^e = \Phi(G_j, C)^e \). By considering (2), we get a polynomial \( Q_n \) given by (6)

\[
Q_n = \Phi(G_j, C_i) - \Phi(G_j, C_i)
\]

If \( \Delta \notin \mathbb{H} \), the determination of \( Q_n \) and \( \Phi(G_j, C) \in \mathbb{H} \) is not so easy. In this case, there exists a polynomial \( Q_n \) such that (2) iff there exists a stable polynomial \( \Phi(G_j, C) \) such that

\[
\Phi(G_j, C) = \Phi(G_j, \widetilde{C}_i) + Q_n \Delta
\]
Consequently \( \Phi(G_j, C) - \Phi(G_j, \tilde{C}_i) \) must be divisible by \( \Delta \) where \( \tilde{C}_i = Q_d X_i/Q_d Y_i \). Equivalently, this means that there exists \( Q_n \) iff there exists a stable polynomial \( \Phi(G_j, C) \) that interpolates the values of \( \Phi(G_j, \tilde{C}_i) \) at the zeros of \( \Delta \) with its multiplicities. This provides the following theorem.

**Theorem 3:** There exists a simultaneous compensator \( C \) given in (1) for the two systems \( G_i \) and \( G_j \) satisfying \( \Phi(G_j, C)^e = \Phi(G_i, C)^e \) iff there exists a stable polynomial \( \Phi(G_j, C) \) with \( \Phi(G_j, C)^e = \Phi(G_i, C)^e \) that interpolates the values of \( \Phi(G_j, \tilde{C}_i) \) at the zeros of \( \Delta \) and its multiplicities.

**Proof:** To show this result, it suffices to write \( \Phi(G_j, C) \) as (7) and to consider the two given polynomials \( \Delta \) and \( \Phi(G_j, \tilde{C}_i) \) as the divisor and the remainder respectively of an euclidean division in the ring of polynomials. Note that the unknown pair \( (\Phi(G_j, C), Q_n) \) representing the dividend and the quotient of (7) is not necessarily unique. We deduce that \( \Delta \) must be a factor of \( \Phi(G_j, \tilde{C}_i) - \Phi(G_j, C) \) to assure that \( Q_n \) is a polynomial where \( \Phi(G_j, \tilde{C}_i) \in H \) such that \( \Phi(G_j, C)^e \) is given (i.e \( \Phi(G_j, C)^e = \Phi(G_i, \tilde{C}_i)^e \) and \( \Phi(G_j, C)^e \) is a polynomial to determine. In this case, there exists a simultaneous compensator \( C \) for \( G_i \) and \( G_j \) such that \( \Phi(G_i, C) \) and \( \Phi(G_j, C) \) have the same even part (or odd part). Moreover, \( \Phi(G_j, C) \) interpolates the values of \( \Phi(G_i, C) \) and its multiplicities at the zeros of \( \Delta \). Conversely, if \( \Phi(G_j, C) \) is a stable polynomial that interpolates the values of \( \Phi(G_j, \tilde{C}_i) \) at the zeros of \( \Delta \) with its multiplicities and satisfies \( \Phi(G_i, C)^e = \Phi(G_j, C)^e \) then there exists a polynomial \( Q_n \) verifying (7). We deduce that there exists a simultaneous compensator \( C \) for \( G_i \) and \( G_j \).

**Corollary 4:** If there exists a compensator \( C \) such that the two following conditions i) and ii) hold then \( C \) stabilizes the segment of systems \( G_i, G_j \) given by (3).

i) \( \Phi(G_j, C) \) and \( \Phi(G_i, C) \) are two stable polynomials with their leading coefficient of same sign.

ii) \( \Phi(G_i, C) \in H \) interpolates the values of \( \Phi(G_j, \tilde{C}_i) \) at the zeros of \( \Delta \) with its multiplicities and verifies \( \Phi(G_i, C)^e = \Phi(G_j, C)^e \).

**Proof:** It is an immediate consequence of Corollary 3 and Theorem 3. Note that according to the parametrization of \( C \), we have \( \Phi(G_i, C) = \Phi(G_i, C_i) \) then \( \Phi(G_i, C)^e = \Phi(G_i, C_i)^e \). From now on, we detail the conditions of Corollary 4.

Let \( \sigma_\nu \) be the zeros of \( \Delta \) and \( \mu_\nu \) the multiplicity of these zeros. Now define \( \beta_\nu, \sigma \) as

\[
\frac{d}{ds} \Phi(G_j, \tilde{C}_i)(\sigma_\nu) = \beta_\nu, \sigma
\]

with \( \nu = 0, \ldots, \mu_\nu - 1 \) and \( \nu = 1, \ldots, n \) and where the derivative of order zero of \( \Phi(G_j, \tilde{C}_i) \) is taken as \( \Phi(G_j, \tilde{C}_i) \) itself. The problem amounts to find a stable polynomial \( \Phi(G_j, C) \) such that

\[
\begin{align*}
\Phi(G_j, C)^e & = \Phi(G_i, \tilde{C}_i)^e \\
\frac{d}{ds} \Phi(G_j, C)(\sigma_\nu) & = \beta_\nu, \sigma
\end{align*}
\]

(8)

In other words, we search to know whether or not there exists a stable polynomial \( \Phi(G_j, C) \) with \( \Phi(G_j, C)^e = \Phi(G_i, \tilde{C}_i)^e \) such that its functional and derivative values at specified point \( \sigma_\nu \), in the complex plane are equal to specified values \( \beta_\nu, \sigma \). Some existence conditions of stable interpolation polynomial have been shown in [9].

IV. INTERPOLATION CONDITIONS OF A STABLE POLYNOMIAL WITH A FIXED EVEN PART.

The purpose of this paper is not to examine the numerous necessary conditions that preserve polynomial stability for this interpolation problem but to derive a general approach to construct stable interpolating polynomials \( \Phi(G_j, C) \) such that \( \Phi(G_j, C)^e = \Phi(G_i, \tilde{C}_i)^e \). In this sense, are given existence conditions of a stable polynomial \( \Phi(G_j, C) \) such that \( \Phi(G_j, C)^e = \Phi(G_i, \tilde{C}_i)^e \) satisfying the interpolation constraints \( \Phi(G_j, C)(\sigma_\nu) = \beta_\nu \) where \( \sigma_\nu \) are distinct zeros of \( \Delta \) with \( \nu \in \{1, \ldots, n\} \).

A. Zeros Interlacing Property and stable polynomials

In this first part, a link between roots interlacing and interpolation of stable polynomials is established by using the results of polynomial stability developed by Hermite-Biehler, see [17]. Before to expand this approach, recall the “Zeros Interlacing Property” of two real polynomials \( f^e \) and \( f^o \) where \( f(s) = f^e(s^2) + sf^o(s^2) \) and define the Cauchy index of a real rational function \( R \).

**Definition 4:** [17], Zeros Interlacing Property.

Let \( f^e(u) \) be of degree \( k \) and \( f^o(u) \) of degree \( k - 1 \) (or \( k \), two real polynomials. Let us assume the roots of these polynomials defined by the following sets

\[
\begin{align*}
\text{root}(f^e(u)) & = \{a_1, \ldots, a_k\} \\
\text{root}(f^o(u)) & = \{b_1, \ldots, b_{k-1}\} \\
(\text{or root}(f^o(u)) & = \{b_1, \ldots, b_k\}
\end{align*}
\]

Then \( f^e(u) \) and \( f^o(u) \) interlace iff

- The roots of \( f^e(u) \) and \( f^o(u) \) are real, negative and simple.
- The leading coefficients of \( f^e(u) \) and \( f^o(u) \) have the same sign.
- The \( k \) roots of \( f^e(u) \) alternate with the \( k - 1 \) (or \( k \)) roots of \( f^o(u) \) as follows

\[
\begin{align*}
a_1 < b_1 < a_2 < b_2 \ldots a_{k-1} < b_{k-1} < a_k < 0 \\
(\text{or} \ b_1 < a_1 < b_2 \ldots < b_k < a_k < 0)
\end{align*}
\]

**Definition 5:** Cauchy index.

The Cauchy index for a pole \( u_p \) of a real rational function \( R \) is defined as the number \( I_{u_p}(R) \) such that

\[
I_{u_p}(R) = \begin{cases} +1, & \text{if } \lim_{u \to u_p^-} R(u) = -\infty \land \lim_{u \to u_p^+} R(u) = +\infty, \\
-1, & \text{if } \lim_{u \to u_p^-} R(u) = +\infty \land \lim_{u \to u_p^+} R(u) = -\infty, \\
0, & \text{otherwise.} \end{cases}
\]

A generalization over the compact interval \((a, b)\) is direct. We have

\[
I_{u_p}^b(a) := \sum_{u_p \in (a, b)} I_{u_p}(R)
\]

The relationship between Zeros Interlacing Property and Hurwitz stability is emphasized by the Hermite-Biehler’s Theorem.
Theorem 4: [17], The three following assertions are equivalent:

i) The real polynomial $\Phi(G_j, C)$ is Hurwitz (or stable),

ii) All the real parts of the roots of $\Phi(G_j, C)$ are strictly negative,

iii) The polynomials $\Phi(G_j, C)^c(u)$ and $\Phi(G_j, C)^o(u)$ verify the Zeros Interlacing Property.

Consequently, by using the Cauchy index previously defined, a rational function is associated to $\Phi(G_j, C)^c(u)$ and $\Phi(G_j, C)^o(u)$ and a simple condition is given for testing the stability of $\Phi(G_j, C)$.

Theorem 5: The real polynomial $\Phi(G_j, C)$ of degree $m$ ($m = 2k$ or $m = 2k + 1$) is stable iff the roots $\{a_1, \ldots, a_k\}$ of $\Phi(G_j, C)^c$ are real, simple and negative and all real $c_l$ in (9) are positive with in the case $m = 2k$ $c_l = 0$.

$$\Phi(G_j, C)^o(u) = c_0\Phi(G_j, C)^c(u) + \sum_{l=1}^{k} c_l \frac{\Phi(G_j, C)^c(u)}{u - a_l}$$

(9)

Proof: By the Hermite-Bielih theorem, we know that if $\Phi(G_j, C)(u)$ is stable then $\Phi(G_j, C)^c(u)$ has real, negative and simple roots $a_l$. Assume $\delta(\Phi(G_j, C)^o(u)) \leq \delta(\Phi(G_j, C)^c(u))$ then $R(u) = \Phi(G_j, C)^c(u)$ is a proper rational function that may be expressed as a partial fraction decomposition as it follows

$$\left\{ \begin{array}{l}
R(u) = c_0 + \sum_{l=1}^{k} \frac{c_l}{u - a_l}, \\

c_l = \frac{\Phi(G_j, C)^o(a_l)}{\delta(\Phi(G_j, C)^o)}(a_l)
\end{array} \right.$$

The polynomials $\Phi(G_j, C)^o(u)$ and $\Phi(G_j, C)^c(u)$ verify the Zeros Interlacing Property iff the Cauchy index of the rational function $R(u)$ is equal to the degree of $\Phi(G_j, C)^c(u)$ ($+1$ if $\delta(\Phi(G_j, C)^c(u)) = \delta(\Phi(G_j, C)^o(u))$), see [17]. Consequently $\Phi(G_j, C)^c(u)$ and $\Phi(G_j, C)^o(u)$ interlace iff for any $l$, $c_l > 0$.

B. Main result

Relation $\Phi(G_j, C)(s) = \Phi(G_j, C)^c(s^2) + s \Phi(G_j, C)^o(s^2)$ yields to expression (10) with $c_0 = 0$ in the case $m = 2k$.

$$\beta_\nu = \Phi(G_j, C)^c(\sigma^2_\nu) + \sigma_\nu \left(c_0\Phi(G_j, C)^c(\sigma^2_\nu) + \sum_{l=1}^{k} c_l \frac{\Phi(G_j, C)^c(\sigma^2_\nu)}{\sigma^2_\nu - a_l} \right)$$

(10)

Relation (10) implies (11).

$$\left\{ \begin{array}{l}
\beta_1 = \prod_{l=1}^{k}(\sigma^2_1 - a_l) - 1 = c_0\sigma_1 + c_1 \frac{\sigma_1}{\sigma^2_1 - a_1} + \ldots + c_k \frac{\sigma_1}{\sigma^2_k - a_k} \\
\vdots \\
\beta_n = \prod_{l=1}^{k}(\sigma^2_n - a_l) - 1 = c_0\sigma_n + c_1 \frac{\sigma_n}{\sigma^2_1 - a_1} + \ldots + c_k \frac{\sigma_n}{\sigma^2_k - a_k}
\end{array} \right.$$

(11)

Therefore, the interpolation problem of a stable polynomial $\Phi(G_j, C)$ is expressed as a system of equations (11) to solve where the unknown are the set of positive parameters $c_l, l \in \{1 \ldots k\}$. By hypothesis, the set of distinct negative roots $a_i, l \in \{1 \ldots k\}$ are known and are given by the even part of the stable polynomials $\Phi(G_i, \tilde{C}_i)^c = \Phi(G_i, \tilde{C}_i)^c$.

Relationship (11) is written equivalently as

$$\Lambda(a_1, \ldots, a_k) = \Psi(a_1, \ldots, a_k) \Gamma$$

(12)

with

$$\Lambda^T(a_1, \ldots, a_k) = \left[ \begin{array}{l}
\frac{\sigma_1}{\sigma^2_1 - a_1} \ldots \frac{\sigma_1}{\sigma^2_k - a_k} \\
\vdots \\
\frac{\sigma_n}{\sigma^2_1 - a_1} \ldots \frac{\sigma_n}{\sigma^2_k - a_k}
\end{array} \right]$$

For $m = 2k$ and $m = 2k + 1$, we have respectively

$$\Psi(a_1, \ldots, a_k) = \left[ \begin{array}{l}
\frac{\sigma_1}{\sigma^2_1 - a_1} \ldots \frac{\sigma_1}{\sigma^2_k - a_k} \\
\vdots \\
\frac{\sigma_n}{\sigma^2_1 - a_1} \ldots \frac{\sigma_n}{\sigma^2_k - a_k}
\end{array} \right]$$

By considering equation (12), the next theorem presents a necessary and sufficient condition that satisfies the three following interpolation constraints

1) $\Phi(G_j, C)(\sigma_\nu) = \beta_\nu$ where $\sigma_\nu, (\nu \in \{1 \ldots n\})$ are distinct zeros of $\Delta$,

2) $\Phi(G_j, C)^c$ and $\Phi(G_j, C)^o$ verify the Zeros Interlacing Property,

3) $\Phi(G_j, C)^c = \Phi(G_i, \tilde{C}_i)^c$.

Theorem 6: Let $n$ pairs of numbers be $(\sigma_\nu, \beta_\nu)$, with $\nu \in \{1 \ldots n\}$. Then $\Phi(G_j, C)$ is a stable real polynomial that interpolates all $(\sigma_\nu, \beta_\nu)$ iff there exists $\Gamma > 0$ (i.e $\forall l \in \{0, \ldots, k\}, c_l > 0$) satisfying (12) where $a_i, l \in \{1 \ldots k\}$ is a set of given negative distinct real numbers representing the roots of $\Phi(G_j, C)^c = \Phi(G_i, \tilde{C}_i)^c$.

Proof: That is an immediate consequence of Theorem 5 and equation (10).

V. Existence conditions of a proper controller.

Lemma 1: If one of the four relationships (13) is satisfied then the simultaneous compensator $C$ defined by (1) is proper for any degree of the polynomials $Q_n$ and $Q_d$ and for any proper compensator $C_i$ stabilizing $G_i$.

$$\left\{ \begin{array}{l}
\delta(D_i) = \delta(D_j) = \delta(N_j) \\
\delta(D_i) = \delta(D_j) = \delta(N_i) \\
\delta(D_j) \geq \delta(D_i) \text{ and } \delta(D_j) = \delta(N_j) \\
\delta(D_j) \geq \delta(D_i) \text{ and } \delta(D_i) = \delta(N_i)
\end{array} \right.$$

(13)

Proof: According to relation (1), the compensator $C$ is proper iff we have

$$\max(\delta(Q_d) + \delta(Y_i), \delta(Q_n) + \delta(N_i)) \geq \max(\delta(Q_d) + \delta(X_i), \delta(Q_n) + \delta(D_i))$$
Moreover, we can set
\[ \delta(Q_d) + \delta(Y_i) \geq \delta(Q_d) + \delta(X_i) \]
\[ \delta(Q_n) + \delta(D_i) \geq \delta(Q_n) + \delta(N_i) \]
 Consequently, \( C \) is proper if
\[ \delta(Q_d) + \ldots + 0.787i, -0.7579 - 1.0787i \]
Let us note that \( \Delta(s) \) has two unstable roots. We obtain
\[ \Delta(s) = s^2 + 1.5158s + 1.7380 \]

and we can state

Four cases can be distinguished:

1) If we have
\[
\left\{ \begin{array}{l}
\delta(D_i) + \delta(N_i) \geq \delta(D_j) + \delta(N_j), \\
\delta(D_i) \geq \delta(D_j)
\end{array} \right.
\]
then in this case condition (16) becomes
\[ \delta(Q_d) + \delta(Y_i) \geq \delta(Q_d) + \delta(Y_i) - \delta(D_j) - \delta(N_i) + \delta(D_j) \]
that implies \( 2\delta(D_i) - \delta(D_j) \leq \delta(N_i) \). This condition is verified iff \( \delta(D_j) = \delta(D_j) = \delta(N_i) \)

2) If we have
\[
\left\{ \begin{array}{l}
\delta(D_j) + \delta(N_i) \geq \delta(D_j) + \delta(N_j), \\
\delta(D_i) \geq \delta(D_j)
\end{array} \right.
\]
then condition (16) becomes for this case
\[ \delta(Q_d) + \delta(Y_i) \geq \delta(Q_d) + \delta(Y_i) - \delta(D_j) - \delta(N_i) + \delta(D_j) \]
That implies \( 2\delta(D_i) - \delta(D_j) \leq \delta(N_i) \). This condition is verified iff \( \delta(D_j) = \delta(D_j) = \delta(N_i) \)

3) If we have
\[
\left\{ \begin{array}{l}
\delta(D_i) + \delta(N_j) \geq \delta(D_j) + \delta(N_i), \\
\delta(D_j) \geq \delta(D_i)
\end{array} \right.
\]
then condition (16) becomes
\[ \delta(Q_d) + \delta(Y_i) \geq \delta(Q_d) + \delta(Y_i) - \delta(D_j) - \delta(N_i) + \delta(D_j) \]
That implies \( 2\delta(D_i) - \delta(D_j) \leq \delta(N_i) \). This condition is verified iff \( \delta(D_i) = \delta(D_i) = \delta(N_i) \)

4) If we have
\[
\left\{ \begin{array}{l}
\delta(D_j) + \delta(N_i) \geq \delta(D_j) + \delta(N_j), \\
\delta(D_i) \geq \delta(D_j)
\end{array} \right.
\]
then relation (16) is written as
\[ \delta(Q_d) + \delta(Y_i) \geq \delta(Q_d) + \delta(Y_i) - \delta(D_j) - \delta(N_i) + \delta(D_j) \]
That implies \( \delta(D_i) \leq \delta(N_i) \). This relation yields to \( \delta(D_i) = \delta(N_i) \)

VI. Simulation Results

A. Design of the simultaneous compensator \( C \)

In this part, from the previous results obtained, we present the different steps permitting to compute a simultaneous compensator \( C \) stabilizing a segment of systems \( G_i \).

1) Calculate \( \Delta = N_j D_i - D_j N_i \) and the roots \( \sigma_u \) of \( \Delta = \Delta - \Delta^+ \) with \( \Delta^- \) and \( \Delta^+ \) respectively the stable part and the unstable part of \( \Delta \). If \( \Delta \in H \) then the solution is obvious, see section III.

2) Find a proper controller \( \bar{C}_i = X_i / Y_i \) stabilizing \( G_i \).

3) Compute \( Q_{d,2} = P \Delta^- \) with \( P \) any stable polynomial.

4) Compute \( C_{d,2} = Q_{d,2} X_i / Q_{d,2} Y_i \) and \( \Phi(G_i, C_{d,2}) \) and extract the set of real distinct roots \( \omega_l \) of \( \Phi(G_i, C_{d,2}) \).

5) Calculate \( c_l \) from relation \( \Lambda = \Psi \Gamma \). If at least one \( c_l \) is negative, then repeat the procedure from 2) by changing the regulator \( C_{d,2} \).

6) Compute \( \Phi(G_j, C) \) and determine \( Q_n \) from relation \( \Phi(G_j, C_{d,2}) + Q_n \Delta = \Phi(G_j, C) \).

7) Finally, compute \( C \) given by (1).

At the moment, if the conditions to be verified fail in step 5, we do not know on the way to modify the controller at step 2. The existence of solutions is dependent on the choice of the initial compensator \( C_{d,2} \). Future works are needed to help in the selection of \( C_{d,2} \) and \( Q_{d,2} \). An outline would be to study the feasibility in order to guarantee the existence of negative distinct real numbers \( \omega_l \).

B. Example

An illustrative example is given hereafter.

Let \( G_i(s) \) and \( G_j(s) \) be the two endpoints of a segment of systems \( G_{\lambda}(s) \).

\[ G_i(s) = \frac{s^2 + s + 2}{s^2 + 2s + 1}, \quad G_j(s) = \frac{3s + 2}{2s^2 - s + 2}. \]

We get \( \Delta(s) = -2s^4 + 2s^3 + 3s^2 + 7s - 2 \). The roots \( \sigma_u \) of \( \Delta(s) \) are given by the set \( E \)
\[ E = \{ 2.2613, 0.2545, -0.7579 + 1.0787i, -0.7579 - 1.0787i \} \]

Let us note that \( \Delta(s) \) has two unstable roots. We obtain
\[ \Delta^- (s) = s^2 + 1.5158s + 1.7380 \]
Let $C_i(s)$ be a compensator that stabilizes $G_i$
\[
C_i(s) = \frac{s + 1}{s^2 + s + 1}
\]
Let $P(s) = s^2 + 1.02s + 0.02$. As $Q_d(s) = \Delta - \ldots$
We conclude that $\Phi(G_i, C_i(s))$ and $\Phi(G_j, C_j(s))$ are two stable polynomials of same degree with their leading coefficient of same sign and with $\Phi(G_i, C)^e(s) = \Phi(G_j, C)^e(s)$.

Consequently, the compensator $C(s)$ stabilizes the segment of systems $G_{\lambda}(s)$ defined by $G_i(s)$ and $G_j(s)$.

Let us compare these results with those of other literature methods. By considering the approach described in [4] which is inspired of [3] and adapted to the case of the simultaneous stabilization of a segment of systems, then this control problem is formulated as a BMI and approximated as a rank-one LMI optimization constraint. Unfortunately, if we apply this method to this example, we have to manage a computational complexity that does not allow to process and to look for solutions. This complexity is due to the number and to the size of the Hermite-Fujiwara matrices to take into account. In our case, their size is 8 and their number is 2*14^2.

VII. CONCLUSION

In this paper, an interpolation approach with stable polynomial interpolants has been applied to the simultaneous stabilization of a segment of systems. The proposed framework has permitted to design simultaneous controllers without conservatism for this family of systems.

REFERENCES